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THE RANGE OF A DERIVATION AND IDEALS

ROBERT EARL WEBER

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R. E. WEBER

When A is in the Banach algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space \mathcal{H} , the derivation generated by A is the bounded operator Δ_A on $\mathcal{B}(\mathcal{H})$ defined by $\Delta_A(X) = AX - XA$. It is shown that the range of a derivation generated by a Hilbert-Schmidt or a diagonal operator contains no nonzero one-sided ideals of $\mathcal{B}(\mathcal{H})$. Also, for a two-sided ideal \mathcal{I} of $\mathcal{B}(\mathcal{H})$, necessary and sufficient conditions on an operator A are given in order that the range of Δ_A equals the range of Δ_A restricted to \mathcal{I} .

1. In the following \mathcal{H} will denote an infinite dimensional complex Hilbert space.

For a fixed $A \in \mathcal{B}(\mathcal{H})$, we will concern ourselves with the following problems:

(a) For what $B \in \mathcal{B}(\mathcal{H})$ is $B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$ or $\mathcal{R}(\Delta_A)B \subset \mathcal{R}(\Delta_A)$.

(b) For what $B \in \mathcal{B}(\mathcal{H})$ is $B\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ or $\mathcal{B}(\mathcal{H})B \subset \mathcal{R}(\Delta_A)$.

(c) For what $B \in \mathcal{B}(\mathcal{H})$ is $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$.

It is easy to verify that for $A, X, Y \in \mathcal{B}(\mathcal{H})$.

(i) $\Delta_A = \Delta_{A+\lambda}$ for all $\lambda \in \mathcal{C}$

and

(ii) $\Delta_A(XY) = X\Delta_A(Y) + \Delta_A(X)Y$.

The identity (ii) yields some simple facts about the range of a derivation which show the interrelation of the above problems. (For a proof see [8].)

LEMMA 1. Let $A, B \in \mathcal{B}(\mathcal{H})$ and let A' belong to the commutant $\{A\}'$ of A . Then

(a) both $A'\mathcal{R}(\Delta_A)$ and $\mathcal{R}(\Delta_A)A'$ are contained in $\mathcal{R}(\Delta_A)$.

(b) if $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$, then both $\Delta_{A'}(B)\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})\Delta_{A'}(B)$ are contained in $\mathcal{R}(\Delta_A)$.

(c) $B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$ if and only if $\Delta_A(B)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$.

(d) $\mathcal{R}(\Delta_A)B \subset \mathcal{R}(\Delta_A)$ if and only if $\mathcal{B}(\mathcal{H})\Delta_A(B) \subset \mathcal{R}(\Delta_A)$.

From (b) of Lemma 1 it follows that if $\mathcal{R}(\Delta_A)$ does not contain left- or right-ideals, then a necessary condition for $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ is that $B \in \{A\}''$. In fact, more is true:

LEMMA 2. Let $A \in \mathcal{B}(\mathcal{H})$. If $\mathcal{R}(\Delta_A)$ contains either no nonzero left-ideals or no nonzero right-ideals, then $\Delta_B(\mathcal{F}) \subset \mathcal{R}(\Delta_A)$ implies

$B \in \{A\}''$. (\mathcal{F} denotes the ideal of finite rank operators.)

Proof. Assume that $\mathcal{R}(\Delta_A)$ contains no nonzero left-ideals (the argument for the other assumption is similar). Let P be a finite rank projection. If $A' \in \{A\}'$, then

$$\Delta_{A'}(B)PX = A'\Delta_B(PX) - \Delta_B(A'PX)$$

is in $\mathcal{R}(\Delta_A)$ for all $X \in \mathcal{B}(\mathcal{H})$. Therefore, $\Delta_{A'}(B)P\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ and hence $\Delta_{A'}(B)P = 0$. However, this is true for any such P and hence $\Delta_{A'}(B) = 0$.

For the sake of completeness we include a somewhat simpler proof of a theorem of Stampfli [6]. In the proof, $\sigma_l(A)$ denotes the left essential spectrum of A and is defined to be the set of those λ for which the coset of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ (where \mathcal{K} is the ideal of compact operators) containing $A - \lambda$ fails to have a left inverse. The right essential spectrum $\sigma_r(A)$ is defined in the obvious way.

