Pacific Journal of Mathematics

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ROBERT EARL WEBER

Vol. 50, No. 2 October 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 2, 1974

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R. E. WEBER

When A is in the Banach algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space $\mathscr{H}$, the derivation generated by A is the bounded operator Δ_A on $\mathscr{B}(\mathscr{H})$ defined by $\Delta_A(X) = AX - XA$. It is shown that the range of a derivation generated by a Hilbert-Schmidt or a diagonal operator contains no nonzero one-sided ideals of $\mathscr{B}(\mathscr{H})$. Also, for a two-sided ideal $\mathscr{I}$ of $\mathscr{B}(\mathscr{H})$, necessary and sufficient condition on an operator A are given in order that the range of Δ_A equals the range of Δ_A restricted to $\mathscr{I}$.

1. In the following $\mathscr{H}$ will denote an infinite dimensional complex Hilbert space.

For a fixed $A \in \mathscr{B}(\mathscr{H})$, we will concern ourselves with the following problems:

(a) For what $B \in \mathscr{B}(\mathscr{H})$ is $B\mathscr{R}(\Delta_A) \subset \mathscr{R}(\Delta_A)$ or $\mathscr{R}(\Delta_A)B \subset \mathscr{R}(\Delta_A)$.

(b) For what $B \in \mathscr{B}(\mathscr{H})$ is $B\,\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\Delta_A)$ or $\mathscr{B}(\mathscr{H})B \subset \mathscr{R}(\Delta_A)$.

(c) For what $B \in \mathscr{B}(\mathscr{H})$ is $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$.

It is easy to verify that for $A, X, Y \in \mathscr{B}(\mathscr{H})$.

(i) $\Delta_A = \Delta_{A+\lambda}$ for all $\lambda \in \mathscr{C}$

and

(ii) $\Delta_A(XY) = X\Delta_A(Y) + \Delta_A(X)Y$.

The identity (ii) yields some simple facts about the range of a derivation which show the interrelation of the above problems. (For a proof see [8].)

LEMMA 1. *Let $A, B \in \mathscr{B}(\mathscr{H})$ and let A' belong to the commutant $\{A\}'$ of A. Then*

(a) *both $A'\mathscr{R}(\Delta_A)$ and $\mathscr{R}(\Delta_A)A'$ are contained in $\mathscr{R}(\Delta_A)$.*

(b) *if $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$, then both $\Delta_{A'}(B)\mathscr{B}(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})\Delta_{A'}(B)$ are contained in $\mathscr{R}(\Delta_A)$.*

(c) *$B\mathscr{R}(\Delta_A) \subset \mathscr{R}(\Delta_A)$ if and only if $\Delta_A(B)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\Delta_A)$.*

(d) *$\mathscr{R}(\Delta_A)B \subset \mathscr{R}(\Delta_A)$ if and only if $\mathscr{B}(\mathscr{H})\Delta_A(B) \subset \mathscr{R}(\Delta_A)$.*

From (b) of Lemma 1 it follows that if $\mathscr{R}(\Delta_A)$ does not contain left- or right-ideals, then a necessary condition for $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$ is that $B \in \{A\}''$. In fact, more is true:

LEMMA 2. *Let $A \in \mathscr{B}(\mathscr{H})$. If $\mathscr{R}(\Delta_A)$ contains either no nonzero left-ideals or no nonzero right-ideals, then $\Delta_B(\mathscr{F}) \subset \mathscr{R}(\Delta_A)$ implies*

$B \in \{A\}''$. *($\mathscr{F}$ denotes the ideal of finite rank operators.)*

Proof. Assume that $\mathscr{R}(\Delta_A)$ contains no nonzero left-ideals (the argument for the other assumption is similar). Let P be a finite rank projection. If $A' \in \{A\}'$, then

$$\Delta_{A'}(B)PX = A'\Delta_B(PX) - \Delta_B(A'PX)$$

is in $\mathscr{R}(\Delta_A)$ for all $X \in \mathscr{B}(\mathscr{H})$. Therefore, $\Delta_{A'}(B)P\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\Delta_A)$ and hence $\Delta_{A'}(B)P = 0$. However, this is true for any such P and hence $\Delta_{A'}(B) = 0$.

