

# Pacific Journal of Mathematics

**VARIOUS TYPES OF LOCAL HOMOGENEITY**

DORON RAVDIN

## VARIOUS TYPES OF LOCAL HOMOGENEITY

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**In this paper some notions of local homogeneity for metric space are investigated. A theorem on convergence of a sequence of homeomorphisms to a homeomorphism is proved and applied i.e., to show that for every two countable and dense subsets  $A$  and  $B$  of the Hilbert space  $l_2$  there exists a homeomorphism  $H$  of  $l_2$  onto itself such that  $H(A) = B$ .**

Let  $\{h_n\}_{n=1}^{\infty}$  be a sequence of homeomorphisms of a space  $X$  onto itself and let

$$(1) \quad H_n = h_n \circ h_{n-1} \circ \cdots \circ h_1$$

denote their composition.

In this note conditions—similar to those obtained by J. L. Paul in [4]—are given under which  $H = \lim_{n \rightarrow \infty} H_n$  exists and is a homeomorphism of  $X$  onto itself. The obtained results are applied to investigate various types of local homogeneity and to prove that if  $X = l_2$  is the Hilbert space then  $X$  has the following property: For every two countable and dense subsets  $A$  and  $B$  of  $X$  there exists a homeomorphism  $H$  of  $X$  onto itself such that  $H(A) = B$ .

A space  $X$  having this property was called by R. B. Bennett countable dense homogeneous (see [1]).

A proof that the Euclidean  $n$ -space  $E^n$  is countable dense homogeneous can be found in the book of W. Hurewicz and H. Wallmann "Dimension Theory" on p. 44.

In this paper the notion of countable dense homogeneity in the class of complete metric spaces is investigated and results—similar to those given by R. B. Bennett in [1] for locally compact spaces—are obtained.

In the sequel  $X$  denotes a metric space with metric  $\rho$ ,  $B(x, r)$ —the open ball with radius  $r$  centered at  $x$ ,  $\delta(V)$ —the diameter of a subset  $V$  of  $X$  and nbd stands for neighborhood. We also always assume that  $\delta(X) \leq 1$ .

1. In this section conditions are given under which a sequence of homeomorphisms  $H_n$  of  $X$  onto itself converges to a homeomorphism  $H$  of  $X$  onto itself. Theorem 1 is a generalization of a similar theorem proved by J. L. Paul, (see [4]). We shall use the following two definitions:

DEFINITION 1. A sequence  $\{U_n\}_{n=1}^{\infty}$  of open subsets of  $X$  is said to be proper if

$$(2) \quad U_i \cap U_j \neq \emptyset \implies U_i \supset \bar{U}_j \quad \text{for } i < j .$$

DEFINITION 2. A homeomorphism  $h$  of  $X$  onto itself will be called of type  $(\xi, r, U)$  where  $0 < \xi < 1, 1 \leq r$  are constants and  $U$  is an open subset of  $X$  if:

$$(3) \quad h|_{X \setminus U} \text{ is the identity mapping}$$

and

$$(4) \quad \rho(h(x), h(y)) > \xi[\rho(x, y)]^r \quad \text{for all } x \neq y \text{ in } X .$$

THEOREM 1. Let  $X$  be a complete metric space and  $\{U_n\}_{n=1}^\infty$  a proper sequence. Let  $\{h_n\}_{n=1}^\infty$  be a sequence of homeomorphisms of  $X$  onto itself where  $h_n$  is of type  $(\xi_n, r_n, U_n)$ . If

$$(5) \quad \delta(U_n) < \xi_{n-1} \cdot \xi_{n-2}^{r_{n-1}} \cdot \dots \cdot \xi_1^{r_{n-1} \dots r_2} \cdot \left(\frac{1}{2^n}\right)^{r_1 r_2 \dots r_n}$$

then  $H = \lim_{n \rightarrow \infty} H_n$  exists and is a homeomorphism of  $X$  onto itself.

*Proof.* The proof being similar to Theorem 1 in [4] we shall confine ourselves to showing only that  $H$  is continuous and one-to-one.

