THE RANGE OF A DERIVATION AND IDEALS

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When \( A \) is in the Banach algebra \( \mathcal{B}(\mathcal{H}) \) of all bounded linear operators on a Hilbert space \( \mathcal{H} \), the derivation generated by \( A \) is the bounded operator \( \Delta_A \) on \( \mathcal{B}(\mathcal{H}) \) defined by \( \Delta_A(X) = AX -XA \). It is shown that the range of a derivation generated by a Hilbert-Schmidt or a diagonal operator contains no nonzero one-sided ideals of \( \mathcal{B}(\mathcal{H}) \). Also, for a two-sided ideal \( \mathcal{I} \) of \( \mathcal{B}(\mathcal{H}) \), necessary and sufficient condition on an operator \( A \) are given in order that the range of \( \Delta_A \) equals the range of \( \Delta_A \) restricted to \( \mathcal{I} \).

1. In the following \( \mathcal{H} \) will denote an infinite dimensional complex Hilbert space.

For a fixed \( A \in \mathcal{B}(\mathcal{H}) \), we will concern ourselves with the following problems:

(a) For what \( B \in \mathcal{B}(\mathcal{H}) \) is \( B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A) \) or \( \mathcal{R}(\Delta_A)B \subset \mathcal{R}(\Delta_A) \).

(b) For what \( B \in \mathcal{B}(\mathcal{H}) \) is \( B\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \) or \( \mathcal{B}(\mathcal{H})B \subset \mathcal{R}(\Delta_A) \).

(c) For what \( B \in \mathcal{B}(\mathcal{H}) \) is \( \mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A) \).

It is easy to verify that for \( \lambda \in \mathbb{C} \)

(i) \( \Delta_A = \Delta_{A+\lambda} \) for all \( \lambda \in \mathbb{C} \)
and

(ii) \( \Delta_A(XY) = X\Delta_A(Y) + \Delta_A(X)Y \).

The identity (ii) yields some simple facts about the range of a derivation which show the interrelation of the above problems. (For a proof see [8].)

LEMMA 1. Let \( A, B \in \mathcal{B}(\mathcal{H}) \) and let \( A' \) belong to the commutant \( \{A\}' \) of \( A \). Then

(a) both \( A'\mathcal{R}(\Delta_A) \) and \( \mathcal{R}(\Delta_A)A' \) are contained in \( \mathcal{R}(\Delta_A) \).

(b) if \( \mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A) \), then both \( A'\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \) and \( \mathcal{B}(\mathcal{H})\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A) \) are contained in \( \mathcal{R}(\Delta_A) \).

(c) \( \mathcal{R}(\Delta_A)B \subset \mathcal{R}(\Delta_A) \) if and only if \( \Delta_A(B)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \).

(d) \( \mathcal{R}(\Delta_A)B \subset \mathcal{R}(\Delta_A) \) if and only if \( \mathcal{B}(\mathcal{H})\Delta_A(B) \subset \mathcal{R}(\Delta_A) \).

From (b) of Lemma 1 it follows that if \( \mathcal{R}(\Delta_A) \) does not contain left- or right-ideals, then a necessary condition for \( \mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A) \) is that \( B \in \{A\}' \). In fact, more is true:

LEMMA 2. Let \( A \in \mathcal{B}(\mathcal{H}) \). If \( \mathcal{R}(\Delta_A) \) contains either no nonzero left-ideals or no nonzero right-ideals, then \( \Delta_B(\mathcal{I}) \subset \mathcal{R}(\Delta_A) \) implies
Proof. Assume that $\mathcal{R}(\Delta_A)$ contains no nonzero left-ideals (the argument for the other assumption is similar). Let $P$ be a finite rank projection. If $A' \in [A]'$, then

$$\Delta_A(B)PX = A'\Delta_B(PX) - \Delta_B(A'PX)$$

is in $\mathcal{R}(\Delta_A)$ for all $X \in \mathcal{B}(\mathcal{H})$. Therefore, $\Delta_A(B)P\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ and hence $\Delta_A(B)P = 0$. However, this is true for any such $P$ and hence $\Delta_A(B) = 0$.

