VERTICALLY COUNTABLE SPHERES AND THEIR WILD SETS

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A 2-sphere $S$ in $E^3$ is said to have vertical order $n$ if the intersection of each vertical line with $S$ contains no more than $n$ points. It is shown that $S \cup \text{Int } S$ is a 3-cell that is locally tame from $\text{Ext } S$ modulo a 0-dimensional set if $S$ has vertical order 5. A subset $X$ of $E^3$ is said to have countable (finite) vertical order if the intersection of $X$ with each vertical line consists of countably (finitely) many points. A 2-sphere in $E^3$ with countable vertical order can have a wild set of dimension no larger than one.

For each 2-sphere $S$ in $E^3$ there is a homeomorphism $h: E^3 \rightarrow E^3$ such that each vertical line intersecting $h(S)$ does so in a 0-dimensional set [2, Theorem 10.1]; thus the condition that a 2-sphere be "vertically 0-dimensional" imposes no restriction on the wildness of the 2-sphere. A study of vertically finite 2-spheres (spheres with finite vertical order) was begun in [10] where it was proven that a 2-sphere in $E^3$ having vertical order 3 is tame. Even though there are wild 2-spheres having vertical order 4, it is known that $S \cup \text{Int } S$ is a 3-cell if $S$ has vertical order 5 [11]. We extend this result to show that the set $W(S)$ of points where the 2-sphere $S$ fails to be locally tame must be 0-dimensional if $S$ has vertical order 5. An example is given at the end of the paper to show that 5 is the largest integer for which this result is true. We also show that the wildness of a vertically countable sphere is limited to a 1-dimensional set.

In the remainder of the paper we use $\pi: E^3 \rightarrow E^2$ to denote the vertical projection of $E^3$ onto the horizontal plane $E^2$. For convenience, we always assume that $E^2$ is located vertically below the sphere or cube under investigation. We use $L(x)$ to denote the vertical line containing the point $x$.

A vertical line $L$ is said to pierce a subdisk $D$ of a 2-sphere $S$ if there is an interval $I$ in $L$ such that $I \cap S$ is a point $p \in D$ and $I$ intersects both $\text{Int } S$ and $\text{Ext } S$. We say that $L$ links the boundary $\text{Bd } D$ of a disk $D$ if $L$ intersects every disk bounded by $\text{Bd } D$.

2. Spheres having countable vertical order.

**Theorem 2.1.** If $S$ is a 2-sphere in $E^3$ having countable vertical order, then $W(S)$ contains no open subset of $S$.

**Proof.** Suppose that $W(S)$ contains a disk $D$ in $S$. We shall
produce a contradiction by exhibiting a vertical line \( L \) whose intersection with \( D \) contains a Cantor set.

**Assertion A.** If \( D' \) is a subdisk of \( D \), then there is an open subset \( U \) of \( E^3 \) such that \( \pi(U) \subseteq \pi(D') \).

To prove Assertion A it suffices to show that \( \pi(D') \) is not one-dimensional. This follows from [9, Theorem VI.7, p. 91] since the map \( \pi \mid D' : D' \to \pi(D) \) is closed.

**Assertion B.** If \( D' \) is a subdisk of \( D \) and \( U \) is an open subset of \( E^3 \) such that \( \pi(U) \subseteq \pi(D') \), then there exist disjoint disks \( D_1 \) and \( D_2 \) in \( D' \) and an open subset \( N \) of \( U \) such that each vertical line through \( \text{cl}(N) \) intersects both \( D_1 \) and \( D_2 \).

In order to select the disks \( D_i \) in Assertion B we first show the existence of a vertical line \( L \) containing two points \( r \) and \( t \) in \( U \) and containing two sequences \( \{u_i \} \) and \( \{l_i \} \) of points such that

1. \( \{u_i \} \) converges to \( r \) from above,
2. \( \{l_i \} \) converges to \( r \) from below,
3. there is a component \( V_1 \) of \( E^3 - S \) containing every \( u_i \), and
4. \( E^3 - (S \cup V_1) = V_2 \) contains every \( l_i \).