By (3), (4), and (5)  $\rho(h_n(x), x) < 1/2^n$ . Hence  $\rho(H_n(x), H_{n-1}(x)) < 1/2^n$ . The space  $X$  being complete it follows that  $H(x) = \lim_{n \rightarrow \infty} H_n(x)$  exists for all  $x$  and that  $H$  is continuous. To show that  $H$  is one-to-one let  $x \neq y$  be points of  $X$ . There are two cases:

Case 1. There exist integers  $j(x)$  and  $k(y)$  such that

$$H_l(x) = H_{j(x)}(x) \text{ for } l > j(x) \text{ and } H_m(y) = H_{k(y)}(y) \text{ for } m > k(y) .$$

Then for  $g = \max \{j(x), k(y)\}$  one has

$$H(x) = H_g(x) \neq H_g(y) = H(y) .$$

Case 2. For at least one of the points  $x$  or  $y$ , (say  $x$ ), there exists a sequence  $m_1 < m_2 < \dots$  such that  $H_{m_i-1}(x) \in U_{m_i}$ . Let  $p = m_i$  be an integer of this sequence such that

$$(6) \quad \left(\frac{1}{2^p}\right)^{r_1 r_2 \dots r_{p-1}} < [\rho(x, y)]^{r_1 r_2 \dots r_{p-1}} .$$

By (2) and (3) the points  $H_{p-1}(x)$  and  $H(x)$  belong to  $U_p$ . By (4) we have

$$\rho(H_{p-1}(x), H_{p-1}(y)) > \xi_{p-1} \xi_{p-2}^{r_{p-1}} \dots \xi_1^{r_2 r_3 \dots r_{p-1}} [\rho(x, y)]^{r_1 r_2 \dots r_{p-1}} .$$

Now by (5) and (6)  $\delta(U_p) < \rho(H_{p-1}(x), H_{p-1}(y))$ .

Therefore,  $H_{p-1}(y) \notin U_{p-1}$ , and thus also  $H(y) \notin U_{p-1}$ . Since  $H(x) \in U_{p-1}$  it follows that  $H(x) \neq H(y)$

REMARK 1. Theorem 1 remains true if the requirement that  $X$  is complete is replaced by requiring the relative compactness of the  $U_i$ . The special case where  $r_n = 1$  ( $n = 1, 2 \dots$ ) is Theorem 1 of [4] and the proof is the same.

The following theorem shows a relation between the types (see Def. 2) of the homeomorphisms  $\{h_n\}_{n=1}^\infty$  and of the limit  $H = \lim_{n \rightarrow \infty} H_n$  of their superposition.

THEOREM 2. Let  $\{U_n\}$  be a sequence of open subsets of  $X$  such that  $U_n \supset \bar{U}_{n+1}$  ( $n = 1, 2 \dots$ ) and let  $\{h_n\}_{n=1}^\infty$  be a sequence of homeomorphisms of  $X$  onto itself. If  $h_n$  is of type  $(1/2, 1, U_n)$  and

$$(7) \quad \delta(U_n) < \frac{1}{4^n}.$$

Then  $H = \lim_{n \rightarrow \infty} H_n$  is a homeomorphism of  $X$  onto itself of type  $(1/4, 2, U_1)$ .

Proof. By Theorem 1,  $H$  is a homeomorphism of  $X$  onto itself and it suffices to show that

$$(8) \quad \rho(H(x), H(y)) > \frac{1}{4}[\rho(x, y)]^2 \text{ for all } x \neq y.$$

Since by assumption  $\delta(X) \leq 1$  there exists an integer  $n_0 \geq 1$  such that  $(1/2)^{n_0} < \rho(x, y) = d \leq (1/2)^{n_0-1}$ . By (4) used for  $\xi_n = 1/2, r_n = 1$  we have

$$\rho(H_{n_0}(x), H_{n_0}(y)) > \left(\frac{1}{2}\right)^{n_0} d.$$

Since by (7)  $\delta(U_{n_0+1}) < 1/4^{n_0+1}$ , the points  $H_{n_0}(x)$  and  $H_{n_0}(y)$  are not both in  $U_{n_0+1}$ .