For the sake of completeness we include a somewhat simpler proof of a theorem of Stampfli [6]. In the proof, $\sigma_l(A)$ denotes the left essential spectrum of $A$ and is defined to be the set of those $\lambda$ for which the coset of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ (where $\mathcal{K}$ is the ideal of compact operators) containing $A - \lambda$ fails to have a left inverse. The right essential spectrum $\sigma_r(A)$ is defined in the obvious way.

**Theorem 1.** Let $A \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{R}(\Delta_A)$ contains no nonzero two-sided ideals of $\mathcal{B}(\mathcal{H})$.

Proof. Replace $A$ by $A - \lambda$ where $\lambda \in \sigma_l(A) \cap \sigma_r(A)$ if necessary in order to assume that there exist orthonormal sequences $\{f_n\}$ and $\{g_n\}$ such that $\sum ||A f_n||_2 < \infty$ and $\sum ||A^* g_n||_2 < \infty$. (See [6].) Then for all $X \in \mathcal{B}(\mathcal{H})$,

$$\sum ||(AX -XA)f_n, g_n||_2 \leq \sum ||X||_2^2(||A^* g_n||_2^2 + ||Af_n||_2^2) < \infty.$$ 

If $\mathcal{R}(\Delta_A)$ contains a two-sided ideal, then it contains all finite rank operators. In particular, if $f \otimes g$ denotes the rank one operator $f \otimes g(x) = (x, g)f$, then $(f \otimes f)X \in \mathcal{R}(\Delta_A)$ for all $f \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. Hence

$$\sum ||(f \otimes f)Xf_n, g_n||_2 \leq \sum \langle (Xf_n, f^*)(\overline{g_n}, f) \rangle < \infty$$

for all $f \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. However, if we choose $X$ such that $Xf_n = g_n$ and $f$ such that $\{(g_n, f)\}$ is not summable, we have a contradiction.
2. Let $\mathcal{S}$ denote the set of Hilbert-Schmidt operators on $\mathcal{H}$. Equipped with the trace inner product $(A, B) = \text{tr}(AB^*)$, $\mathcal{S}$ is a Hilbert space [5]. If $A \in \mathcal{B}(\mathcal{H})$, then the restriction of $\Delta_A$ to $\mathcal{S}$ is a bounded operator on $\mathcal{S}$ with adjoint $(\Delta_A|_{\mathcal{S}})^* = \Delta_A|_{\mathcal{S}}$. Hence $\mathcal{S} = \mathcal{B}(\mathcal{S} | \mathcal{S}) = (A^*)' \cap \mathcal{S}$ where the double bar indicates closure with respect to the topology on $\mathcal{S}$.

**Theorem 2.** Let $A \in \mathcal{S}$. Then $\mathcal{R}(\Delta_A)^{-} = \mathcal{B}(\Delta_A|_{\mathcal{S}})^{-}$.

*Proof.* It follows from the above remarks that $\mathcal{R}(\Delta_A)^{-} \subset \mathcal{B}(\Delta_A|_{\mathcal{S}})^{-} = (A^*)' \cap \mathcal{S}$. It remains to show the reverse inclusion. Let $T \in (A^*)' \cap \mathcal{S}$. Then for $X \in \mathcal{B}(\mathcal{H})$

$$(\Delta_A(X), T) = \text{tr}(T^*\Delta_A(X)) = \text{tr}(T^*AX) - \text{tr}(T^*XA) = \text{tr}(AT^*X) - \text{tr}(T^*X) \Delta_A = \text{tr}(T^*X) - \text{tr}(T^*X) = 0 .$$

Therefore $T \in \mathcal{B}(\Delta_A)^{-}$.