Notice that some vertical line \( L' \) intersects \( D' \) in more than two points [7, Theorem 2.3], so we may choose two points \( r' \) and \( t' \) in \( L' \cap D' \). Let \( B \) be an open ball centered at \( r' \) such that \( B \cap S \subseteq D' \). If \( r' \) does not satisfy the four conditions above relative to \( L' \), it must be because some interval \( I \) in \( L' \cap B \) has \( r' \) as its midpoint and lies, except for \( r' \), in a single component, say \( V_1 \), of \( E^3 - S \). Let \( B_1 \) and \( B_2 \) be disjoint round open balls of equal radius centered at points of \( L' \) above and below \( r' \), respectively such that \( B_1 \cup B_2 \subseteq V_1 \cap B \). Now close to \( r' \) and vertically between \( B_1 \) and \( B_2 \), there must exist a point \( e \) of \( V_2 \). Then \( L = L(e) \) intersects \( V_2 \) between its two intersections with \( V_1 \cap (B_1 \cup B_2) \), so \( L \) intersects \( D' \) at least twice. Let \( r \) be the lowest point of the component of \( L \cap (S \cup V_1) \) containing \( L \cap B_1 \), and choose \( t \) to be some other point of \( L \cap S \). Since \( S \) has countable vertical order it is clear that \( r \) is a limit point of \( L \cap V_1 \) from above and of \( L \cap V_2 \) from below. Thus conditions (1), (2), (3), and (4) are satisfied.

Choose a disk \( D_1 \) in \( D' \) such that \( r \in \text{Int} \, D_1 \) and \( t \notin D_1 \). We claim that there is an open set \( U_1 \) containing \( r \) such that every vertical line through \( U_1 \) intersects \( D_1 \). Suppose there is no such open set, and for each \( i \) let \( E_i \) be a horizontal disk centered at \( l_i \) and lying in \( V_2 \). There must be a sequence \( \{x_i \} \) such that \( x_i \in E_i \), for each \( i \), no \( L(x_i) \) intersects \( D_1 \), and \( \{L(x_i)\} \) converges to \( L(r) \). For each \( i \) let \( y_i \) be the first point of \( S \) above \( x_i \) on \( L(x_i) \) (such a point will exist for suf-
ficiently large integers \( i \) since \( u_i \) and \( l_i \) are different components of \( E^3 - S \), and let \( I_i \) be the vertical interval \([x_i, y_i]\) in \( S \cup V_i \). Since some subsequence of \( \{y_i\} \) converges, we assume for notational convenience that \( \{y_i\} \) converges to a point \( y \). Of course \( y \in L(r) \cap S \). It is clear that \( y \) is not above \( r \) on \( L(r) \) because \( \{r, y\} \subset \text{lim inf } I_i \subset S \cup V_i \), whereas \( \{u_i \rightarrow r \} \) and \( u_i \in V_i \). Nor is \( y \) below \( r \) on \( L(r) \) because \( \{l_i \rightarrow r, \} \subset E_i \), and \( x_i \) lies vertically below \( y_i \). Thus \( \{y_i\} \) converges to \( r \), and we have the contradiction that most of the \( y_i \)’s must belong to \( D' \) while \( L(y_i) \cap D' \) was supposed to be empty. The existence of \( U_1 \) is established.

Now choose a disk \( D_2 \) such that \( D_1 \cap D_2 = \emptyset, t \in \text{Int } D_2, D_2 \subset D', \) and \( \pi(D_2) \subset \pi(U_1) \). From Assertion A there is an open set \( U_2 \) such that every vertical line through \( U_2 \) intersects \( D_2 \). Such a line will also intersect \( U_1 \) and hence \( D_1 \). Choose \( N \) to be any open subset of \( U \) such that \( \pi(\text{cl}(N)) \subset \pi(U_1) \cap \pi(U_2) \).