For the sake of completeness we include a somewhat simpler proof of a theorem of Stampfli [6]. In the proof, $\sigma_l(A)$ denotes the left essential spectrum of A and is defined to be the set of those λ for which the coset of the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$ (where $\mathscr{K}$ is the ideal of compact operators) containing $A - \lambda$ fails to have a left inverse. The right essential spectrum $\sigma_r(A)$ is defined in the obvious way.

THEOREM 1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then $\mathscr{R}(\Delta_A)$ contains no nonzero two-sided ideals of $\mathscr{B}(\mathscr{H})$.*

Proof. Replace A by $A - \lambda$ where $\lambda \in \sigma_l(A) \cap \sigma_r(A)$ if necessary in order to assume that there exist orthonormal sequences $\{f_n\}$ and $\{g_n\}$ such that $\sum \|Af_n\|^{1/2} < \infty$ and $\sum \|A^*g_n\|^{1/2} < \infty$. (See [6].) Then for all $X \in \mathscr{B}(\mathscr{H})$,

$$\sum |((AX - XA)f_n, g_n)|^{1/2} \leqq \sum \|X\|^{1/2}(\|A^*g_n\|^{1/2} + \|Af_n\|^{1/2}) < \infty .$$

If $\mathscr{R}(\Delta_A)$ contains a two-sided ideal, then it contains all finite rank operators. In particular, if $f \otimes g$ denotes the rank one operator $f \otimes g(x) = (x, g)f$, then $(f \otimes f)X \in \mathscr{R}(\Delta_A)$ for all $f \in \mathscr{H}$ and $X \in \mathscr{B}(\mathscr{H})$. Hence

$$\sum |((f \otimes f)Xf_n, g_n)|^{1/2} < \infty .$$

Since

$$\begin{aligned}\sum |((f \otimes f)Xf_n, g_n)|^{1/2} &= \sum |(Xf_n, (f \otimes f)g_n)|^{1/2} \\ &= \sum |(Xf_n, f)\overline{(g_n, f)}|^{1/2} ,\end{aligned}$$

then

$$\sum |(Xf_n, f)\overline{(g_n, f)}|^{1/2} < \infty$$

for all $f \in \mathscr{H}$ and $X \in \mathscr{B}(\mathscr{H})$. However, if we choose X such that $Xf_n = g_n$ and f such that $\{|(g_n, f)|\}$ is not summable, we have a contradiction.

2. Let $\mathscr{S}$ denote the set of Hilbert-Schmidt operators on $\mathscr{H}$. Equipped with the trace inner product $(A, B) = \operatorname{tr}(AB^*)$, $\mathscr{S}$ is a Hilbert space [5]. If $A \in \mathscr{B}(\mathscr{H})$, then the restriction of Δ_A to $\mathscr{S}$ is a bounded operator on $\mathscr{S}$ with adjoint $(\Delta_A \mid \mathscr{S})^* = \Delta_{A^*} \mid \mathscr{S}$. Hence $\mathscr{S} = \mathscr{R}(\Delta_A \mid \mathscr{S})^= \oplus (\{A^*\}' \cap \mathscr{S})$ where the double bar indicates closure with respect to the topology on $\mathscr{S}$.

THEOREM 2. *Let* $A \in \mathscr{S}$. *Then* $\mathscr{R}(\Delta_A)^= = \mathscr{R}(\Delta_A \mid \mathscr{S})^=$.

Proof. It follows from the above remarks that $\mathscr{R}(\Delta_A)^\perp \subset \mathscr{R}(\Delta_A \mid \mathscr{S})^\perp = \{A^*\}' \cap \mathscr{S}$. It remains to show the reverse inclusion. Let $T \in \{A^*\}' \cap \mathscr{S}$. Then for $X \in \mathscr{B}(\mathscr{H})$

$$\begin{aligned}(\Delta_A(X), T) &= \operatorname{tr}(T^*\Delta_A(X)) = \operatorname{tr}(T^*AX) - \operatorname{tr}(T^*XA)\\ &= \operatorname{tr}(AT^*X) - \operatorname{tr}(T^*XA) = \operatorname{tr}(T^*XA) - \operatorname{tr}(T^*XA) = 0 .\end{aligned}$$

Therefore $T \in \mathscr{R}(\Delta_A)^\perp$.

