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ON A PROBLEM OF HURWITZ

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A. Hurwitz proposed the problem of finding all the positive integers $z, \mathbf{x} = (x_1, \dots, x_n)$ satisfying the diophantine equation $x_1^2 + \dots + x_n^2 = z \cdot x_1, \dots, x_n$. This paper investigates the question of which values of z can occur, using only the most elementary techniques. An algorithm is given for determining all permissible values of (z, n) for all n below a given bound. As an application it is established that the only possible values in the range $z \geq (n + 15)/4$ are $z = n, z = (n + 8)/3$ when n is odd, and $z = (n + 15)/4$. As another application the fifteen values of $n \leq 131,020$ for which the only permissible value of z is n have been found.

2. The problem of finding all the integer solutions $z, \mathbf{x} = (x_1, \dots, x_n)$ of the equation

$$(1) \quad x_1^2 + \dots + x_n^2 = z \cdot x_1, \dots, x_n$$

was raised by A. Hurwitz in [1]. In that paper he showed that for $n > z$ there are no solutions. This is an easy consequence of Theorem 1 (see §3) and will be replaced by the stronger result in Theorem 3. To keep this paper self-contained, let us recall the following facts from [1].

For $n = 2$, the only solutions are $z = 2, x_1 = x_2$; for upon setting $x_1 = dy_1, x_2 = dy_2$ with $(y_1, y_2) = 1, y_1^2 + y_2^2 = zy_1y_2$, and so $z = 2, x_1 = x_2 = d$.

If $z, x_1, \dots, x_j, \dots, x_n$ is a solution, then so is $z, x_1, \dots, x'_j, \dots, x_n$ where x'_j satisfies

$$x_j + x'_j = z \prod_{i \neq j} x_i.$$

The n solutions derived in this way are called the *neighbors* of z, \mathbf{x} . Define the *height* of a solution to be simply $x_1 + \dots + x_n$, and call a solution *fundamental* if its height is no greater than the height of any of its neighbors. If a solution is not fundamental, it has a neighbor of strictly smaller height, and since the heights are all positive integers, in a finite number of steps we arrive at a fundamental solution. So we see that *it suffices to study fundamental solutions*. Moreover, it obviously suffices to study solutions that satisfy

$$(2) \quad x_1 \geq x_2 \geq \dots \geq x_n \geq 1.$$

Also, as Hurwitz point out, it is easy to see that fundamental solutions

satisfying (2) are characterized by

$$(3) \quad 2x_1 \leq z \prod_{i=2}^n x_i.$$

We now propose to study the system of Equations (1), (2), (3), and shall regard n as well as z and x_1, \dots, x_n as variables. By the first remark in this section we may also assume

$$(4) \quad n \geq 3.$$

3. In this section we state our basic theorem. First some notation.

The *trivial solution* of (1)-(4) is $x_1 = \dots = x_n = 1, z = n$. Call a nontrivial solution of (1)-(4) a SOL. For any SOL, we define

$$\chi(\mathbf{x}) = \text{the largest index } i \text{ for which } x_i > 1.$$

THEOREM 1. *Let n, z, \mathbf{x} be a SOL with $k = \chi(\mathbf{x})$. There is a chain of SOLs $n^{(i)}, z, \mathbf{x}^{(i)}, i = 0, \dots, t$ such that*

(a) $\chi(\mathbf{x}) = \chi(\mathbf{x}^{(i)})$ for all i .

(b) If $z = 1$ and $k = 3$, then $n^{(0)}, \mathbf{x}^{(0)} = 3, (3, 3, 3)$. Otherwise, $n^{(0)} = 2^k \cdot z - 3k, x_1^{(0)} = \dots = x_k^{(0)} = 2$.

(c) $n^{(i)} > n^{(i-1)}$ for $i = 1, \dots, t$.

(d) $n^{(t)}, \mathbf{x}^{(t)} = n, \mathbf{x}$.

The proof is in the next section. Below we give some immediate corollaries of the theorem, using the same notation.

COROLLARY 1. k must satisfy $2^k - 3k \leq n$. [By (b), since $z \geq 1$.]

COROLLARY 2. z must satisfy $z \leq (n + 3)/2$. [By (b), since $k \geq 1$.]

