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**THE RANGE OF A DERIVATION AND IDEALS**

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## THE RANGE OF A DERIVATION AND IDEALS

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When  $A$  is in the Banach algebra  $\mathcal{B}(\mathcal{H})$  of all bounded linear operators on a Hilbert space  $\mathcal{H}$ , the derivation generated by  $A$  is the bounded operator  $\Delta_A$  on  $\mathcal{B}(\mathcal{H})$  defined by  $\Delta_A(X) = AX - XA$ . It is shown that the range of a derivation generated by a Hilbert-Schmidt or a diagonal operator contains no nonzero one-sided ideals of  $\mathcal{B}(\mathcal{H})$ . Also, for a two-sided ideal  $\mathcal{I}$  of  $\mathcal{B}(\mathcal{H})$ , necessary and sufficient condition on an operator  $A$  are given in order that the range of  $\Delta_A$  equals the range of  $\Delta_A$  restricted to  $\mathcal{I}$ .

1. In the following  $\mathcal{H}$  will denote an infinite dimensional complex Hilbert space.

For a fixed  $A \in \mathcal{B}(\mathcal{H})$ , we will concern ourselves with the following problems:

(a) For what  $B \in \mathcal{B}(\mathcal{H})$  is  $B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$  or  $\mathcal{R}(\Delta_A)B \subset \mathcal{R}(\Delta_A)$ .

(b) For what  $B \in \mathcal{B}(\mathcal{H})$  is  $B\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$  or  $\mathcal{B}(\mathcal{H})B \subset \mathcal{R}(\Delta_A)$ .

(c) For what  $B \in \mathcal{B}(\mathcal{H})$  is  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ .

It is easy to verify that for  $A, X, Y \in \mathcal{B}(\mathcal{H})$ .

(i)  $\Delta_A = \Delta_{A+\lambda}$  for all  $\lambda \in \mathcal{C}$

and

(ii)  $\Delta_A(XY) = X\Delta_A(Y) + \Delta_A(X)Y$ .

The identity (ii) yields some simple facts about the range of a derivation which show the interrelation of the above problems. (For a proof see [8].)

LEMMA 1. Let  $A, B \in \mathcal{B}(\mathcal{H})$  and let  $A'$  belong to the commutant  $\{A\}'$  of  $A$ . Then

(a) both  $A'\mathcal{R}(\Delta_A)$  and  $\mathcal{R}(\Delta_A)A'$  are contained in  $\mathcal{R}(\Delta_A)$ .

(b) if  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ , then both  $\Delta_{A'}(B)\mathcal{B}(\mathcal{H})$  and  $\mathcal{B}(\mathcal{H})\Delta_{A'}(B)$  are contained in  $\mathcal{R}(\Delta_A)$ .

(c)  $B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$  if and only if  $\Delta_A(B)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ .

(d)  $\mathcal{R}(\Delta_A)B \subset \mathcal{R}(\Delta_A)$  if and only if  $\mathcal{B}(\mathcal{H})\Delta_A(B) \subset \mathcal{R}(\Delta_A)$ .

From (b) of Lemma 1 it follows that if  $\mathcal{R}(\Delta_A)$  does not contain left- or right-ideals, then a necessary condition for  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$  is that  $B \in \{A\}''$ . In fact, more is true:

LEMMA 2. Let  $A \in \mathcal{B}(\mathcal{H})$ . If  $\mathcal{R}(\Delta_A)$  contains either no nonzero left-ideals or no nonzero right-ideals, then  $\Delta_B(\mathcal{I}) \subset \mathcal{R}(\Delta_A)$  implies

$B \in \{A\}''$ . ( $\mathcal{F}$  denotes the ideal of finite rank operators.)

*Proof.* Assume that  $\mathcal{R}(\Delta_A)$  contains no nonzero left-ideals (the argument for the other assumption is similar). Let  $P$  be a finite rank projection. If  $A' \in \{A\}'$ , then

$$\Delta_{A'}(B)PX = A'\Delta_B(PX) - \Delta_B(A'PX)$$

is in  $\mathcal{R}(\Delta_A)$  for all  $X \in \mathcal{B}(\mathcal{H})$ . Therefore,  $\Delta_{A'}(B)P\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$  and hence  $\Delta_{A'}(B)P = 0$ . However, this is true for any such  $P$  and hence  $\Delta_{A'}(B) = 0$ .