THEOREM 1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{R}(\Delta_A)$ contains no nonzero two-sided ideals of $\mathcal{B}(\mathcal{H})$.*

Proof. Replace A by $A - \lambda$ where $\lambda \in \sigma_l(A) \cap \sigma_r(A)$ if necessary in order to assume that there exist orthonormal sequences $\{f_n\}$ and $\{g_n\}$ such that $\sum \|Af_n\|^{1/2} < \infty$ and $\sum \|A^*g_n\|^{1/2} < \infty$. (See [6].) Then for all $X \in \mathcal{B}(\mathcal{H})$,

$$\sum |((AX - XA)f_n, g_n)|^{1/2} \leq \sum \|X\|^{1/2} (\|A^*g_n\|^{1/2} + \|Af_n\|^{1/2}) < \infty .$$

If $\mathcal{R}(\Delta_A)$ contains a two-sided ideal, then it contains all finite rank operators. In particular, if $f \otimes g$ denotes the rank one operator $f \otimes g(x) = (x, g)f$, then $(f \otimes f)X \in \mathcal{R}(\Delta_A)$ for all $f \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. Hence

$$\sum |((f \otimes f)Xf_n, g_n)|^{1/2} < \infty .$$

Since

$$\begin{aligned} \sum |((f \otimes f)Xf_n, g_n)|^{1/2} &= \sum |(Xf_n, (f \otimes f)g_n)|^{1/2} \\ &= \sum |(Xf_n, f)(\overline{g_n, f})|^{1/2} , \end{aligned}$$

then

$$\sum |(Xf_n, f)(\overline{g_n, f})|^{1/2} < \infty$$

for all $f \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. However, if we choose X such that $Xf_n = g_n$ and f such that $\{|(g_n, f)|\}$ is not summable, we have a contradiction.

2. Let \mathcal{S} denote the set of Hilbert-Schmidt operators on \mathcal{H} . Equipped with the trace inner product $(A, B) = \text{tr}(AB^*)$, \mathcal{S} is a Hilbert space [5]. If $A \in \mathcal{B}(\mathcal{H})$, then the restriction of Δ_A to \mathcal{S} is a bounded operator on \mathcal{S} with adjoint $(\Delta_A|_{\mathcal{S}})^* = \Delta_{A^*}|_{\mathcal{S}}$. Hence $\mathcal{S} = \mathcal{R}(\Delta_A|_{\mathcal{S}})^\perp \oplus (\{A^*\}' \cap \mathcal{S})$ where the double bar indicates closure with respect to the topology on \mathcal{S} .

THEOREM 2. *Let $A \in \mathcal{S}$. Then $\mathcal{R}(\Delta_A)^\perp = \mathcal{R}(\Delta_A|_{\mathcal{S}})^\perp$.*

Proof. It follows from the above remarks that $\mathcal{R}(\Delta_A)^\perp \subset \mathcal{R}(\Delta_A|_{\mathcal{S}})^\perp = \{A^*\}' \cap \mathcal{S}$. It remains to show the reverse inclusion. Let $T \in \{A^*\}' \cap \mathcal{S}$. Then for $X \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} (\Delta_A(X), T) &= \text{tr}(T^* \Delta_A(X)) = \text{tr}(T^*AX) - \text{tr}(T^*XA) \\ &= \text{tr}(AT^*X) - \text{tr}(T^*XA) = \text{tr}(T^*XA) - \text{tr}(T^*XA) = 0. \end{aligned}$$

Therefore $T \in \mathcal{R}(\Delta_A)^\perp$.

COROLLARY. *Let $A \in \mathcal{S}$. Then $\mathcal{R}(\Delta_A)^\perp \oplus (\{A^*\}' \cap \mathcal{S}) = \mathcal{S}$.*

THEOREM 3. *If $A \in \mathcal{S}$, then $\mathcal{R}(\Delta_A)$ does not contain any nonzero left- or right-ideals.*

In the proof of Theorem 3 we will make use of the following result.

LEMMA 3. *Let $A \in \mathcal{S}$. If $(f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$, then $Af = 0$.*

Proof. Since $\mathcal{R}(\Delta_A) \perp \{A^*\}' \cap \mathcal{S}$, then $0 = \text{tr}(A(f \otimes f)X) = \text{tr}(Af \otimes X^*f) = (Af, X^*f)$ for all $X \in \mathcal{B}(\mathcal{H})$. Hence $Af = 0$.

Proof of Theorem 3. Suppose that $(f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$. Then $f \otimes f = \Delta_A(X)$ for some $X \in \mathcal{B}(\mathcal{H})$ and by Lemma 3, $f = (f \otimes f)f = AXf - XAf = AXf$. Since $(f \otimes f)\mathcal{B}(\mathcal{H}) = \Delta_A(X)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$, then by Lemma 1, $X\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$. Therefore, $((Xf) \otimes (Xf))\mathcal{B}(\mathcal{H}) \subset X(f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A)$ and by Lemma 3, $Xf \in \ker(A)$. Hence $f = AXf = 0$. The remainder follows by taking adjoints.