If both points are outside  $U_{n_0+1}$  then by  $U_n \supset \bar{U}_{n+1}$  and by (3) we have  $H(x) = H_{n_0}(x)$  and  $H(y) = H_{n_0}(y)$ . Hence

$$\rho(H(x), H(y)) > \frac{1}{2^{n_0}} \cdot d > \frac{1}{4}[\rho(x, y)]^2.$$

Thus we may assume that  $H_{n_0}(x) \in U_{n_0+1}$  and  $H_{n_0}(y) \notin U_{n_0+1}$ . By (3)  $\rho(H_{n_0}(x), H(x)) < \delta(U_{n_0+1}) < 1/4^{n_0+1}$ , whence

$$\begin{aligned} \rho(H(x), H(y)) &= \rho(H(x), H_{n_0}(y)) \geq \rho(H_{n_0}(x), H_{n_0}(y)) - \rho(H_{n_0}(x), H(x)) \\ &> \left(\frac{1}{2}\right)^{n_0} \cdot d - \left(\frac{1}{4}\right)^{n_0+1} > \frac{1}{4}d^2. \end{aligned}$$

REMARK 2. Clearly the types  $(1/2, 1, U_n)$  and  $(1/4, 2, U_1)$  can be replaced by other appropriate combinations.

2. In this section various notions of local homogeneity are defined. It is remarked that the Hilbert space  $l_2$  is locally homogeneous of Type 1. It is then proved that for separable and complete metric spaces local homogeneity of a variable type implies countable dense homogeneity. Some problems and examples are also given.

DEFINITION 3. Let  $r \geq 1$ . A metric space  $X$  will be called locally homogeneous of type  $r$  if for every  $x$  and every nbd.  $U$  of  $x$  there exists a nbd.  $V$  of  $x$  such that  $V \subset U$  and such that for each two points  $y_1$  and  $y_2$  of  $V$  there exists a homeomorphism  $h$  of  $X$  onto itself of type  $(\xi, r, U)$  satisfying  $h(y_1) = y_2$ .

Let us note that if  $X$  is locally homogeneous of type  $r$  then  $X$  is strongly locally homogeneous according to [1]. Furthermore, if  $X$  is locally homogeneous of type  $r$  then clearly  $X$  is locally homogeneous of type  $s$  for  $s > r$  (we assume  $\delta(X) \leq 1$ ). The following example shows the converse does not hold.

EXAMPLE 1. Let  $X$  be the subset of  $E^2$  defined as follows: For any real number  $r \geq 1$  denote by  $L_n$  the segment in  $E^2$  whose endpoints are  $(1/n, 0)$  and  $(1/n, (1/n)^{1/r})$ , and by  $M_n$  the segment in  $E^2$  whose endpoints are  $(1/n, (1/n)^{1/r})$  and  $(1/(n+1), 0)$ , where  $n = 1, 2 \dots$ . Define

$$X = \bigcup_{n=1}^{\infty} (L_n \cup M_n) \cup \{(x, 0) | x \leq 0 \text{ or } x \geq 1\}.$$

Let  $\tilde{d}$  be the metric on  $X$  induced by  $\tilde{d}(x, y) = \min\{d(x, y), 1\}$  where  $d$  is the usual Euclidean metric in  $E^2$ . Then  $(X, \tilde{d})$  is *not* locally homogeneous of type  $r$  but *is* locally homogeneous of type  $s$  for every  $s > r$ .

DEFINITION 4. A space  $X$  will be called locally homogeneous of variable type if for every  $x \in X$  and every nbd  $U$  of  $x$  there exists a nbd.  $V$  of  $x$  such that  $V \subset U$  and such that for each two points  $y_1$  and  $y_2$  of  $V$  there exists a homeomorphism  $h$  of  $X$  onto itself of type  $(\xi, r(x, U), U)$  satisfying  $h(y_1) = y_2$ . ( $r$  depends on  $x$  and  $U$ .) A space which is locally homogeneous of strictly variable type is a space which is locally homogeneous of variable type but not of any fixed type  $r$ .

Clearly a space which is locally homogeneous of type  $r$  is locally homogeneous of variable type. For an example of a space which is locally homogeneous of strictly variable type we give

EXAMPLE 2. By modifying Example 1 we obtain an example of a locally homogeneous space of strictly variable type. Define

$$X = \bigcup_{n,m=1}^{\infty} (L_{n,m} \cup M_{n,m}) \cup \{(x, 0) | x \leq 0\}$$

where  $L_{n,m}$  is the segment in  $E^2$  whose endpoints are  $(1/n + m - 1, 0)$  and  $(1/n + m - 1, (1/n + m - 1)^{1/m})$  and  $M_{n,m}$  is the segment in  $E^2$  whose endpoints are  $(1/n + m - 1, (1/n + m - 1)^{1/m})$  and  $(1/(n + 1) + m - 1, 0)$ .