COROLLARY. *Let* $A \in \mathscr{S}$. *Then* $\mathscr{R}(\Delta_A)^= \oplus (\{A^*\}' \cap \mathscr{S}) = \mathscr{S}$.

THEOREM 3. *If* $A \in \mathscr{S}$, *then* $\mathscr{R}(\Delta_A)$ *does not contain any nonzero left- or right-ideals.*

In the proof of Theorem 3 we will make use of the following result.

LEMMA 3. *Let* $A \in \mathscr{S}$. *If* $(f \otimes f)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\Delta_A)$, *then* $Af = 0$.

Proof. Since $\mathscr{R}(\Delta_A) \perp \{A^*\}' \cap \mathscr{S}$, then $0 = \operatorname{tr}(A(f \otimes f)X) = \operatorname{tr}(Af \otimes X^*f) = (Af, X^*f)$ for all $X \in \mathscr{B}(\mathscr{H})$. Hence $Af = 0$.

Proof of Theorem 3. Suppose that $(f \otimes f)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\Delta_A)$. Then $f \otimes f = \Delta_A(X)$ for some $X \in \mathscr{B}(\mathscr{H})$ and by Lemma 3, $f = (f \otimes f)f = AXf - XAf = AXf$. Since $(f \otimes f)\mathscr{B}(\mathscr{H}) = \Delta_A(X)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\Delta_A)$, then by Lemma 1, $X\mathscr{R}(\Delta_A) \subset \mathscr{R}(\Delta_A)$. Therefore, $((Xf) \otimes (Xf))\mathscr{B}(\mathscr{H}) \subset X(f \otimes f)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\Delta_A)$ and by Lemma 3, $Xf \in \ker(A)$. Hence $f = AXf = 0$. The remainder follows by taking adjoints.

COROLLARY 1. *Let* $A \in \mathscr{S}$ *and* $B \in \mathscr{B}(\mathscr{H})$. *Then* $B\mathscr{R}(\Delta_A) \subset \mathscr{R}(\Delta_A)$ *if and only if* $B \in \{A\}'$.

Proof. This follows from Lemma 1 and the theorem.

COROLLARY 2. *Let* $A \in \mathscr{S}$. *If* $\Delta_B(\mathscr{F}) \subset \mathscr{R}(\Delta_A)$ *then* $B \in \{A\}''$.

Proof. This follows from Lemma 2 and the theorem.

3. We now turn our attention to diagonal operators. When expressing a diagonal operator as the sum $A = \sum \alpha_n P_n$, unless otherwise stated we shall assume that P_n is the rank one projection onto the subspace spanned by e_n, where $\{e_n\}$ is an orthonormal basis. (However, we do not require that the α_n's be distinct.) Each operator X has a matrix (x_{ij}) with respect to this fixed basis.

The principle result of this section is that the range of a derivation generated by a diagonal operator contains no nonzero left- or right-ideals. The theorem is slightly more general.

THEOREM 4. *Let* $A \in \mathscr{B}(\mathscr{H})$ *have the property that there exist reducing subspaces* $\mathscr{M}_n$ *of* A, *each finite dimensional, such that* $\mathscr{H} = \sum \oplus \mathscr{M}_n$. *Then* $\mathscr{R}(\Delta_A)$ *contains no nonzero positive operators.*

Proof. Let $P = \Delta_A(X)$ where P is positive. If P_n is the orthogonal projection onto $\mathscr{M}_n$, then $P_n P \mid \mathscr{M}_n = A_n X_n - X_n A_n$ where $A_n = A \mid \mathscr{M}_n$ and X_n is the compression of X to $\mathscr{M}_n$. Since $\mathscr{M}_n$ is finite dimensional, then $\operatorname{tr}(P_n P \mid \mathscr{M}_n) = 0$. Hence $P_n P \mid \mathscr{M}_n$ being a positive operator with zero trace, must be 0. Therefore, $P_n P P_n = 0$ (on $\mathscr{H}$). Hence $P^{1/2} P_n = 0$ and $P^{1/2} = 0$.