COROLLARY 1. *Let $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathcal{H})$. Then $B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A)$ if and only if $B \in \{A\}'$.*

Proof. This follows from Lemma 1 and the theorem.

COROLLARY 2. *Let $A \in \mathcal{S}$. If $\Delta_B(\mathcal{F}) \subset \mathcal{R}(\Delta_A)$ then $B \in \{A\}''$.*

Proof. This follows from Lemma 2 and the theorem.

3. We now turn our attention to diagonal operators. When expressing a diagonal operator as the sum $A = \sum \alpha_n P_n$, unless otherwise stated we shall assume that P_n is the rank one projection onto the subspace spanned by e_n , where $\{e_n\}$ is an orthonormal basis. (However, we do not require that the α_n 's be distinct.) Each operator X has a matrix (x_{ij}) with respect to this fixed basis.

The principle result of this section is that the range of a derivation generated by a diagonal operator contains no nonzero left- or right-ideals. The theorem is slightly more general.

THEOREM 4. *Let $A \in \mathcal{B}(\mathcal{H})$ have the property that there exist reducing subspaces \mathcal{M}_n of A , each finite dimensional, such that $\mathcal{H} = \sum \bigoplus \mathcal{M}_n$. Then $\mathcal{R}(\Delta_A)$ contains no nonzero positive operators.*

Proof. Let $P = \Delta_A(X)$ where P is positive. If P_n is the orthogonal projection onto \mathcal{M}_n , then $P_n P | \mathcal{M}_n = A_n X_n - X_n A_n$ where $A_n = A | \mathcal{M}_n$ and X_n is the compression of X to \mathcal{M}_n . Since \mathcal{M}_n is finite dimensional, then $\text{tr}(P_n P | \mathcal{M}_n) = 0$. Hence $P_n P | \mathcal{M}_n$ being a positive operator with zero trace, must be 0. Therefore, $P_n P P_n = 0$ (on \mathcal{H}). Hence $P^{1/2} P_n = 0$ and $P^{1/2} = 0$.

COROLLARY 1. *If A satisfies the hypothesis of the theorem and if either $B\mathcal{R}(\Delta_A)$ or $\mathcal{R}(\Delta_A)B$ is contained in $\mathcal{R}(\Delta_A)$, then $B \in \{A\}'$.*

COROLLARY 2. *If A satisfies the hypothesis of the theorem and $\Delta_B(\mathcal{F}) \subset \mathcal{R}(\Delta_A)$, then $B \in \{A\}''$.*

COROLLARY 3. *Let A be normal with finite spectrum. Then for $B \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ if and only if $B \in \{A\}''$.*

Proof. If $B \in \{A\}''$ then B is a polynomial of A and hence $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$. (See [1, p. 79].) The converse follows from Corollary 2.

LEMMA 4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ where $A = \sum \alpha_i P_i$. Then $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$ if and only if $B = \sum \beta_i P_i$ for some set of scalars β_0, β_1, \dots and for every operator $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$ there exists an operator $Y = (y_{ij}) \in \mathcal{B}(\mathcal{H})$ such that $(\alpha_i - \alpha_j) = (\beta_i - \beta_j)x_{ij}$ for all i, j .*

Proof. This follows from Corollary 2 and the fact that $[\Delta_A(X)]_{ij} = (\alpha_i - \alpha_j)x_{ij}$ if $X = (x_{ij})$.

THEOREM 5. *Let $A \in \mathcal{B}(\mathcal{H})$ be diagonal. If for $B \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$, then $B = f(A)$ for some function f which is Lipschitz on the spectrum of A .*

Proof. Let $A = \sum \alpha_i P_i$. If $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$, then by Corollary 2, $B = \sum \beta_i P_i$ for some sequence of scalars $\{\beta_i\}$ and for any $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$, there exists a $Y = (y_{ij}) \in \mathcal{B}(\mathcal{H})$ such that $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$. It follows that $((\beta_i - \beta_j)/(\alpha_i - \alpha_j))$ is bounded by some positive number M . Define f such that $f(\alpha_i) = \beta_i$. Then f is a Lipschitz function defined on a dense subset of $\sigma(A)$ onto a dense subset of $\sigma(B)$. Therefore, we can extend f to be Lipschitz on $\sigma(A)$ onto $\sigma(B)$.