If  $X$  is topologized by  $\tilde{d}$  of Example 1 then  $(X, \tilde{d})$  is locally homogeneous of strictly variable type.

DEFINITION 5. A separable metric space  $X$  will be called (see [1]) countable dense homogeneous if for every two countable and dense subsets  $A$  and  $B$  of  $X$  there exists a homeomorphism of  $(X, A)$  onto  $(X, B)$ . The proof of the following theorem is not difficult and will be omitted.

THEOREM 3. *The Euclidean  $n$ -space  $E^n$ , the Hilbert space  $l_2$  and the Cantor set  $C$  are locally homogeneous of Type 1.*

We now prove

THEOREM 4. *A complete separable metric space  $X$  which is locally homogeneous of variable type is countable dense homogeneous.*

*Proof.* Let  $A = \{a_i\}_{i=1}^{\infty}$  and  $B = \{b_i\}_{i=1}^{\infty}$  be two countable dense subsets of  $X$ . We shall construct by induction a sequence  $\{h_n\}_{n=1}^{\infty}$  of homeomorphisms of  $X$  onto itself such that  $H = \lim_{n \rightarrow \infty} H_n$  is a homeomorphism of  $X$  onto itself satisfying  $H(A) = B$ .

(a) Let  $U = U_1 = B(a_1, 1)$  and  $V = V_1 \subset U_1$  be the subset of  $U$  given by Definition 4. Let  $b_{i_0}$  be an arbitrary point of  $V_1 \cap B$ . By hypothesis there exists a homeomorphism  $h_1$  of  $X$  onto itself of type  $(\xi_1, r_1, U_1)$  such that  $h_1(a_1) = b_{i_0}$ . Put  $a_1^* = a_1$  and  $b_1^* = b_{i_0}$ .

Since  $A \cup B$  is countable we can assume that  $(A \cup B) \cap \text{Bd}(U_1) = \emptyset$ .

(b) Define  $b_2^* = b_j$  where  $j$  is the first integer such that  $b_j \in B \setminus \{b_1^*\}$ . Let  $\varepsilon_2 = \min \{\rho(b_2^*, a_1^*), \rho(b_2^*, b_1^*), \rho(b_2^*, \text{Bd } U_1), \xi_1(1/2^2)^{r_1}\}$ .

Put  $U = U_2 = B(b_2^*, \varepsilon_2/2)$  and let  $V = V_2 \subset U_2$  be the subset of  $U_2$  given by Definition 4. Let  $h_1(a_2^*)$  be an arbitrary point of  $h_1(A) \cap V_2$  (evidently  $a_2^* \in A \setminus \{a_1^*\}$ ).

By hypothesis there exists a homeomorphism  $h_2$  of type  $(\xi_2, r_2, U_2)$  of  $X$  onto itself such that  $h_2 \circ h_1(a_2^*) = b_2^*$ . Again we can assume that  $(h_1(A) \cup B) \cap \text{Bd}(U_2) = \emptyset$ .

(c) Define  $a_3^* = a_i$  where  $i$  is the first integer such that  $a_i \in A \setminus \{a_1^*, a_2^*\}$  and let

$$\varepsilon_3 = \min_{i \leq 2} \left\{ \rho(h_2 \circ h_1(a_3^*), h_2 \circ h_1(a_i^*)), \rho(h_2 \circ h_1(a_3^*), \text{Bd}(U_i)), \rho(h_2 \circ h_1(a_3^*), b_i^*), \xi_1 \xi_2 \left( \frac{1}{2^3} \right)^{r_1 r_2} \right\}.$$

Put  $U_3 = B(h_2 \circ h_1(a_3^*), \varepsilon_3/2)$  and let  $V_3 \subset U_3$  be the subset of  $U_3$  given by Definition 4. We can assume that  $(h_2(h_1(A)) \cup B) \cap \text{Bd}(U_3) = \emptyset$ . Let  $b_3^*$  be an arbitrary point of  $B \cap V_3$  (evidently  $b_3^* \in B \setminus \{b_1^*, b_2^*\}$ ). Again by hypothesis there exists a homeomorphism  $h_3$  of  $X$  onto itself of type  $(\xi_3, r_3, U_3)$  such that  $h_3 \circ h_2 \circ h_1(a_3^*) = b_3^*$ .