For the sake of completeness we include a somewhat simpler proof of a theorem of Stampfli [6]. In the proof,  $\sigma_l(A)$  denotes the left essential spectrum of  $A$  and is defined to be the set of those  $\lambda$  for which the coset of the Calkin algebra  $\mathcal{B}(\mathcal{H})/\mathcal{K}$  (where  $\mathcal{K}$  is the ideal of compact operators) containing  $A - \lambda$  fails to have a left inverse. The right essential spectrum  $\sigma_r(A)$  is defined in the obvious way.

**THEOREM 1.** *Let  $A \in \mathcal{B}(\mathcal{H})$ . Then  $\mathcal{R}(\Delta_A)$  contains no nonzero two-sided ideals of  $\mathcal{B}(\mathcal{H})$ .*

*Proof.* Replace  $A$  by  $A - \lambda$  where  $\lambda \in \sigma_l(A) \cap \sigma_r(A)$  if necessary in order to assume that there exist orthonormal sequences  $\{f_n\}$  and  $\{g_n\}$  such that  $\sum \|Af_n\|^{1/2} < \infty$  and  $\sum \|A^*g_n\|^{1/2} < \infty$ . (See [6].) Then for all  $X \in \mathcal{B}(\mathcal{H})$ ,

$$\sum |((AX - XA)f_n, g_n)|^{1/2} \leq \sum \|X\|^{1/2}(\|A^*g_n\|^{1/2} + \|Af_n\|^{1/2}) < \infty .$$

If  $\mathcal{R}(\Delta_A)$  contains a two-sided ideal, then it contains all finite rank operators. In particular, if  $f \otimes g$  denotes the rank one operator  $f \otimes g(x) = (x, g)f$ , then  $(f \otimes f)X \in \mathcal{R}(\Delta_A)$  for all  $f \in \mathcal{H}$  and  $X \in \mathcal{B}(\mathcal{H})$ . Hence

$$\sum |((f \otimes f)Xf_n, g_n)|^{1/2} < \infty .$$

Since

$$\begin{aligned} \sum |((f \otimes f)Xf_n, g_n)|^{1/2} &= \sum |(Xf_n, (f \otimes f)g_n)|^{1/2} \\ &= \sum |(Xf_n, f)(\overline{g_n, f})|^{1/2} , \end{aligned}$$

then

$$\sum |(Xf_n, f)(\overline{g_n, f})|^{1/2} < \infty$$

for all  $f \in \mathcal{H}$  and  $X \in \mathcal{B}(\mathcal{H})$ . However, if we choose  $X$  such that  $Xf_n = g_n$  and  $f$  such that  $\{|(g_n, f)|\}$  is not summable, we have a contradiction.

2. Let  $\mathcal{S}$  denote the set of Hilbert-Schmidt operators on  $\mathcal{H}$ . Equipped with the trace inner product  $(A, B) = \text{tr}(AB^*)$ ,  $\mathcal{S}$  is a Hilbert space [5]. If  $A \in \mathcal{B}(\mathcal{H})$ , then the restriction of  $\Delta_A$  to  $\mathcal{S}$  is a bounded operator on  $\mathcal{S}$  with adjoint  $(\Delta_A|_{\mathcal{S}})^* = \Delta_{A^*}|_{\mathcal{S}}$ . Hence  $\mathcal{S} = \mathcal{R}(\Delta_A|_{\mathcal{S}})^\ominus = \bigoplus (\{A^*\}' \cap \mathcal{S})$  where the double bar indicates closure with respect to the topology on  $\mathcal{S}$ .