**Corollary.** Let $A \in \mathcal{S}$. Then $\mathcal{R}(\Delta_A)^{-} = \mathcal{B}(\Delta_A|_{\mathcal{S}})^{-} = \mathcal{S}$.

**Theorem 3.** If $A \in \mathcal{S}$, then $\mathcal{R}(\Delta_A)$ does not contain any nonzero left- or right-ideals.

In the proof of Theorem 3 we will make use of the following result.

**Lemma 3.** Let $A \in \mathcal{S}$. If $(f \otimes f) \mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$, then $Af = 0$.

*Proof.* Since $\mathcal{R}(\Delta_A) \perp (A^*)' \cap \mathcal{S}$, then $0 = \text{tr}(A(f \otimes f)X) = \text{tr}(Af \otimes X^*f) = (Af, X^*f)$ for all $X \in \mathcal{B}(\mathcal{H})$. Hence $Af = 0$.

*Proof of Theorem 3.* Suppose that $(f \otimes f) \mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$. Then $f \otimes f = \Delta_A(X)$ for some $X \in \mathcal{B}(\mathcal{H})$ and by Lemma 3, $f = (f \otimes f)f = AXf - XAf = AXf$. Since $(f \otimes f) \mathcal{B}(\mathcal{H}) = \Delta_A(X) \mathcal{B}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, then by Lemma 1, $X \mathcal{B}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. Therefore, $(Xf \otimes (Xf)) \mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ and by Lemma 3, $Xf \in \ker(A)$. Hence $f = AXf = 0$. The remainder follows by taking adjoints.

**Corollary 1.** Let $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathcal{H})$. Then $B \mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$ if and only if $B \in (A)'$.

*Proof.* This follows from Lemma 1 and the theorem.

**Corollary 2.** Let $A \in \mathcal{S}$. If $\Delta_B(\mathcal{S}) \subset \mathcal{R}(\Delta_A)$ then $B \in (A)'$.

*Proof.* This follows from Lemma 2 and the theorem.
3. We now turn our attention to diagonal operators. When expressing a diagonal operator as the sum $A = \sum \alpha_n P_n$, unless otherwise stated we shall assume that $P_n$ is the rank one projection onto the subspace spanned by $e_n$, where $\{e_n\}$ is an orthonormal basis. (However, we do not require that the $\alpha_n$'s be distinct.) Each operator $X$ has a matrix $(x_{ij})$ with respect to this fixed basis.

The principle result of this section is that the range of a derivation generated by a diagonal operator contains no nonzero left- or right-ideals. The theorem is slightly more general.

**Theorem 4.** Let $A \in B(H)$ have the property that there exist reducing subspaces $M_n$ of $A$, each finite dimensional, such that $H = \bigoplus M_n$. Then $\mathcal{R}(A)_{\mathbb{R}}$ contains no nonzero positive operators.

**Proof.** Let $P = A(X)$ where $P$ is positive. If $P_n$ is the orthogonal projection onto $M_n$, then $P_nP|_{M_n} = A_nX_n - X_nA_n$ where $A_n = A|_{M_n}$ and $X_n$ is the compression of $X$ to $M_n$. Since $M_n$ is finite dimensional, then $\text{tr}(P_nP|_{M_n}) = 0$. Hence $P_nP|_{M_n}$ being a positive operator with zero trace, must be 0. Therefore, $P_nPP_n = 0$ (on $H$). Hence $P^{1/2}P_n = 0$ and $P^{1/2} = 0$.

**Corollary 1.** If $A$ satisfies the hypothesis of the theorem and if either $B\mathcal{R}(A)$ or $\mathcal{R}(A)B$ is contained in $\mathcal{R}(A)_{\mathbb{R}}$, then $B \in \{A\}'$.

**Corollary 2.** If $A$ satisfies the hypothesis of the theorem and $A_{\mathbb{H}} \subset \mathcal{R}(A)_{\mathbb{R}}$, then $B \in \{A\}''$.