COROLLARY 3. *The only fundamental solution to Equation (1) with $z \geq n$ is the trivial solution.*

4. In this section, we prove Theorem 1. First we state and prove some simple lemmas.

LEMMA 1. *Let n, z, \mathbf{x} be a SOL.*

If $z = 3$, then $\chi(\mathbf{x}) \geq 2$.

If $z \leq 2$, then $\chi(\mathbf{x}) \geq 3$.

If $z = 1$, and $\chi(\mathbf{x}) = 3$, then $x_3 \geq 3$.

Proof. If $z \leq 3$ and $\chi(\mathbf{x}) = 1$, then by (3) $2x_1 \leq 3$ which contradicts $x_1 > 1$. Hence $\chi(\mathbf{x}) \geq 2$. If $z \leq 2$ and $\chi(\mathbf{x}) \leq 2$, then by (1) $x_1^2 + x_2^2 + (n - 2) \cdot 1^2 \leq 2 \cdot x_1 \cdot x_2 \cdot 1$. Thus $(x_1 - x_2)^2 \leq 2 - n$. This contradicts (4). Finally suppose $z = 1, \chi(\mathbf{x}) = 3$ and $x_3 = 2$. Then by (1)

$x_1^2 + x_2^2 + (n + 1) = 2x_1x_2$, a contradiction.

LEMMA 2. Let n, z, x be a SOL with $k = \chi(x)$. When $z = 1$ and $k = 3$, $n^{(0)}, z, x^{(0)} = 3, 1, (3, 3, 3)$ is a SOL. Otherwise if $n^{(0)} = z \cdot 2^k - 3k$ and $x_1^{(0)} = \dots = x_k^{(0)} = 2, x_i^{(0)} = 1$ for $i = k + 1, \dots, n^{(0)}$, then $n^{(0)}, z, x$ is a SOL with $\chi(x^{(0)}) = k$.

Proof. Obviously $n^{(0)}, z, x^{(0)} = 1, 3, (3, 3, 3)$ is a SOL. As for the other cases: $\sum x_i^2 = 4k + n^{(0)} - k$ while $z \prod x_i = z \cdot 2^k$, thus the definition of $n^{(0)}$ guarantees (1). (2) and (4) are trivial while to verify (3) we must check that

$$4 \leq z \prod_{i>1} x_i$$

which is obvious when $z \geq 4$ and true for $z \leq 3$ by the constraints imposed by Lemma 1.

LEMMA 3. Let n, z, x and N, z, X be two SOLs such that

- (a) $\chi(x) = \chi(X) = k$
- (b) $X_1 > x_1$
- (c) $X_j \geq x_j$ for $j = 2, \dots, k$.

Let r be the last index j for which $X_j > x_j$. Let s' be the first index j for which $x_j < x_1$, and define $s = s'$ if $s' \leq r, s = 1$ if $s' > r$. Then m, z, w is a SOL if

$$m = n - 2x_s - 1 + z \prod_{i \neq s} x_i$$

$$w_i = x_i \text{ for } i \leq k, i \neq s$$

$$w_s = x_s + 1$$

$$w_i = 1 \text{ for } i > k.$$

Moreover $m > n$.

Proof. We use the notation \sum and \prod to denote sums and products for which the index i runs from 1 to k , and append a prime to mean that $i \neq s$.

To check that w really is a SOL we must check (1) and (3). Now by (1) $\sum x_i^2 = z \prod x_i - (n - k)$. Thus $\sum w_i^2 = z \prod w_i = z \prod' x_i - (n - k) + 2x_s + 1$. So by the definition of m , (1) is satisfied.

If $s > 1$, then since x satisfies (3) so will w . We may therefore assume $s = 1$. By the definition of $s, x_1 = \dots = x_r$ and $x_{r+1} = X_{r+1}, \dots, x_k = X_k$. Thus either (i) $r = 1$ or (ii) $r \geq 2$ and $x_1 = x_2$. In case (i) we note that N, z, X satisfies (3), that $z \prod' w_i = z \prod' X_i$, and that $X_1 > x_1$ implies $2X_1 \geq 2(x_1 + 1) = 2w_1$. Thus w satisfies (3). In case (ii) we must check that $z \prod' x_i \leq 2x_1 + 2$. Dividing by $x_1 = x_2$ and

recalling that $x_1 \geq 2$ we see that it suffices to know that

$$z \prod_{i=3}^k x_i \geq 3. \quad (\text{The empty product equals 1.})$$

This is certainly true if $z \geq 3$ and easily checked via the constraints of Lemma 1 when $z < 3$.