**THEOREM 2.** *Let  $A \in \mathcal{S}$ . Then  $\mathcal{R}(\Delta_A)^\ominus = \mathcal{R}(\Delta_A|_{\mathcal{S}})^\ominus$ .*

*Proof.* It follows from the above remarks that  $\mathcal{R}(\Delta_A)^\perp \subset \mathcal{R}(\Delta_A|_{\mathcal{S}})^\perp = \{A^*\}' \cap \mathcal{S}$ . It remains to show the reverse inclusion. Let  $T \in \{A^*\}' \cap \mathcal{S}$ . Then for  $X \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} (\Delta_A(X), T) &= \text{tr}(T^* \Delta_A(X)) = \text{tr}(T^* AX) - \text{tr}(T^* XA) \\ &= \text{tr}(AT^* X) - \text{tr}(T^* XA) = \text{tr}(T^* XA) - \text{tr}(T^* XA) = 0. \end{aligned}$$

Therefore  $T \in \mathcal{R}(\Delta_A)^\perp$ .

**COROLLARY.** *Let  $A \in \mathcal{S}$ . Then  $\mathcal{R}(\Delta_A)^\ominus = \bigoplus (\{A^*\}' \cap \mathcal{S}) = \mathcal{S}$ .*

**THEOREM 3.** *If  $A \in \mathcal{S}$ , then  $\mathcal{R}(\Delta_A)$  does not contain any nonzero left- or right-ideals.*

In the proof of Theorem 3 we will make use of the following result.

**LEMMA 3.** *Let  $A \in \mathcal{S}$ . If  $(f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ , then  $Af = 0$ .*

*Proof.* Since  $\mathcal{R}(\Delta_A)^\perp = \{A^*\}' \cap \mathcal{S}$ , then  $0 = \text{tr}(A(f \otimes f)X) = \text{tr}(Af \otimes X^*f) = (Af, X^*f)$  for all  $X \in \mathcal{B}(\mathcal{H})$ . Hence  $Af = 0$ .

*Proof of Theorem 3.* Suppose that  $(f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ . Then  $f \otimes f = \Delta_A(X)$  for some  $X \in \mathcal{B}(\mathcal{H})$  and by Lemma 3,  $f = (f \otimes f)f = AXf - XAf = AXf$ . Since  $(f \otimes f)\mathcal{B}(\mathcal{H}) = \Delta_A(X)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ , then by Lemma 1,  $X\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$ . Therefore,  $((Xf) \otimes (Xf))\mathcal{B}(\mathcal{H}) \subset X(f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$  and by Lemma 3,  $Xf \in \ker(A)$ . Hence  $f = AXf = 0$ . The remainder follows by taking adjoints.

**COROLLARY 1.** *Let  $A \in \mathcal{S}$  and  $B \in \mathcal{B}(\mathcal{H})$ . Then  $B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$  if and only if  $B \in \{A\}'$ .*

*Proof.* This follows from Lemma 1 and the theorem.

**COROLLARY 2.** *Let  $A \in \mathcal{S}$ . If  $\Delta_B(\mathcal{F}) \subset \mathcal{R}(\Delta_A)$  then  $B \in \{A\}''$ .*

*Proof.* This follows from Lemma 2 and the theorem.

3. We now turn our attention to diagonal operators. When expressing a diagonal operator as the sum  $A = \sum \alpha_n P_n$ , unless otherwise stated we shall assume that  $P_n$  is the rank one projection onto the subspace spanned by  $e_n$ , where  $\{e_n\}$  is an orthonormal basis. (However, we do not require that the  $\alpha_n$ 's be distinct.) Each operator  $X$  has a matrix  $(x_{ij})$  with respect to this fixed basis.

The principle result of this section is that the range of a derivation generated by a diagonal operator contains no nonzero left- or right-ideals. The theorem is slightly more general.

**THEOREM 4.** *Let  $A \in \mathcal{B}(\mathcal{H})$  have the property that there exist reducing subspaces  $\mathcal{M}_n$  of  $A$ , each finite dimensional, such that  $\mathcal{H} = \sum \bigoplus \mathcal{M}_n$ . Then  $\mathcal{R}(\Delta_A)$  contains no nonzero positive operators.*

*Proof.* Let  $P = \Delta_A(X)$  where  $P$  is positive. If  $P_n$  is the orthogonal projection onto  $\mathcal{M}_n$ , then  $P_n P | \mathcal{M}_n = A_n X_n - X_n A_n$  where  $A_n = A | \mathcal{M}_n$  and  $X_n$  is the compression of  $X$  to  $\mathcal{M}_n$ . Since  $\mathcal{M}_n$  is finite dimensional, then  $\text{tr}(P_n P | \mathcal{M}_n) = 0$ . Hence  $P_n P | \mathcal{M}_n$  being a positive operator with zero trace, must be 0. Therefore,  $P_n P P_n = 0$  (on  $\mathcal{H}$ ). Hence  $P^{1/2} P_n = 0$  and  $P^{1/2} = 0$ .