COROLLARY 1. *If* A *satisfies the hypothesis of the theorem and if either* $B\mathscr{R}(\Delta_A)$ *or* $\mathscr{R}(\Delta_A)B$ *is contained in* $\mathscr{R}(\Delta_A)$, *then* $B \in \{A\}'$.

COROLLARY 2. *If* A *satisfies the hypothesis of the theorem and* $\Delta_B(\mathscr{T}) \subset \mathscr{R}(\Delta_A)$, *then* $B \in \{A\}''$.

COROLLARY 3. *Let* A *be normal with finite spectrum. Then for* $B \in \mathscr{B}(\mathscr{H})$, $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$ *if and only if* $B \in \{A\}''$.

Proof. If $B \in \{A\}''$ then B is a polynomial of A and hence $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$. (See [1, p. 79].) The converse follows from Corollary 2.

LEMMA 4. *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *where* $A = \sum \alpha_i P_i$. *Then* $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$ *if and only if* $B = \sum \beta_i P_i$ *for some set of scalars* $\beta_0, \beta_1 \cdots$ *and for every operator* $X = (x_{ij}) \in \mathscr{B}(\mathscr{H})$ *there exists an operator* $Y = (y_{ij}) \in \mathscr{B}(\mathscr{H})$ *such that* $(\alpha_i - \alpha_j) = (\beta_i - \beta_j)x_{ij}$ *for all* i, j.

Proof. This follows from Corollary 2 and the fact that $[\Delta_A(X)]_{ij} = (\alpha_i - \alpha_j)x_{ij}$ if $X = (x_{ij})$.

THEOREM 5. *Let* $A \in \mathscr{B}(\mathscr{H})$ *be diagonal. If for* $B \in \mathscr{B}(\mathscr{H})$, $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$, *then* $B = f(A)$ *for some function* f *which is Lipschitz on the spectrum of* A.

Proof. Let $A = \sum \alpha_i P_i$. If $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$, then by Corollary 2, $B = \sum \beta_i P_i$ for some sequence of scalars $\{\beta_i\}$ and for any $X = (x_{ij}) \in \mathscr{B}(\mathscr{H})$, there exists a $Y = (y_{ij}) \in \mathscr{B}(\mathscr{H})$ such that $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$. It follows that $((\beta_i - \beta_j)/(\alpha_i - \alpha_j))$ is bounded by some positive number M. Define f such that $f(\alpha_i) = \beta_i$. Then f is a Lipschitz function defined on a dense subset of $\sigma(A)$ onto a dense subset of $\sigma(B)$. Therefore, we can extend f to be Lipschitz on $\sigma(A)$ onto $\sigma(B)$.

It was shown in [7] that if B is an analytic function of A, then $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$. To have range inclusion it is neither necessary that B be an analytic function of A nor sufficient that B be a continuous function of A as seen in the next two examples.

EXAMPLE 1. Let $A = \sum \alpha_n P_n$ where $\dim P_n = 1$, $\alpha_0 = 0$, and

$$\alpha_n = \begin{cases} i/n & \text{for } n \text{ even} \\ 1/n & \text{for } n \text{ odd}\,. \end{cases}$$

Let $B = \sum \beta_n P_n$ where $\beta_0 = 0$ and $\beta_n = -i/n^2$ for $n \geqq 1$. A direct computation shows that if $n < m$, then $|(\beta_n - \beta_m)/(\alpha_n - \alpha_m)| \leqq 2/n$. Now, for any $X = (x_{ij}) \in \mathscr{B}(\mathscr{H})$, consider the matrix $Y = (y_{ij})$ where $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$ and zero otherwise. Then

$$\sum_{i,j} |y_{ij}|^2 = \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} |y_{nj}|^2 + \sum_{m=0}^{\infty} \sum_{i=m}^{\infty} |y_{im}|^2 \,.$$