Now that the two assertions have been proven it might be clear how to proceed inductively to produce a vertical line containing uncountably many points of \( S \); nevertheless, we give a brief outline. From Assertion A there is an open set \( U \) such that every vertical line through \( U \) intersects \( D \). Now we apply Assertion B to obtain an open set \( U_1 \), whose closure lies in \( U \), and two disjoint disks \( D_1 \) and \( D_2 \) in \( D \) such that every vertical line through \( \text{cl}(U_1) \) intersects both \( D_1 \) and \( D_2 \). This ends the first step in the construction. Assertion B can now be applied to \( D_1 \) to obtain two disjoint disks \( D_{11} \) and \( D_{12} \) in \( D \), and an open set \( N_2 \) such that vertical lines through \( \text{cl}(N_2) \) intersect both \( D_{11} \) and \( D_{12} \). Now B is applied to \( D_2 \) and \( N_2 \) so that at the completion of step 2 we have an open set \( U_2 \) whose closure lies in \( U_2 \) and four disjoint disks \( D_{11}, D_{12}, D_{21}, \) and \( D_{22} \), in \( D \) where each vertical line through \( \text{cl}(U_2) \) intersects each of the four disks. When the construction is finished it is clear that a vertical line through \( \bigcap_i \text{cl}(U_i) \) will intersect each of the \( 2^n \) disks at the \( n \)th step. Thus such a line contains an uncountable set of points of \( S \). This contradiction establishes the theorem.

**Corollary 2.2.** If \( S \) is a 2-sphere in \( E^3 \) having countable vertical order, then \( S \) is locally tame modulo a 1-dimensional subset.

3. Spheres of vertical order order 5. The following four lemmas are used to establish the main result (Theorem 3.5).

**Lemma 3.1.** If \( S \) has vertical 5, then \( S \) is locally tame at each point of \( S \) that is vertically above or below a point of \( \text{Int } S \); that is, \( \pi(\text{Int } S) \cap \pi(W(S)) = \emptyset \).

**Proof.** Let \( p \) be a point of \( S \) such that \( L(p) \cap \text{Int } S \neq \emptyset \). Thus
$L(p)$ must link the boundaries of each of two disjoint disks $D_1$ and $D_2$ in $S$. Let $B$ be a ball lying in $\text{Int} S$ such that each vertical line through $B$ links both $\text{Bd} D_1$ and $\text{Bd} D_2$. If $p \in D_1 \cup D_2$, then there is a disk $D_3$ in $S$ such that $p \in \text{Int} D_3$, $D_3 \cap (D_1 \cup D_2) = \emptyset$, and $\pi(D_3) \subset \pi(B)$. Then each vertical line intersecting $D_3$ also intersects both $D_1$ and $D_2$. Since $D$ has vertical order 5 it is clear that $D_3$ has vertical order 3. Thus $D$ is locally tame at $p$ [7, Theorem 2.3] and so is $S$.

We may now assume that $p \in \text{Int} D_1$. Let $D'_1$ be a subdisk of $D_1$ such that $\pi(D'_1) \subset \pi(B)$, and, for each $\xi > 0$, let $X^\xi$ be the union of all vertical intervals of diameter no less than $\xi$ in $S \cup \text{Int} S$ that intersect $D'_1$. It is an exercise to see that $X^\xi$ is closed, and it follows from [6, Theorem 5] that $X^\xi$ is a $*$-taming set. Now consider a point $q$ in $D'_1$ but not in $X^{1/\xi}$ for any $\xi$. It follows that $q$ lies in no vertical interval in $S \cup \text{Int} S$. Thus $L(q)$ does not pierce $D'_1$ at $q$, and $L(q)$ must pierce $D'_1$ at some other point $t$ by the choice of $B$. Let $D$ be a disk in $D'_1$ with $t$ in its interior such that $q \in D$ and $L(q)$ links $\text{Bd} D$. Then there is a disk $D_q$ in $D'_1 - D$ such that $q \in \text{Int} D_q$ and each vertical line through $D_q$ links $\text{Bd} D$. Thus such a line intersects both $D$ and $D_q$. This means that $D_q$ has vertical order 3 and is tame [7, Theorem 2.3]. Now we see that each point of $D'_1$ either lies in the interior of a tame disk in $D'_1$ or lies in $\bigcup_i X^{1/i}$. Since a tame disk is a $*$-taming set and a countable number of tame disks suffice to cover $D'_1 - \bigcup_i X^{1/i}$, we see that $D'_1$ lies in a $*$-taming set of the form $(\bigcup_i X^{1/i}) \cup (a \text{ countable collection of tame disks})$ in $S \cup \text{Int} S$ [5, Theorem 3.7 and Corollary 3.8]. Thus $S$ is locally tame at $p$ from $E^3 - (S \cup \text{Int} S)$ by the definition of a $*$-taming set. Since $S$ is locally tame from $\text{Int} S$ [11], it follows that $S$ is locally tame at $p$.