**COROLLARY 1.** *If  $A$  satisfies the hypothesis of the theorem and if either  $B \mathcal{R}(\Delta_A)$  or  $\mathcal{R}(\Delta_A) B$  is contained in  $\mathcal{R}(\Delta_A)$ , then  $B \in \{A\}'$ .*

**COROLLARY 2.** *If  $A$  satisfies the hypothesis of the theorem and  $\Delta_B(\mathcal{F}) \subset \mathcal{R}(\Delta_A)$ , then  $B \in \{A\}''$ .*

**COROLLARY 3.** *Let  $A$  be normal with finite spectrum. Then for  $B \in \mathcal{B}(\mathcal{H})$ ,  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$  if and only if  $B \in \{A\}''$ .*

*Proof.* If  $B \in \{A\}''$  then  $B$  is a polynomial of  $A$  and hence  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ . (See [1, p. 79].) The converse follows from Corollary 2.

**LEMMA 4.** *Let  $A, B \in \mathcal{B}(\mathcal{H})$  where  $A = \sum \alpha_i P_i$ . Then  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$  if and only if  $B = \sum \beta_i P_i$  for some set of scalars  $\beta_0, \beta_1, \dots$  and for every operator  $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$  there exists an operator  $Y = (y_{ij}) \in \mathcal{B}(\mathcal{H})$  such that  $(\alpha_i - \alpha_j) = (\beta_i - \beta_j)x_{ij}$  for all  $i, j$ .*

*Proof.* This follows from Corollary 2 and the fact that  $[\Delta_A(X)]_{ij} = (\alpha_i - \alpha_j)x_{ij}$  if  $X = (x_{ij})$ .

**THEOREM 5.** *Let  $A \in \mathcal{B}(\mathcal{H})$  be diagonal. If for  $B \in \mathcal{B}(\mathcal{H})$ ,  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ , then  $B = f(A)$  for some function  $f$  which is Lipschitz on the spectrum of  $A$ .*

*Proof.* Let  $A = \sum \alpha_i P_i$ . If  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ , then by Corollary 2,  $B = \sum \beta_i P_i$  for some sequence of scalars  $\{\beta_i\}$  and for any  $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$ , there exists a  $Y = (y_{ij}) \in \mathcal{B}(\mathcal{H})$  such that  $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$  whenever  $\alpha_i \neq \alpha_j$ . It follows that  $((\beta_i - \beta_j)/(\alpha_i - \alpha_j))$  is bounded by some positive number  $M$ . Define  $f$  such that  $f(\alpha_i) = \beta_i$ . Then  $f$  is a Lipschitz function defined on a dense subset of  $\sigma(A)$  onto a dense subset of  $\sigma(B)$ . Therefore, we can extend  $f$  to be Lipschitz on  $\sigma(A)$  onto  $\sigma(B)$ .

It was shown in [7] that if  $B$  is an analytic function of  $A$ , then  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ . To have range inclusion it is neither necessary that  $B$  be an analytic function of  $A$  nor sufficient that  $B$  be a continuous function of  $A$  as seen in the next two examples.

EXAMPLE 1. Let  $A = \sum \alpha_n P_n$  where  $\dim P_n = 1$ ,  $\alpha_0 = 0$ , and

$$\alpha_n = \begin{cases} i/n & \text{for } n \text{ even} \\ 1/n & \text{for } n \text{ odd.} \end{cases}$$

Let  $B = \sum \beta_n P_n$  where  $\beta_0 = 0$  and  $\beta_n = -i/n^2$  for  $n \geq 1$ . A direct computation shows that if  $n < m$ , then  $|(\beta_n - \beta_m)/(\alpha_n - \alpha_m)| \leq 2/n$ . Now, for any  $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$ , consider the matrix  $Y = (y_{ij})$  where  $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$  whenever  $\alpha_i \neq \alpha_j$  and zero otherwise. Then

$$\sum_{i,j} |y_{ij}|^2 = \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} |y_{nj}|^2 + \sum_{m=0}^{\infty} \sum_{i=m}^{\infty} |y_{im}|^2.$$