Finally we note that $m > n$ is equivalent to $z \prod x_i \geq 2(x_s + 1)$. Multiplying by x_s we see that it suffices to show $z \prod x_i \geq 2(x_s^2 + x_s)$ and since $x_1 \geq x_s$ it suffices to prove this when $s = 1$. Dividing by x_1 we obtain Equation (3) for w , which was verified above.

Proof of Theorem 1. The $n^{(0)}$, z , $\mathbf{x}^{(0)}$ defined in (b) is a SOL by Lemma 2. If $(x_1, \dots, x_k) \neq (x_1^{(0)}, \dots, x_k^{(0)})$, we apply Lemma 3 (with $s = 1$) to obtain a SOL $n^{(1)}$, z , $\mathbf{x}^{(1)}$, with $n^{(1)} > n^{(0)}$. By induction: At step i , if $r > 1$, we will have either $x_1^{(i)} = \dots = x_{s-1}^{(i)} > x_s^{(i)} = \dots = x_r^{(i)}$ where $x_{s-1}^{(i)} = x_s^{(i)} + 1$, or $x_1^{(i)} = x_r^{(i)}$ and $s = 1$. Hence we will be able to apply Lemma 3. When $r = 1$, at $i = t$ say, we have $(x_1, \dots, x_k) = (x_1^{(t)}, \dots, x_k^{(t)})$ and by (1) both n and $n^{(t)}$ equal

$$k + z \prod_{i=1}^k x_i - \sum_{i=1}^k x_i^2.$$

Hence, $n^{(t)}$, z , $\mathbf{x}^{(t)} = n$, z , \mathbf{x} .

5. The following corollary is an easy consequence of the proof of Theorem 1.

COROLLARY 4. *Every SOL n , z , \mathbf{x} satisfies $n \geq x_1$.*

Proof. (We use the notation of Theorem 1.) To construct $\mathbf{x}^{(i+1)}$ from $\mathbf{x}^{(i)}$ we applied Lemma 3. Thus for $1 \leq j \leq k$

$$x_j^{(i+1)} - x_j^{(i)} = \begin{cases} 0 & \text{if } j \neq s \\ 1 & \text{if } j = s. \end{cases}$$

Since $n^{(i+1)} > n^{(i)}$,

$$\sum_{j=1}^k x_j^{(i+1)} - x_j^{(i)} \leq n^{(i+1)} - n^{(i)}.$$

Summing these equations for $i = v, \dots, t$ we get

$$(5) \quad n^{(t)} = n \geq n^{(v)} + \sum_{j=1}^k x_j - x_j^{(v)} \geq n^{(v)} + x_1 - x_1^{(v)}.$$

If $z \neq 1$ or $\chi(\mathbf{x}) \neq 3$, then $x_1^{(0)} = 2$ and $n^{(0)} \geq 4$. Thus by (5), $n \geq x_1 + 2$. If $z = 1$ and $\chi(\mathbf{x}) = 3$, then $n^{(0)}$, $\mathbf{x}^{(0)} = 3, (3, 3, 3)$; $n^{(1)}$, $\mathbf{x}^{(1)} = 5, (4, 3, 3, 1, 1)$; and $n^{(2)}$, $\mathbf{x}^{(2)} = 10, (4, 4, 3, 1, \dots, 1)$. Thus the

corollary is true for $x = \mathbf{x}^{(0)}$ or $\mathbf{x}^{(1)}$. Setting $\nu = 2$ in (5), we have $n \geq x_1 + 6$ otherwise.

6. Lemma 3 and Theorem 1 yield an algorithm that produces only SOLs, and each only once.

THEOREM 2. *The following seven step algorithm constructs all SOLs n, z, \mathbf{x} with $n \leq M$.*

Let A be a list of SOLs, initially empty. The set of SOLs put into A will be the SOLs sought.

(1) *Set $k = 1$ and $z = 4$.*

(2) *Using the current values of z and k , put the SOL constructed in Lemma 2 on the bottom of the list A .*

(3) *If A is empty, go to Step 6, otherwise remove the top SOL n, z, \mathbf{x} from A .*

(4) *Define $w_1 = x_1 + 1, w_i = x_i$ for $i \geq 2, k = \chi(\mathbf{x})$ and*

$$\nu = z \prod_{i=2}^k w_i - 2w_1 + 1.$$

Let $m = n + \nu$. If $n < m < M$ define $w_i = 1$ for $i = n + 1, \dots, m$. m, z, \mathbf{w} is a new SOL. Put it on the bottom of A . (If m is not between n and M we do nothing.)