For $m > 0$,

$$\sum_{i=m}^{\infty} |y_{im}|^2 \leqq 4/m^2 \sum_{i=m}^{\infty} |x_{im}|^2 \leqq 4/m^2 \,\|X\|^2$$

and for $n > 0$,

$$\sum_{j=n}^{\infty} |y_{nj}|^2 \leqq 4/n^2 \,\|X\|^2 \,.$$

Hence

$$\textstyle\sum_{i,j} |y_{ij}|^2 \leqq \|X\|^2 + \sum_{m=1}^{\infty} 4/n^2 \,\|X\|^2 + \|X\|^2 + \sum_{m=1}^{\infty} 4/m^2 \,\|X\|^2 \,.$$

Therefore, $Y \in \mathscr{B}(\mathscr{H})$ and by Lemma 4, $\mathscr{R}(\Delta_B) \subset \mathscr{R}(\Delta_A)$. Now, assume f is an analytic function on $\sigma(A)$ such that for even n, $f(i/n) = -i/n^2$. Then $f(z) = z^2 i$. Hence for odd n, $f(1/n) = i/n^2 \neq -i/n^2$ and $B \neq f(A)$.

EXAMPLE 2. Let $A = \sum \alpha_n P_n$ where P_n is rank one for all n, $\alpha_0 = 0$, and $\alpha_n = 1/n^2$ for $n > 0$ and let $B = \sum \beta_n P_n$ where $\beta_0 = 0$

and $\beta_n = 1/n$ for $n > 0$. Then B is a continuous function of A, in fact $B = f(A)$ where $f(z) = z^{1/2}$. Let $X = (x_{ij}) \in \mathscr{B}(\mathscr{H})$ where

$$x_{nj} = \begin{cases} 1/n & \text{for } n > 0 \text{ and } j = 0 \\ 0 & \text{otherwise} . \end{cases}$$

If $\Delta_B(X) = \Delta_A(Y)$ where $Y = (y_{ij})$, then

$$y_{n0} = x_{n0}(\beta_n - \beta_0)/(\alpha_n - \alpha_0) = (1/n)(1/n)/(1/n^2) = 1$$

for all n. Hence $Y \notin \mathscr{B}(\mathscr{H})$ and $\mathscr{R}(\Delta_B) \not\subset \mathscr{R}(\Delta_A)$.

Other derivations whose ranges do not contain any nonzero one-sided ideals are those generated by unitary and self-adjoint operators. (See [9].)

It was shown in [7] that the range of a derivation generated by a nonunitary isometry *does* contain nonzero left-ideals. Other operators which possess this property are some of the weighted shifts.

4. Another question concerning the range of a derivation and, in this case, a two-sided ideal $\mathscr{J}$ of $\mathscr{B}(\mathscr{H})$ is whether $\mathscr{R}(\Delta_A) = \Delta_A(\mathscr{J})$.

THEOREM 6. *Let* $A \in \mathscr{B}(\mathscr{H})$ *and let* $\mathscr{J}$ *be a proper two-sided ideal of* $\mathscr{B}(\mathscr{H})$. *Consider the following conditions*:

(a) $\{A\}' + \mathscr{J} = \mathscr{B}(\mathscr{H})$.

(b) $\mathscr{R}(\Delta_A) = \Delta_A(\mathscr{J})$.

(c) $\mathscr{R}(\Delta_A) \subset \mathscr{J}$.

(d) $A = T - \lambda$ *for some* $T \in \mathscr{J}$ *and* $\lambda \in \mathscr{C}$.

(a) *is equivalent to* (b), (c) *is equivalent to* (d), *and* (b) *implies* (c).

Proof. That (a) is equivalent to (b) is a consequence of the fact that $X = T + A'$ for some $T \in \mathscr{J}$ and $A' \in \{A\}'$ if and only if $\Delta_A(X) \in \Delta_A(\mathscr{J})$. That (c) is equivalent to (d) is a consequence of a theorem of Calkin [2] where he shows that the center of $\mathscr{B}(\mathscr{H})/\mathscr{J}$ consists of scalars. It is immediate that (b) implies (c).

REMARK. An example to show that (c) does not imply (b) for the case when $\mathscr{J}$ is the ideal of compact operators can be obtained by letting A be the adjoint of the weighted shift with weights $\{2, 1, 1/2, 1/3, \cdots\}$ and showing that each element of $\{A\}'$ is the translate of a Hilbert-Schmidt operator. (See [8].)