For  $m > 0$ ,

$$\sum_{i=m}^{\infty} |y_{im}|^2 \leq 4/m^2 \sum_{i=m}^{\infty} |x_{im}|^2 \leq 4/m^2 \|X\|^2$$

and for  $n > 0$ ,

$$\sum_{j=n}^{\infty} |y_{nj}|^2 \leq 4/n^2 \|X\|^2.$$

Hence

$$\sum_{i,j} |y_{ij}|^2 \leq \|X\|^2 + \sum_{m=1}^{\infty} 4/m^2 \|X\|^2 + \|X\|^2 + \sum_{m=1}^{\infty} 4/m^2 \|X\|^2.$$

Therefore,  $Y \in \mathcal{B}(\mathcal{H})$  and by Lemma 4,  $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ . Now, assume  $f$  is an analytic function on  $\sigma(A)$  such that for even  $n$ ,  $f(i/n) = -i/n^2$ . Then  $f(z) = z^2 i$ . Hence for odd  $n$ ,  $f(1/n) = i/n^2 \neq -i/n^2$  and  $B \neq f(A)$ .

EXAMPLE 2. Let  $A = \sum \alpha_n P_n$  where  $P_n$  is rank one for all  $n$ ,  $\alpha_0 = 0$ , and  $\alpha_n = 1/n^2$  for  $n > 0$  and let  $B = \sum \beta_n P_n$  where  $\beta_0 = 0$

and  $\beta_n = 1/n$  for  $n > 0$ . Then  $B$  is a continuous function of  $A$ , in fact  $B = f(A)$  where  $f(z) = z^{1/2}$ . Let  $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$  where

$$x_{nj} = \begin{cases} 1/n & \text{for } n > 0 \text{ and } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If  $\Delta_B(X) = \Delta_A(Y)$  where  $Y = (y_{ij})$ , then

$$y_{n0} = x_{n0}(\beta_n - \beta_0)/(\alpha_n - \alpha_0) = (1/n)(1/n)/(1/n^2) = 1$$

for all  $n$ . Hence  $Y \notin \mathcal{B}(\mathcal{H})$  and  $\mathcal{R}(\Delta_B) \not\subset \mathcal{R}(\Delta_A)$ .

Other derivations whose ranges do not contain any nonzero one-sided ideals are those generated by unitary and self-adjoint operators. (See [9].)

It was shown in [7] that the range of a derivation generated by a nonunitary isometry *does* contain nonzero left-ideals. Other operators which possess this property are some of the weighted shifts.

4. Another question concerning the range of a derivation and, in this case, a two-sided ideal  $\mathcal{I}$  of  $\mathcal{B}(\mathcal{H})$  is whether  $\mathcal{R}(\Delta_A) = \Delta_A(\mathcal{I})$ .

**THEOREM 6.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and let  $\mathcal{I}$  be a proper two-sided ideal of  $\mathcal{B}(\mathcal{H})$ . Consider the following conditions:*

- (a)  $\{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H})$ .
- (b)  $\mathcal{R}(\Delta_A) = \Delta_A(\mathcal{I})$ .
- (c)  $\mathcal{R}(\Delta_A) \subset \mathcal{I}$ .
- (d)  $A = T - \lambda$  for some  $T \in \mathcal{I}$  and  $\lambda \in \mathcal{C}$ .

(a) is equivalent to (b), (c) is equivalent to (d), and (b) implies (c).

*Proof.* That (a) is equivalent to (b) is a consequence of the fact that  $X = T + A'$  for some  $T \in \mathcal{I}$  and  $A' \in \{A\}'$  if and only if  $\Delta_A(X) \in \Delta_A(\mathcal{I})$ . That (c) is equivalent to (d) is a consequence of a theorem of Calkin [2] where he shows that the center of  $\mathcal{B}(\mathcal{H})/\mathcal{I}$  consists of scalars. It is immediate that (b) implies (c).