(5) *Find the smallest index $s \leq k$ satisfying $x_1 - x_s = 1$. If no such s exists, go to Step 3; otherwise define $w_s = x_s + 1, w_i = x_i$ for $i \neq s, k = \chi(\mathbf{x})$ and*

$$\nu = z \prod_{i \neq s}^k w_i - 2w_s + 1.$$

Let $m = n + \nu$. If $m > M$ go to Step 3. If $m \leq M$ define $w_i = 1$ for $i = n + 1, \dots, m$. m, z, \mathbf{w} is a new SOL (since $n < m$ is always true). Put it on the bottom of A and go to Step 3.

(6) *Increase z by 1 and set $\nu = z \cdot 2^k - 3k$. If $\nu \leq M$ go to Step 3, otherwise go to Step 7.*

(7) *Increase k by 1. If $k = 2$, set $z = 3$, otherwise set $z = 1$. Set $\nu = z \cdot 2^k - 3k$. If $\nu \leq M$ go to Step 2, otherwise stop.*

Proof. Every SOL n, z, \mathbf{x} satisfying $n \leq M$ eventually is put on A because the algorithm produces a unique sequence of SOLs passing through the $\chi(\mathbf{x}) = k$ SOLs of the form $m, z, \mathbf{w}^{(j)}$ where $\chi(\mathbf{w}^{(j)}) = k$ and

$$\mathbf{w}^{(j)} = (x_j, \dots, x_j, x_{j+1}, x_{j+2}, \dots, x_k, 1, \dots, 1).$$

(Uniqueness is guaranteed by Step 5.)

Theorem 2 is extremely powerful, and it is no trouble to produce

a table of SOLs by hand for moderately large n . The Appendix lists all solutions of (1)-(4) with $n \leq 45$ except the trivial solution (when $z = n$). We have omitted those x_i which equal 1.

7. In this section, we will apply Theorem 2 to get a better bound on z than that given by Corollary 2.

Suppose n, z, \mathbf{x} is a SOL with $k = \chi(\mathbf{x})$, and suppose $n \neq 2z - 3$. In particular, if $k = 1$, then $n \neq n^{(0)}$. Hence either (i) $k \geq 2$ or (ii) $k = 1$ and $n \geq n^{(1)}$. In case (i) by Theorem 1 (b)

$$z \leq (n + 3k)/2^k \leq (n + 6)/4 .$$

In case (ii) since $n^{(0)} = 2z - 3$ and $n^{(1)} = n^{(0)} + z - 5$, we see that $z \leq (n + 8)/3$. Now if $n \geq 14$, $(n + 8)/3 \leq (n + 6)/4$, while for $n \leq 14$, $z \leq (n + 8)/3$ by inspection.

THEOREM 3. *The only SOLs n, z, \mathbf{x} with $z > (n + 8)/3$ are the SOLs with n odd, $z = (n + 3)/2$, $\mathbf{x} = (2, 1, \dots, 1)$.*

Proof. Since n even implies $n \neq 2z - 3$, there are no SOLs with $z > (n + 8)/3$. If n is odd and $n = 2z - 3$, then $\chi(\mathbf{x}) = 1$ and $n = n^{(0)}$, $\mathbf{x} = \mathbf{x}^{(0)}$ of Theorem 1 (b).

Theorem 3 is hardly the best possible. For any n , each SOL n, z, \mathbf{x} is the end point of one of the chains described in Theorem 1, and in general, the longer the chain, the larger n must be compared to z . So for example if $n \geq n^{(2)}$, $z \leq (n + 15)/4$ when $\chi(\mathbf{x}) = 1$ and if $\chi(\mathbf{x}) \geq 2$ and $z \geq 3$, then $z \leq (n + 10)/8$. Thus there are no solutions to (1) when $(n + 8)/3 > z > (n + 15)/4$, etc. .