**Corollary 3.** Let $A$ be normal with finite spectrum. Then for $B \in B(H)$, $\mathcal{R}(A)_{\mathbb{R}} \subset \mathcal{R}(A)$ if and only if $B \in \{A\}''$.

**Proof.** If $B \in \{A\}''$ then $B$ is a polynomial of $A$ and hence $\mathcal{R}(A)_{\mathbb{R}} \subset \mathcal{R}(A)$. (See [1, p. 79].) The converse follows from Corollary 2.

**Lemma 4.** Let $A, B \in B(H)$ where $A = \sum \alpha_i P_i$. Then $\mathcal{R}(A)_{\mathbb{R}} \subset \mathcal{R}(A)$ if and only if $B = \sum \beta_i P_i$ for some set of scalars $\beta_0, \beta_1, \cdots$ and for every operator $X = (x_{ij}) \in B(H)$ there exists an operator $Y = (y_{ij}) \in B(H)$ such that $(\alpha_i - \beta_i)y_{ij} = x_{ij}$ for all $i, j$.

**Proof.** This follows from Corollary 2 and the fact that $[J_A(X)]_{ij} = (\alpha_i - \beta_i)x_{ij}$ if $X = (x_{ij})$.

**Theorem 5.** Let $A \in B(H)$ be diagonal. If for $B \in B(H)$, $\mathcal{R}(A)_{\mathbb{R}} \subset \mathcal{R}(A)$, then $B = f(A)$ for some function $f$ which is Lipschitz on the spectrum of $A$. 

Proof. Let $A = \sum \alpha_i P_i$. If $\mathcal{B}(A_B) \subset \mathcal{B}(A_A)$, then by Corollary 2, $B = \sum \beta_i P_i$ for some sequence of scalars $\{\beta_i\}$ and for any $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$, there exists a $Y = (y_{ij}) \in \mathcal{B}(\mathcal{H})$ such that $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$. It follows that $((\beta_i - \beta_j)/(\alpha_i - \alpha_j))$ is bounded by some positive number $M$. Define $f$ such that $f(\alpha_i) = \beta_i$. Then $f$ is a Lipschitz function defined on a dense subset of $\sigma(A)$ onto a dense subset of $\sigma(B)$. Therefore, we can extend $f$ to be Lipschitz on $\sigma(A)$ onto $\sigma(B)$.

It was shown in [7] that if $B$ is an analytic function of $A$, then $\mathcal{B}(A_B) \subset \mathcal{B}(A_A)$. To have range inclusion it is neither necessary that $B$ be an analytic function of $A$ nor sufficient that $B$ be a continuous function of $A$ as seen in the next two examples.

**Example 1.** Let $A = \sum \alpha_n P_n$ where $\dim P_n = 1$, $\alpha_0 = 0$, and

$$\alpha_n = \begin{cases} \frac{i}{n} & \text{for } n \text{ even} \\ \frac{1}{n} & \text{for } n \text{ odd} \end{cases}$$

Let $B = \sum \beta_n P_n$ where $\beta_0 = 0$ and $\beta_n = -i/n^2$ for $n \geq 1$. A direct computation shows that if $n < m$, then $|(\beta_n - \beta_m)/(\alpha_n - \alpha_m)| \leq 2/n$.

Now, for any $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$, consider the matrix $Y = (y_{ij})$ where $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$ and zero otherwise.