It was shown in [7] that if B is an analytic function of A , then $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$. To have range inclusion it is neither necessary that B be an analytic function of A nor sufficient that B be a continuous function of A as seen in the next two examples.

EXAMPLE 1. Let $A = \sum \alpha_n P_n$ where $\dim P_n = 1$, $\alpha_0 = 0$, and

$$\alpha_n = \begin{cases} i/n & \text{for } n \text{ even} \\ 1/n & \text{for } n \text{ odd.} \end{cases}$$

Let $B = \sum \beta_n P_n$ where $\beta_0 = 0$ and $\beta_n = -i/n^2$ for $n \geq 1$. A direct computation shows that if $n < m$, then $|(\beta_n - \beta_m)/(\alpha_n - \alpha_m)| \leq 2/n$. Now, for any $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$, consider the matrix $Y = (y_{ij})$ where $y_{ij} = ((\beta_i - \beta_j)/(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$ and zero otherwise. Then

$$\sum_{i,j} |y_{ij}|^2 = \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} |y_{nj}|^2 + \sum_{m=0}^{\infty} \sum_{i=m}^{\infty} |y_{im}|^2.$$

For $m > 0$,

$$\sum_{i=m}^{\infty} |y_{im}|^2 \leq 4/m^2 \sum_{i=m}^{\infty} |x_{im}|^2 \leq 4/m^2 \|X\|^2$$

and for $n > 0$,

$$\sum_{j=n}^{\infty} |y_{nj}|^2 \leq 4/n^2 \|X\|^2.$$

Hence

$$\sum_{i,j} |y_{ij}|^2 \leq \|X\|^2 + \sum_{m=1}^{\infty} 4/m^2 \|X\|^2 + \|X\|^2 + \sum_{m=1}^{\infty} 4/m^2 \|X\|^2.$$

Therefore, $Y \in \mathcal{B}(\mathcal{H})$ and by Lemma 4, $\mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A)$. Now, assume f is an analytic function on $\sigma(A)$ such that for even n , $f(i/n) = -i/n^2$. Then $f(z) = z^2 i$. Hence for odd n , $f(1/n) = i/n^2 \neq -i/n^2$ and $B \neq f(A)$.

EXAMPLE 2. Let $A = \sum \alpha_n P_n$ where P_n is rank one for all n , $\alpha_0 = 0$, and $\alpha_n = 1/n^2$ for $n > 0$ and let $B = \sum \beta_n P_n$ where $\beta_0 = 0$

and $\beta_n = 1/n$ for $n > 0$. Then B is a continuous function of A , in fact $B = f(A)$ where $f(z) = z^{1/2}$. Let $X = (x_{ij}) \in \mathcal{B}(\mathcal{H})$ where

$$x_{nj} = \begin{cases} 1/n & \text{for } n > 0 \text{ and } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $\Delta_B(X) = \Delta_A(Y)$ where $Y = (y_{ij})$, then

$$y_{n0} = x_{n0}(\beta_n - \beta_0)/(\alpha_n - \alpha_0) = (1/n)(1/n)/(1/n^2) = 1$$

for all n . Hence $Y \notin \mathcal{B}(\mathcal{H})$ and $\mathcal{R}(\Delta_B) \not\subset \mathcal{R}(\Delta_A)$.

Other derivations whose ranges do not contain any nonzero one-sided ideals are those generated by unitary and self-adjoint operators. (See [9].)

It was shown in [7] that the range of a derivation generated by a nonunitary isometry *does* contain nonzero left-ideals. Other operators which possess this property are some of the weighted shifts.

4. Another question concerning the range of a derivation and, in this case, a two-sided ideal \mathcal{I} of $\mathcal{B}(\mathcal{H})$ is whether $\mathcal{R}(\Delta_A) = \Delta_A(\mathcal{I})$.

THEOREM 6. *Let $A \in \mathcal{B}(\mathcal{H})$ and let \mathcal{I} be a proper two-sided ideal of $\mathcal{B}(\mathcal{H})$. Consider the following conditions:*

- (a) $\{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H})$.
- (b) $\mathcal{R}(\Delta_A) = \Delta_A(\mathcal{I})$.
- (c) $\mathcal{R}(\Delta_A) \subset \mathcal{I}$.
- (d) $A = T - \lambda$ for some $T \in \mathcal{I}$ and $\lambda \in \mathcal{C}$.

(a) is equivalent to (b), (c) is equivalent to (d), and (b) implies (c).