If we require only that the closures be equal, we have the following;

THEOREM 7. *Let* $A \in \mathscr{B}(\mathscr{H})$ *be compact and let* $\mathscr{F}$ *be the ideal of finite rank operators. Then* $\mathscr{R}(\Delta_A)^- = \Delta_A(\mathscr{F})^-$.

Proof. Let $f \in \mathscr{B}(\mathscr{H})^*$. Then $f = f_0 + f_T$ for some trace-class operator T where $f_T(X) = \operatorname{tr}(XT)$ and where f_0 annihilates the compact operators. (See Dixmier [3].) If f annihilates $\Delta_A(\mathscr{F})$ then $f_T(\Delta_A(F)) = f(\Delta_A(F)) = 0$ for all $F \in \mathscr{F}$. However,

$$\begin{aligned} f_T(\Delta_A(F)) &= \operatorname{tr}((AF - FA)T) = \operatorname{tr}(AFT - FAT) \\ &= \operatorname{tr}(FTA - FAT) = \operatorname{tr}(F\Delta_A(-T)) \end{aligned}$$

for all $F \in \mathscr{F}$. Since $\mathscr{F}$ is dense in the trace-class operators, then $\Delta_A(-T) = 0$ and $T \in \{A\}'$. Hence f_T annihilates the range of Δ_A and since A is compact, $f(\Delta_A(X)) = f_T(\Delta_A(X)) = 0$ for all $X \in \mathscr{B}(\mathscr{H})$.

If A is normal then Theorem 6 can be improved;

THEOREM 8. *Let $A \in \mathscr{B}(\mathscr{H})$ be normal and let $\mathscr{J}$ be a proper two-sided ideal of $\mathscr{B}(\mathscr{H})$. The following are equivalent:*

(a) $\{A\}' + \mathscr{J} = \mathscr{B}(\mathscr{H})$.

(b) $\mathscr{R}(\Delta_A) = \Delta_A(\mathscr{J})$.

(c) $\mathscr{R}(\Delta_A) \subset \mathscr{J}$ *and* $\sigma(A)$ *is finite.*

(d) $A = T - \lambda$ *for some* $T \in \mathscr{J}$, *some* $\lambda \in \mathscr{C}$ *and* $\sigma(A)$ *is finite.*

Proof. That (a) is equivalent to (b) and (c) is equivalent to (d) follows from Theorem 6. If A is normal with finite spectrum, then by a theorem of Anderson [1, p. 96] $\mathscr{R}(\Delta_A) + \{A\}' = \mathscr{B}(\mathscr{H})$. Hence, if $A = T - \lambda$ for some $T \in \mathscr{J}$ and $\lambda \in \mathscr{C}$ then $\mathscr{R}(\Delta_A) \subset \mathscr{J}$ and (d) implies (a). To show that (a) implies (d), assume that $\sigma(A)$ is infinite and that $\{A\}' + \mathscr{J} = \mathscr{B}(\mathscr{H})$. Then by Theorem 6, $A - \lambda \in \mathscr{J}$ for some $\lambda \in \mathscr{C}$. Since $\mathscr{J}$ is contained in the ideal of compact operators, we can assume that A is compact. Let $A = A_1 \oplus A_2$ on $\mathscr{M} \oplus \mathscr{M}^\perp$ where A_1 is an infinite dimensional diagonal operator with distinct eigenvalues and let P be the orthogonal projection onto $\mathscr{M}$. Hence, if $X \in \{A\}'$, then PXP is diagonal. However, if we let U be the unilateral shift on $\mathscr{M}$, then $\{A\}' + \mathscr{J} = \mathscr{B}(\mathscr{H})$ implies that $U = D + K$ for some diagonal operator D and some compact operator K. This is clearly a contradiction (let $\{e_n\}$ be an orthonormal basis for $\mathscr{M}$ by which U is the shift, then $((D - U)e_n, e_{n+1}) = 1$ for all n).

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Received November 28, 1972 and in revised form October 10, 1973. This paper contains part of a doctoral dissertation written under the direction of Professor J. P. Williams at Indiana University.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $60.00 a year (6 Vols., 12 issues). Special rate: $30.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan

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Pacific Journal of Mathematics

Vol. 50, No. 2 October, 1974