Then

$$\sum_{i,j} |y_{ij}|^2 = \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} |y_{nj}|^2 + \sum_{m=0}^{\infty} \sum_{i=m}^{\infty} |y_{im}|^2.$$

For $m > 0$,

$$\sum_{i=m}^{\infty} |y_{im}|^2 \leq 4/m^2 \sum_{i=m}^{\infty} |x_{im}|^2 \leq 4/m^2 \|X\|^2$$

and for $n > 0$,

$$\sum_{j=n}^{\infty} |y_{nj}|^2 \leq 4/n^2 \|X\|^2.$$

Hence

$$\sum_{i,j} |y_{ij}|^2 \leq \|X\|^2 + \sum_{m=1}^{\infty} 4/n^2 \|X\|^2 + \|X\|^2 + \sum_{m=1}^{\infty} 4/m^2 \|X\|^2.$$

Therefore, $Y \in \mathcal{B}(\mathcal{H})$ and by Lemma 4, $\mathcal{B}(A_B) \subset \mathcal{B}(A_A)$. Now, assume $f$ is an analytic function on $\sigma(A)$ such that for even $n$, $f(i/n) = -i/n^2$. Then $f(z) = z^2i$. Hence for odd $n$, $f(1/n) = i/n^2 \neq -i/n^2$ and $B \neq f(A)$.

**Example 2.** Let $A = \sum \alpha_n P_n$ where $P_n$ is rank one for all $n$, $\alpha_0 = 0$, and $\alpha_n = 1/n^2$ for $n > 0$ and let $B = \sum \beta_n P_n$ where $\beta_0 = 0$. 
and $\beta_n = 1/n$ for $n > 0$. Then $B$ is a continuous function of $A$, in fact $B = f(A)$ where $f(z) = z^{1/2}$. Let $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$ where

$$x_{nj} = \begin{cases} 1/n & \text{for } n > 0 \text{ and } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $\Delta_B(X) = \Delta_A(Y)$ where $Y = (y_{ij})$, then

$$y_{n0} = x_{n0}(\beta - \beta_0)/(\alpha_n - \alpha_0) = (1/n)(1/n)/(1/n^2) = 1$$

for all $n$. Hence $Y \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\Delta_B) \not\subset \mathcal{B}(\Delta_A)$.

Other derivations whose ranges do not contain any nonzero one-sided ideals are those generated by unitary and self-adjoint operators. (See [9].)

It was shown in [7] that the range of a derivation generated by a nonunitary isometry does contain nonzero left-ideals. Other operators which possess this property are some of the weighted shifts.

4. Another question concerning the range of a derivation and, in this case, a two-sided ideal $\mathcal{I}$ of $\mathcal{B}(\mathcal{H})$ is whether $\mathcal{B}(\Delta_A) = \Delta_A(\mathcal{I})$.

**Theorem 6.** Let $A \in \mathcal{B}(\mathcal{H})$ and let $\mathcal{I}$ be a proper two-sided ideal of $\mathcal{B}(\mathcal{H})$. Consider the following conditions:

(a) $A' + \mathcal{I} = \mathcal{B}(\mathcal{H})$.

(b) $\mathcal{B}(\Delta_A) = \Delta_A(\mathcal{I})$.

(c) $\mathcal{B}(\Delta_A) \subset \mathcal{I}$.

(d) $A = T - \lambda$ for some $T \in \mathcal{I}$ and $\lambda \in \mathcal{C}$.

(a) is equivalent to (b), (c) is equivalent to (d), and (b) implies (c).

**Proof.** That (a) is equivalent to (b) is a consequence of the fact that $X = T + A'$ for some $T \in \mathcal{I}$ and $A' \in (A)'$ if and only if $\Delta_A(X) \in \Delta_A(I)$. That (c) is equivalent to (d) is a consequence of a theorem of Calkin [2] where he shows that the center of $\mathcal{B}(\mathcal{H})/\mathcal{I}$ consists of scalars. It is immediate that (b) implies (c).

**Remark.** An example to show that (c) does not imply (b) for the case when $\mathcal{I}$ is the ideal of compact operators can be obtained by letting $A$ be the adjoint of the weighted shift with weights $\{2, 1, 1/2, 1/3, \cdots\}$ and showing that each element of $(A)'$ is the translate of a Hilbert-Schmidt operator. (See [8].)