**Lemma 3.2.** If $M$ is a continuum in $W(S)$ and $S$ is a 2-sphere having vertical order 5, then $M$ is tame.

**Proof.** We may assume that $M$ is nondegenerate since singleton sets always lie on tame spheres. From the previous lemma it is clear that $\pi(M) \subset \text{Bd} \pi(\text{Int} S)$. Let $U = \text{Int} S$ and let $X$ be the component of $\text{Bd} \pi(U)$ containing $\pi(M)$. We shall show the existence of a space homeomorphism $H: E^3 \to E^3$ such that $\pi(H(M))$ is either an arc or a simple closed curve. Then $H(M)$ is clearly tame since it lies in $\pi^{-1}(\pi(H(M)))$.

The continuum $X$ can be shown locally connected as in [7, Part 0.2]. Notice that $\pi(U)$ is open and connected. We let $U'$ be the component of $E^2 - X$ containing $\pi(U)$ and for convenience in what follows we assume that $U'$ is bounded. Notice that $\text{cl} \, (U') = X \cup U'$ since every point of $S$ is accessible from $\text{Int} S$. Let $B^2 = \{(x, y) \mid x^2 + y^2 \leq 1\} \subset E^2$. There is a continuous function $f: B^2 \to \text{cl} \, (U')$ such that
$f \mid \text{Int } B^2$ is a homeomorphism of $\text{Int } B^2$ onto $U'$ and $f^{-1}(x)$ is a totally disconnected subset of $S' = \text{Bd } B^2$ for each $x \in X$ (see [12, p. 186]). Now we follow [7, §§ 2.1, 2.2, 2.3, and 2.4] to find a homeomorphism $H$ of $E^3$ onto $E^3$ such that $\pi(\pi^{-1}(X) \cap S))$ is a simple closed curve. Thus $\pi(H(M))$ is either an arc or a simple closed curve since $\pi(H(M)) \subset \pi(H(\pi^{-1}(X) \cap S))$.

In the case where $U'$ is not bounded the map $f$ above takes $E^2 - \text{Int } B^2$ onto $\text{cl } (U')$ and causes some notational difficulties when we try to follow [7] as above. However, [7] still serves as an outline and we leave the details to the reader.

**Lemma 3.3.** If $M$ is a nondegenerate continuum in $W(S)$ and $S$ is a 2-sphere having vertical order 5, then each point of $M$ is a limit point of $W(S) - M$.

**Proof.** Suppose some point $p \in M$ is not a limit point of $W(S) - M$, and choose a disk $D$ on $S$ such that $p \in \text{Int } D$, $\text{Bd } D$ is tame [3], and $D \cap W(S) \subset M$. Let $X = M \cup (\text{Bd } D)$, and let $S'$ be a 2-sphere containing $M \cup D$ that is locally tame modulo $X[1]$. From Lemma 3.2 we see that $X$ is a taming set [4, Theorem 8.1.6, p. 320]. Thus $S'$ is tame. This is a contradiction and the result follows.

**Lemma 3.4.** If $D$ is a disk in a 2-sphere $S$, $S$ has vertical order 5, $p \in \text{Int } D$, and $V$ is an open subset of $E^3$ such that $p \in V$ and, for each vertical line $L$ piercing $D$ at a point in $V$, $L \cap \text{Int } S$ has exactly one component whose closure intersects $D$, then $D$ is locally tame at $p$.