Proof. That (a) is equivalent to (b) is a consequence of the fact that $X = T + A'$ for some $T \in \mathcal{I}$ and $A' \in \{A\}'$ if and only if $\Delta_A(X) \in \Delta_A(\mathcal{I})$. That (c) is equivalent to (d) is a consequence of a theorem of Calkin [2] where he shows that the center of $\mathcal{B}(\mathcal{H})/\mathcal{I}$ consists of scalars. It is immediate that (b) implies (c).

REMARK. An example to show that (c) does not imply (b) for the case when \mathcal{I} is the ideal of compact operators can be obtained by letting A be the adjoint of the weighted shift with weights $\{2, 1, 1/2, 1/3, \dots\}$ and showing that each element of $\{A\}'$ is the translate of a Hilbert-Schmidt operator. (See [8].)

If we require only that the closures be equal, we have the following;

THEOREM 7. *Let $A \in \mathcal{B}(\mathcal{H})$ be compact and let \mathcal{I} be the ideal of finite rank operators. Then $\mathcal{R}(\Delta_A)^- = \Delta_A(\mathcal{I})^-$.*

Proof. Let $f \in \mathcal{B}(\mathcal{H})^*$. Then $f = f_0 + f_T$ for some trace-class operator T where $f_T(X) = \text{tr}(XT)$ and where f_0 annihilates the compact operators. (See Dixmier [3].) If f annihilates $\Delta_A(\mathcal{F})$ then $f_T(\Delta_A(F)) = f(\Delta_A(F)) = 0$ for all $F \in \mathcal{F}$. However,

$$\begin{aligned} f_T(\Delta_A(F)) &= \text{tr}((AF - FA)T) = \text{tr}(AFT - FAT) \\ &= \text{tr}(FTA - FAT) = \text{tr}(F\Delta_A(-T)) \end{aligned}$$

for all $F \in \mathcal{F}$. Since \mathcal{F} is dense in the trace-class operators, then $\Delta_A(-T) = 0$ and $T \in \{A\}'$. Hence f_T annihilates the range of Δ_A and since A is compact, $f(\Delta_A(X)) = f_T(\Delta_A(X)) = 0$ for all $X \in \mathcal{B}(\mathcal{H})$.

If A is normal then Theorem 6 can be improved;

THEOREM 8. *Let $A \in \mathcal{B}(\mathcal{H})$ be normal and let \mathcal{F} be a proper two-sided ideal of $\mathcal{B}(\mathcal{H})$. The following are equivalent:*

- (a) $\{A\}' + \mathcal{F} = \mathcal{B}(\mathcal{H})$.
- (b) $\mathcal{R}(\Delta_A) = \Delta_A(\mathcal{F})$.
- (c) $\mathcal{R}(\Delta_A) \subset \mathcal{F}$ and $\sigma(A)$ is finite.
- (d) $A = T - \lambda$ for some $T \in \mathcal{F}$, some $\lambda \in \mathcal{C}$ and $\sigma(A)$ is finite.

Proof. That (a) is equivalent to (b) and (c) is equivalent to (d) follows from Theorem 6. If A is normal with finite spectrum, then by a theorem of Anderson [1, p. 96] $\mathcal{R}(\Delta_A) + \{A\}' = \mathcal{B}(\mathcal{H})$. Hence, if $A = T - \lambda$ for some $T \in \mathcal{F}$ and $\lambda \in \mathcal{C}$ then $\mathcal{R}(\Delta_A) \subset \mathcal{F}$ and (d) implies (a). To show that (a) implies (d), assume that $\sigma(A)$ is infinite and that $\{A\}' + \mathcal{F} = \mathcal{B}(\mathcal{H})$. Then by Theorem 6, $A - \lambda \in \mathcal{F}$ for some $\lambda \in \mathcal{C}$. Since \mathcal{F} is contained in the ideal of compact operators, we can assume that A is compact. Let $A = A_1 \oplus A_2$ on $\mathcal{M} \oplus \mathcal{M}^\perp$ where A_1 is an infinite dimensional diagonal operator with distinct eigenvalues and let P be the orthogonal projection onto \mathcal{M} . Hence, if $X \in \{A\}'$, then PXP is diagonal. However, if we let U be the unilateral shift on \mathcal{M} , then $\{A\}' + \mathcal{F} = \mathcal{B}(\mathcal{H})$ implies that $U = D + K$ for some diagonal operator D and some compact operator K . This is clearly a contradiction (let $\{e_n\}$ be an orthonormal basis for \mathcal{M} by which U is the shift, then $((D - U)e_n, e_{n+1}) = 1$ for all n).

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