8. Hurwitz asked if there exists n for which the only solutions to (1) have $z = n$. There are.

PROPOSITION. *There are 15 values of $n \leq 301020$ for which (1)-(4) has no nontrivial solutions. They occur when $n = 12, 24, 32, 48, 60, 108, 240, 384, 480, 608, 972, 984, 1020$, and 2688.*

This is the result of a computer program implementing Theorem 2. Suppose a computer has b binary bits per word. Since one only wants to remember which n have at least one SOL, this information can be stored in a single bit. Hence at most $[n/b] + 1$ words are needed to keep track of which n have a SOL. Suppose $\chi(\mathbf{x}) = k \geq 17$, then $2^k - 3k \geq 301,021$. Thus all SOLs for which $n \leq 301,020$ have $k \leq 16$. By Theorem 3, $z < 2^{16}$. It is possible to show that for $n \geq 55$, $x_1 < \sqrt{2n}$. Hence $x_1 < 2^9$. Thus, if $b \geq 25$, n, z , and k can be

packed into one computer word, and x_1, \dots, x_k can be packed $[b/8]$ to a computer word. So e.g., if $b = 25$, no more than six computer words are needed for the x_i . The list A of active solutions will not grow too large if the solutions are packed in this way. Finally let me comment that removing SOLs from the end of A , rather than the beginning (see Step 3) will save considerable computing time, since the stack A need not be "pushed down" after a SOL is removed. Moreover, if the last entry for each SOL is the word containing (n, z, k) , then upon removing the last word of A one knows how many words were needed to store x_1, \dots, x_k .

It is tempting to conjecture that there is at least one SOL for all $n > 2688$.

PROPOSITION. *There are nontrivial solutions to (1) whenever $n \equiv 1 \pmod{u}$ and $n > u^2$, or $n \equiv 2 \pmod{u^2}$ for any integer $u > 1$.*

Proof. If n, z, \mathbf{x} is a SOL with $\lambda(\mathbf{x}) = k$, then so is $n' = n + d \prod x_i, z' = z + d, \mathbf{x}' = (x_1, \dots, x_k, 1, \dots, 1)$ for any $d \geq 0$. Apply this fact to the SOLs, $n = u^2 + 2, z = 3, \mathbf{x} = (u, 1, \dots, 1)$ and the SOLs, $n = u^2 + 2, z = 3, \mathbf{x} = (u, u, 1, \dots, 1)$.

COROLLARY. *If (1) has only trivial solutions, then $n \equiv 0$ or $8 \pmod{12}$.*

[Set $u = 2, 3$.]

I take this opportunity to thank Ed Bender for many valuable discussions.

APPENDIX

(See the end of Section 6.)