**Proof.** If $L(p)$ intersects $\text{Int } S$, then the conclusion of Lemma 3.4 follows from Lemma 3.1. Thus we now assume $L(p) \cap \text{Int } S = \emptyset$. Choose a 2-sphere $H$ in the shape of a right circular cylinder such that $p \in \text{Int } H$, $H \cap S \subset D$, $\text{Bd } D \subset \text{Ext } H$, $[L \cap (\text{Int } H)] \cap S = \{p\}$, the top and bottom disks $T$ and $D$ of $H$ lie in $\text{Ext } S$, and each vertical line intersecting $H$ also intersects $V$.

Let $X$ be a component of $(\text{Int } S) \cap H$, and let $K = \text{Bd } X$. We shall show that $X \cup K$ is a disk by showing that $K$ is a simple closed curve. To show that $K$ is connected it suffices to prove that each simple closed curve $J$ in $X$ bounds a disk in $X$. Such a curve $J$ cannot be essential on the annulus $H - D \cup T$ since $J$ would link $L(p)$ while $L(p) \subset (\text{Ext } S) \cup S$ and $J \subset \text{Int } S$. Thus $J$ must bound a disk $E$ in $H - D \cup T$. From the hypothesis of Lemma 3.4 it is clear that $E \subset X$. Thus $K$ is connected. The fact that $K$ has vertical order 5 insures that $K$ is arcwise accessible from both its complementary domains in $H$, and this implies that $K$ is a simple closed curve.

Thus the closure of each component of $(\text{Int } S) \cap H$ is a spanning
disk for the 3-cell \( C = S \cup \text{Int} S \). There can be at most a countable collection \( \{D_1, D_2, \ldots\} \) of these spanning disks since their interiors are pairwise disjoint. The fact that \( D \) has vertical order 5 insures that \( \{D_i\} \) is a null sequence. We use these spanning disks to construct a 2-sphere \( S' \) containing \( p \) and lying in \( D \cup (\bigcup_i D_i) \) and in \( H \cup \text{Int} H \). From the hypothesis on \( D \) we see that the interior of \( S' \) is vertically connected; thus \( S' \) is tame [7, Main Theorem]. This means that \( D \) is locally tame at \( p \).

**Theorem 3.5.** If a 2-sphere \( S \) in \( E^3 \) has vertical order 5, then \( S \cup \text{Int} S \) is a 3-cell and \( S \) is locally tame from \( \text{Ext} S \) modulo a 0-dimensional set.

**Proof.** That \( C = S \cup \text{Int} S \) is a 3-cell follows from [11]. It remains to show that the set \( W \) of wild points of \( S \) is 0-dimensional. Suppose to the contrary that there is a nondegenerate continuum \( M \) lying in \( W \). Since \( C \) is a 3-cell there is an embedding \( g: M \times [0, 1] \rightarrow C \) such that \( G = g(M \times [0, 1]) \subset \text{Int} S \) and \( g(m, 0) = m \) for every \( m \in M \). We let \( F = g(M \times [0, 1]) \), and we note that it follows from Lemma 3.1 that \( \pi(M) \) lies in the boundary of \( \pi(F) \) in \( E^3 \). For the same reason, \( \pi(G) \cap \pi(M) = \emptyset \). Let \( U \) be a disk in \( E^2 \) and let \( p' \) be a point of \( \text{Int} U \) such that \( U \cap (\pi(\text{Bd} F)) \subset \pi(M) \) and \( p' \in \pi(M) \). Choose a point \( p \) in \( M \cap \pi^{-1}(p') \). In the next paragraph we show the existence of a disk \( E \) in \( S \) with \( p \in \text{Int} E \) and \( \pi(E) \subset U \cap \pi(F) \).

The difficulty in choosing \( E \) is the requirement that \( \pi(E) \subset \pi(F) \). If no such \( E \) exists there must exist a sequence \( \{p_i\} \) of points of \( \text{Int} S \) converging to \( p \) such that \( \pi(p_i) \in U - \pi(F) \) for each \( i \). Using the 0-ULC of \( \text{Int} S \) it is easy to select a point \( g \in G \subset \text{Int} S \) close enough to \( p \) and an integer \( N \) large enough that \( g \) and \( p_N \) are the end points of an arc \( A \) in \( \text{Int} S \) where \( \pi(A) \subset U \). Now \( \pi(A) \) contains an arc with one end point \( a \) in \( \pi(G) \) and the other end point \( b \) in \( U - \pi(F) \). If this arc is traversed from \( b \) to \( a \), then there is a first point \( f \) of \( \pi(F) \) encountered. This point \( f \) clearly belongs to \( \text{Bd} \pi(F) \). This contradiction establishes the existence of \( E \).