**REMARK.** An example to show that (c) does not imply (b) for the case when  $\mathcal{I}$  is the ideal of compact operators can be obtained by letting  $A$  be the adjoint of the weighted shift with weights  $\{2, 1, 1/2, 1/3, \dots\}$  and showing that each element of  $\{A\}'$  is the translate of a Hilbert-Schmidt operator. (See [8].)

If we require only that the closures be equal, we have the following;

**THEOREM 7.** *Let  $A \in \mathcal{B}(\mathcal{H})$  be compact and let  $\mathcal{F}$  be the ideal of finite rank operators. Then  $\mathcal{R}(\Delta_A)^- = \Delta_A(\mathcal{F})^-$ .*

*Proof.* Let  $f \in \mathcal{B}(\mathcal{H})^*$ . Then  $f = f_0 + f_T$  for some trace-class operator  $T$  where  $f_T(X) = \text{tr}(XT)$  and where  $f_0$  annihilates the compact operators. (See Dixmier [3].) If  $f$  annihilates  $\Delta_A(\mathcal{I})$  then  $f_T(\Delta_A(F)) = f(\Delta_A(F)) = 0$  for all  $F \in \mathcal{I}$ . However,

$$\begin{aligned} f_T(\Delta_A(F)) &= \text{tr}((AF - FA)T) = \text{tr}(AFT - FAT) \\ &= \text{tr}(FTA - FAT) = \text{tr}(F\Delta_A(-T)) \end{aligned}$$

for all  $F \in \mathcal{I}$ . Since  $\mathcal{I}$  is dense in the trace-class operators, then  $\Delta_A(-T) = 0$  and  $T \in \{A\}'$ . Hence  $f_T$  annihilates the range of  $\Delta_A$  and since  $A$  is compact,  $f(\Delta_A(X)) = f_T(\Delta_A(X)) = 0$  for all  $X \in \mathcal{B}(\mathcal{H})$ .

If  $A$  is normal then Theorem 6 can be improved;

**THEOREM 8.** *Let  $A \in \mathcal{B}(\mathcal{H})$  be normal and let  $\mathcal{I}$  be a proper two-sided ideal of  $\mathcal{B}(\mathcal{H})$ . The following are equivalent:*

- (a)  $\{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H})$ .
- (b)  $\mathcal{R}(\Delta_A) = \Delta_A(\mathcal{I})$ .
- (c)  $\mathcal{R}(\Delta_A) \subset \mathcal{I}$  and  $\sigma(A)$  is finite.
- (d)  $A = T - \lambda$  for some  $T \in \mathcal{I}$ , some  $\lambda \in \mathcal{C}$  and  $\sigma(A)$  is finite.

*Proof.* That (a) is equivalent to (b) and (c) is equivalent to (d) follows from Theorem 6. If  $A$  is normal with finite spectrum, then by a theorem of Anderson [1, p. 96]  $\mathcal{R}(\Delta_A) + \{A\}' = \mathcal{B}(\mathcal{H})$ . Hence, if  $A = T - \lambda$  for some  $T \in \mathcal{I}$  and  $\lambda \in \mathcal{C}$  then  $\mathcal{R}(\Delta_A) \subset \mathcal{I}$  and (d) implies (a). To show that (a) implies (d), assume that  $\sigma(A)$  is infinite and that  $\{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H})$ . Then by Theorem 6,  $A - \lambda \in \mathcal{I}$  for some  $\lambda \in \mathcal{C}$ . Since  $\mathcal{I}$  is contained in the ideal of compact operators, we can assume that  $A$  is compact. Let  $A = A_1 \oplus A_2$  on  $\mathcal{M} \oplus \mathcal{M}^\perp$  where  $A_1$  is an infinite dimensional diagonal operator with distinct eigenvalues and let  $P$  be the orthogonal projection onto  $\mathcal{M}$ . Hence, if  $X \in \{A\}'$ , then  $PXP$  is diagonal. However, if we let  $U$  be the unilateral shift on  $\mathcal{M}$ , then  $\{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H})$  implies that  $U = D + K$  for some diagonal operator  $D$  and some compact operator  $K$ . This is clearly a contradiction (let  $\{e_n\}$  be an orthonormal basis for  $\mathcal{M}$  by which  $U$  is the shift, then  $((D - U)e_n, e_{n+1}) = 1$  for all  $n$ ).

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