(d) It should be clear how steps (b) and (c) can be repeated to obtain a sequence  $\{h_n\}_{n=1}^*$  of homeomorphisms of  $X$  onto itself satisfying the assumptions of Theorem 1. By Theorem 1  $H = \lim_{n \rightarrow \infty} H_n$  is a homeomorphism of  $X$  onto itself and clearly  $H(A) = B$ . Theorem 3 and Theorem 4 imply the following

**COROLLARY 1.** *The Hilbert space  $l_2$  is countable dense homogeneous. This result does not follow from [1].*

The above concepts now lead to the following problems:

*Problem 1.* Let  $r > s \geq 1$  be fixed real numbers. Does there exist a metric space  $(X, \rho)$  which is locally homogeneous of type  $r$ , and if  $\rho_1$  is any metric on  $X$  equivalent to  $\rho$  then the space  $(X, \rho_1)$  is not locally homogeneous of type  $s$ ?

*Problem 2.* Does there exist a metric space  $(X, \rho)$  which is locally homogeneous of strictly variable type for each metric  $\rho_1$  equivalent to  $\rho$ ?

*Problem 3.* Determine conditions which imply that a strongly locally homogeneous space is locally homogeneous of variable type.

The author wishes to thank the referee for his advice especially for supplying Definition 4, Example 2, and problem 2.

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Received November 8, 1972 and in revised form April 8, 1973.

TECHNION, HAIFA, ISRAEL

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$60.00 a year (6 Vols., 12 issues). Special rate: \$30.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

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# Pacific Journal of Mathematics

Vol. 50, No. 2

October, 1974

Mustafa Agah Akcoglu, John Philip Huneke and Hermann Rost, <i>A counter example to the Blum Hanson theorem in general spaces</i> .....	305
Huzihiro Araki, <i>Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule</i> .....	309
E. F. Beckenbach, Fook H. Eng and Richard Edward Tafel, <i>Global properties of rational and logarithmico-rational minimal surfaces</i> .....	355
David W. Boyd, <i>A new class of infinite sphere packings</i> .....	383
K. G. Choo, <i>Whitehead Groups of twisted free associative algebras</i> .....	399
Charles Kam-Tai Chui and Milton N. Parnes, <i>Limit sets of power series outside the circles of convergence</i> .....	403
Allan Clark and John Harwood Ewing, <i>The realization of polynomial algebras as cohomology rings</i> .....	425
Dennis Garbanati, <i>Classes of circulants over the <math>p</math>-adic and rational integers</i> .....	435
Arjun K. Gupta, <i>On a "square" functional equation</i> .....	449
David James Hallenbeck and Thomas Harold MacGregor, <i>Subordination and extreme-point theory</i> .....	455
Douglas Harris, <i>The local compactness of <math>vX</math></i> .....	469
William Emery Haver, <i>Monotone mappings of a two-disk onto itself which fix the disk's boundary can be canonically approximated by homeomorphisms</i> .....	477
Norman Peter Herzberg, <i>On a problem of Hurwitz</i> .....	485
Chin-Shui Hsu, <i>A class of Abelian groups closed under direct limits and subgroups formation</i> .....	495
Bjarni Jónsson and Thomas Paul Whaley, <i>Congruence relations and multiplicity types of algebras</i> .....	505
Lowell Duane Loveland, <i>Vertically countable spheres and their wild sets</i> .....	521
Nimrod Megiddo, <i>Kernels of compound games with simple components</i> .....	531
Russell L. Merris, <i>An identity for matrix functions</i> .....	557
E. O. Milton, <i>Fourier transforms of odd and even tempered distributions</i> .....	563
Dix Hayes Pettey, <i>One-one-mappings onto locally connected generalized continua</i> .....	573
Mark Bernard Ramras, <i>Orders with finite global dimension</i> .....	583
Doron Ravdin, <i>Various types of local homogeneity</i> .....	589
George Michael Reed, <i>On metrizable complete Moore spaces</i> .....	595
Charles Small, <i>Normal bases for quadratic extensions</i> .....	601
Philip C. Tonne, <i>Polynomials and Hausdorff matrices</i> .....	613
Robert Earl Weber, <i>The range of a derivation and ideals</i> .....	617