| N | Z | X1 | X2 | X3 | X4 | X5 | N | Z | X1 | X2 | X3 | X4 | X5 |
|----|------|----|----|----|----|----|----|------|----|----|----|----|----|
| 3 | 1 | 3 | 3 | 3 | | | 22 | 2 | 4 | 3 | 2 | | |
| 4 | 1 | 2 | 2 | 2 | 2 | | | 3 | 3 | 2 | 2 | | |
| 5 | 1 | 4 | 3 | 3 | | | | 3 | 6 | 4 | | | |
| | 4 | 2 | | | | | | 5 | 4 | 2 | | | |
| 6 | 3 | 2 | 2 | | | | | 7 | 2 | 2 | | | |
| 7 | 1 | 3 | 2 | 2 | 2 | | | 10 | 3 | | | | |
| | 2 | 2 | 2 | 2 | | | 23 | 1 | 6 | 3 | 2 | 2 | |
| | 3 | 3 | 2 | | | | | 1 | 6 | 5 | 3 | | |
| | 5 | 2 | | | | | | 4 | 2 | 2 | 2 | | |
| 8 | 1 | 4 | 2 | 2 | 2 | | | 5 | 5 | 2 | | | |
| 9 | 6 | 2 | | | | | | 13 | 2 | | | | |
| 10 | 1 | 4 | 4 | 3 | | | 24 | NONE | | | | | |
| | 2 | 3 | 2 | 2 | | | 25 | 1 | 7 | 5 | 3 | | |
| | 4 | 2 | 2 | | | | | 2 | 5 | 3 | 2 | | |
| | 6 | 3 | | | | | | 4 | 4 | 3 | | | |
| 11 | 2 | 4 | 2 | 2 | | | | 6 | 3 | 2 | | | |
| | 3 | 3 | 3 | | | | | 10 | 4 | | | | |
| | 7 | 2 | | | | | | 11 | 3 | | | | |
| 12 | NONE | | | | | | | 14 | 2 | | | | |
| 13 | 1 | 5 | 4 | 3 | | | 26 | 1 | 5 | 4 | 4 | | |
| | 3 | 4 | 3 | | | | | 2 | 6 | 3 | 2 | | |
| | 4 | 3 | 2 | | | | | 8 | 2 | 2 | | | |
| | 7 | 3 | | | | | | 10 | 5 | | | | |
| | 8 | 2 | | | | | 27 | 1 | 3 | 3 | 3 | 2 | |
| 14 | 1 | 3 | 3 | 2 | 2 | | | 3 | 4 | 2 | 2 | | |
| | 1 | 6 | 4 | 3 | | | | 3 | 5 | 5 | | | |
| | 4 | 4 | 2 | | | | | 15 | 2 | | | | |
| | 5 | 2 | 2 | | | | 28 | 1 | 3 | 2 | 2 | 2 | 2 |
| 15 | 3 | 2 | 2 | 2 | | | | 1 | 4 | 4 | 2 | 2 | |
| | 9 | 2 | | | | | | 4 | 5 | 3 | | | |
| 16 | 8 | 3 | | | | | | 12 | 3 | | | | |
| 17 | 1 | 2 | 2 | 2 | 2 | 2 | 29 | 4 | 6 | 3 | | | |
| | 2 | 3 | 3 | 2 | | | | 5 | 3 | 3 | | | |
| | 8 | 4 | | | | | | 11 | 4 | | | | |
| | 10 | 2 | | | | | | 16 | 2 | | | | |
| 18 | 3 | 4 | 4 | | | | 30 | 1 | 6 | 6 | 3 | | |
| | 6 | 2 | 2 | | | | | 2 | 3 | 3 | 3 | | |
| 19 | 1 | 4 | 3 | 2 | 2 | | | 3 | 5 | 2 | 2 | | |
| | 1 | 5 | 5 | 3 | | | | 6 | 4 | 2 | | | |
| | 1 | 4 | 4 | 4 | | | | 9 | 2 | 2 | | | |
| | 5 | 3 | 2 | | | | 31 | 1 | 6 | 4 | 4 | | |
| | 9 | 3 | | | | | | 2 | 3 | 2 | 2 | 2 | |
| | 11 | 2 | | | | | | 2 | 4 | 4 | 2 | | |
| 20 | 2 | 2 | 2 | 2 | 2 | | | 3 | 6 | 2 | 2 | | |
| | 4 | 3 | 3 | | | | | 3 | 6 | 5 | | | |
| 21 | 3 | 5 | 4 | | | | | 5 | 2 | 2 | 2 | | |
| | 9 | 4 | | | | | | 7 | 3 | 2 | | | |
| | 12 | 2 | | | | | | 11 | 5 | | | | |
| 22 | 1 | 5 | 3 | 2 | 2 | | | 13 | 3 | | | | |