Now that the existence of \( E \) is clear we proceed by using Lemma 3.3 to pick a point \( q \) in \( E \cap (W - M) \). Let \( V \) be an open ball centered at \( q \) such that \( V \cap S \subset E \) and \( V \cap F = \emptyset \). Since \( L(q) \cap \text{Int} S = \emptyset \) (see Lemma 3.1) there are open balls \( B_1 \) and \( B_2 \) centered at points above and below \( q \), respectively, that lie in \( (\text{Ext} S) \cap V \). We choose a disk \( D \) in \( V \cap S \) with \( q \in \text{Int} D \) vertically between \( B_1 \) and \( B_2 \) such that \( \pi(D) \subset \pi(B_1) \cap \pi(B_2) \). We shall show that \( D \) is locally tame at \( q \) to obtain a contradiction to \( q \in W \).

In order to apply Lemma 3.4 we must show that if a vertical line \( L \) pierces \( D \) at a point of \( V \), then \( L \cap \text{Int} S \) has exactly one
component whose closure intersects \( D \). Suppose to the contrary that for some such line \( L \) there are two components \( X \) and \( Y \) of \( L \cap \text{Int} \, S \) whose closures intersect \( D \). Now \( X \cup Y \subset V \) since \( D \) lies between \( B_1 \) and \( B_2 \). Since \( L \cap \text{Int} \, S = \emptyset \) and \( \pi(D) \subset \pi(F) \), we see that \( L \cap G \neq \emptyset \). Thus \( L \cap (\text{Int} \, S) \) has a third component \( Z \), different from both \( X \) and \( Y \) because \( Z \) lies either above \( B_1 \) or below \( B_2 \). Now the only way to avoid there being 6 points in \( L \cap S \) is for \( X \) and \( Y \) to share an end point \( x \). In this case there is a point \( e \) of \( \text{Ext} \, S \) close enough to \( x \) to insure that there are three components of \( L(e) \cap \text{Int} \, S \) with pairwise disjoint closures. Now \( L(e) \cap S \) contains 6 points contrary to the hypothesis.

4. Examples and questions. One can use a countably infinite null sequence of Fox-Artin [8] "feelers" whose wild points form a dense subset of an arc to see that a vertically countable 2-sphere can have an arc in its wild set. Thus Corollary 2.2 cannot be improved in this direction.

**Example 4.1.** A wild 2-sphere \( S \) having vertical order 6 such that \( W(S) \) is not 0-dimensional. In Figure 1 we see an embedding of

![Figure 1.](image)

the Alexander Horned Sphere, having vertical order 4, inside a wedge-shaped 3-cell in \( E^3 \). We attach a null sequence of such wedges to a right circular cone, as indicated in Figure 2, to obtain the desired example \( S \). Notice that \( W(S) \) is the union of a tame simple closed curve with countably infinite number of tame Cantor sets. Furthermore, every point of \( S \) is a piercing point of \( S \).

In Example 4.1 we see that every nondegenerate continuum in \( W(S) \) is tame.
Question 4.2. If $S$ is a 2-sphere in $E^3$ having finite vertical order, then must every nondegenerate continuum in $W(S)$ be tame?

We do not know the answer to Question 4.2 even when “vertical order $n$” replaces “finite vertical order”, unless $n \leq 5$ where Theorem 3.5 applies. The proof of Lemma 3.2 shows an affirmative answer to Question 4.2 if it is also known that $\pi(W(S)) \cap \pi(\text{Int} \ S) = \emptyset$.

REFERENCES

2. ———, Conditions under which a surface in $E^3$ is tame, Fund. Math., 47 (1959), 105-139.

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<td>Dix Hayes Pettey, <em>One-one-mappings onto locally connected generalized continua</em></td>
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<td>Mark Bernard Ramras, <em>Orders with finite global dimension</em></td>
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