If we require only that the closures be equal, we have the following;

**Theorem 7.** Let $A \in \mathcal{B}(\mathcal{H})$ be compact and let $\mathcal{F}$ be the ideal of finite rank operators. Then $\mathcal{B}(\Delta_A) = \Delta_A(\mathcal{F})$. 
Proof. Let $f \in \mathcal{B}(\mathcal{H})$. Then $f = f_0 + f_\tau$ for some trace-class operator $T$ where $f_\tau(X) = \text{tr}(XT)$ and where $f_\tau$ annihilates the compact operators. (See Dixmier [3].) If $f$ annihilates $\Delta_A(\mathcal{I})$ then $f_\tau(\Delta_A(F)) = f(\Delta_A(F)) = 0$ for all $F \in \mathcal{I}$. However,

$$f_\tau(\Delta_A(F)) = \text{tr}((AF - FA)T) = \text{tr}(FAT - FAT)$$

for all $F \in \mathcal{I}$. Since $\mathcal{I}$ is dense in the trace-class operators, then $\Delta_A(-T) = 0$ and $T \in \{A\}'$. Hence $f_\tau$ annihilates the range of $\Delta_A$ and since $A$ is compact, $f(\Delta_A(X)) = f_\tau(\Delta_A(X)) = 0$ for all $X \in \mathcal{B}(\mathcal{H})$.

If $A$ is normal then Theorem 6 can be improved;

**Theorem 8.** Let $A \in \mathcal{B}(\mathcal{H})$ be normal and let $\mathcal{I}$ be a proper two-sided ideal of $\mathcal{B}(\mathcal{H})$. The following are equivalent:

(a) $[A]' + \mathcal{I} = \mathcal{B}(\mathcal{H})$.
(b) $\mathcal{R}(\Delta_A) = \Delta_A(\mathcal{I})$.
(c) $\mathcal{R}(\Delta_A) \subset \mathcal{I}$ and $\sigma(A)$ is finite.
(d) $A = T - \lambda$ for some $T \in \mathcal{I}$, some $\lambda \in \mathbb{C}$ and $\sigma(A)$ is finite.

Proof. That (a) is equivalent to (b) and (c) is equivalent to (d) follows from Theorem 6. If $A$ is normal with finite spectrum, then by a theorem of Anderson [1, p. 96] $\mathcal{R}(\Delta_A) + [A]' = \mathcal{B}(\mathcal{H})$. Hence, if $A = T - \lambda$ for some $T \in \mathcal{I}$ and $\lambda \in \mathbb{C}$ then $\mathcal{R}(\Delta_A) \subset \mathcal{I}$ and (d) implies (a). To show that (a) implies (d), assume that $\sigma(A)$ is infinite and that $[A]' + \mathcal{I} = \mathcal{B}(\mathcal{H})$. Then by Theorem 6, $A - \lambda \in \mathcal{I}$ for some $\lambda \in \mathbb{C}$. Since $\mathcal{I}$ is contained in the ideal of compact operators, we can assume that $A$ is compact. Let $A = A_1 \oplus A_2$ on $\mathcal{M} \bigoplus \mathcal{M}^\perp$ where $A_1$ is an infinite dimensional diagonal operator with distinct eigenvalues and let $P$ be the orthogonal projection onto $\mathcal{M}$. Hence, if $X \in \{A\}'$, then $PXP$ is diagonal. However, if we let $U$ be the unilateral shift on $\mathcal{M}$, then $[A]' + \mathcal{I} = \mathcal{B}(\mathcal{H})$ implies that $U = D + K$ for some diagonal operator $D$ and some compact operator $K$. This is clearly a contradiction (let $\{e_n\}$ be an orthonormal basis for $\mathcal{M}$ by which $U$ is the shift, then $((D - U)e_n, e_{n+1}) = 1$ for all $n$).

**References**


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