| N | Z | X1 | X2 | X3 | X4 | X5 | N | Z | X1 | X2 | X3 | X4 | X5 | |
|----|------|----|----|----|----|----|----|----|----|----|----|----|----|--|
| | 17 | 2 | | | | | 38 | 6 | 3 | 3 | | | | |
| 32 | NONE | | | | | | | 7 | 4 | 2 | | | | |
| 33 | 3 | 7 | 5 | | | | 11 | 2 | 2 | | | | | |
| | 6 | 5 | 2 | | | | 39 | 1 | 9 | 6 | 3 | | | |
| | 12 | 4 | | | | | 6 | 2 | 2 | 2 | | | | |
| | 18 | 2 | | | | | 21 | 2 | | | | | | |
| 34 | 1 | 7 | 4 | 4 | | | 40 | 1 | 6 | 4 | 2 | 2 | | |
| | 4 | 3 | 2 | 2 | | | 2 | 4 | 2 | 2 | 2 | | | |
| | 4 | 4 | 4 | | | | 16 | 3 | | | | | | |
| | 6 | 6 | 2 | | | | 41 | 2 | 4 | 3 | 3 | | | |
| | 10 | 2 | 2 | | | | 4 | 5 | 4 | | | | | |
| | 14 | 3 | | | | | 13 | 5 | | | | | | |
| 35 | 1 | 5 | 4 | 2 | 2 | | 14 | 4 | | | | | | |
| | 1 | 8 | 4 | 4 | | | 22 | 2 | | | | | | |
| | 1 | 7 | 6 | 3 | | | 42 | 12 | 2 | 2 | | | | |
| | 3 | 3 | 3 | 2 | | | 43 | 1 | 7 | 4 | 2 | 2 | | |
| | 19 | 2 | | | | | 1 | 7 | 7 | 3 | | | | |
| 36 | 3 | 2 | 2 | 2 | 2 | | 2 | 6 | 4 | 2 | | | | |
| | 12 | 5 | | | | | 3 | 7 | 6 | | | | | |
| 37 | 1 | 4 | 2 | 2 | 2 | 2 | 4 | 4 | 2 | 2 | | | | |
| | 1 | 5 | 5 | 4 | | | 5 | 5 | 3 | | | | | |
| | 5 | 4 | 3 | | | | 7 | 5 | 2 | | | | | |
| | 8 | 3 | 2 | | | | 9 | 3 | 2 | | | | | |
| | 12 | 6 | | | | | 13 | 6 | | | | | | |
| | 13 | 4 | | | | | 17 | 3 | | | | | | |
| | 15 | 3 | | | | | 23 | 2 | | | | | | |
| | 20 | 2 | | | | | 44 | 1 | 5 | 2 | 2 | 2 | 2 | |
| 38 | 1 | 4 | 3 | 3 | 2 | | 1 | 8 | 4 | 2 | 2 | | | |
| | 1 | 8 | 6 | 3 | | | 45 | 15 | 4 | | | | | |
| | 2 | 5 | 4 | 2 | | | 24 | 2 | | | | | | |
| | 3 | 6 | 6 | | | | | | | | | | | |

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|--|-----|
| Mustafa Agah Akcoglu, John Philip Huneke and Hermann Rost, <i>A counter example to the Blum Hanson theorem in general spaces</i> | 305 |
| Huzihiro Araki, <i>Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule</i> | 309 |
| E. F. Beckenbach, Fook H. Eng and Richard Edward Tafel, <i>Global properties of rational and logarithmico-rational minimal surfaces</i> | 355 |
| David W. Boyd, <i>A new class of infinite sphere packings</i> | 383 |
| K. G. Choo, <i>Whitehead Groups of twisted free associative algebras</i> | 399 |
| Charles Kam-Tai Chui and Milton N. Parnes, <i>Limit sets of power series outside the circles of convergence</i> | 403 |
| Allan Clark and John Harwood Ewing, <i>The realization of polynomial algebras as cohomology rings</i> | 425 |
| Dennis Garbanati, <i>Classes of circulants over the p-adic and rational integers</i> | 435 |
| Arjun K. Gupta, <i>On a "square" functional equation</i> | 449 |
| David James Hallenbeck and Thomas Harold MacGregor, <i>Subordination and extreme-point theory</i> | 455 |
| Douglas Harris, <i>The local compactness of vX</i> | 469 |
| William Emery Haver, <i>Monotone mappings of a two-disk onto itself which fix the disk's boundary can be canonically approximated by homeomorphisms</i> | 477 |
| Norman Peter Herzberg, <i>On a problem of Hurwitz</i> | 485 |
| Chin-Shui Hsu, <i>A class of Abelian groups closed under direct limits and subgroups formation</i> | 495 |
| Bjarni Jónsson and Thomas Paul Whaley, <i>Congruence relations and multiplicity types of algebras</i> | 505 |
| Lowell Duane Loveland, <i>Vertically countable spheres and their wild sets</i> | 521 |
| Nimrod Megiddo, <i>Kernels of compound games with simple components</i> | 531 |
| Russell L. Merris, <i>An identity for matrix functions</i> | 557 |
| E. O. Milton, <i>Fourier transforms of odd and even tempered distributions</i> | 563 |
| Dix Hayes Pettey, <i>One-one-mappings onto locally connected generalized continua</i> | 573 |
| Mark Bernard Ramras, <i>Orders with finite global dimension</i> | 583 |
| Doron Ravdin, <i>Various types of local homogeneity</i> | 589 |
| George Michael Reed, <i>On metrizable complete Moore spaces</i> | 595 |
| Charles Small, <i>Normal bases for quadratic extensions</i> | 601 |
| Philip C. Tonne, <i>Polynomials and Hausdorff matrices</i> | 613 |
| Robert Earl Weber, <i>The range of a derivation and ideals</i> | 617 |