THE RANGE OF A DERIVATION AND IDEALS

ROBERT EARL WEBER
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When $A$ is in the Banach algebra $B(H)$ of all bounded linear operators on a Hilbert space $H$, the derivation generated by $A$ is the bounded operator $\Delta_A$ on $B(H)$ defined by $\Delta_A(X) = AX - XA$. It is shown that the range of a derivation generated by a Hilbert-Schmidt or a diagonal operator contains no nonzero one-sided ideals of $B(H)$. Also, for a two-sided ideal $I$ of $B(H)$, necessary and sufficient condition on an operator $A$ are given in order that the range of $\Delta_A$ equals the range of $\Delta_A$ restricted to $I$.

1. In the following $H$ will denote an infinite dimensional complex Hilbert space.

For a fixed $A \in B(H)$, we will concern ourselves with the following problems:

(a) For what $B \in B(H)$ is $B B(\Delta_A) \subset B(\Delta_A)$ or $B(\Delta_A) B \subset B(\Delta_A)$.

(b) For what $B \in B(H)$ is $B B(H) \subset B(\Delta_A)$ or $B(H) B \subset B(\Delta_A)$.

(c) For what $B \in B(H)$ is $B(\Delta_B) \subset B(\Delta_A)$.

It is easy to verify that for $A, X, Y \in B(H)$.

(i) $\Delta_A = \Delta_{A+\lambda}$ for all $\lambda \in \mathbb{C}$

and

(ii) $\Delta_A(XY) = X\Delta_A(Y) + \Delta_A(X)Y$.

The identity (ii) yields some simple facts about the range of a derivation which show the interrelation of the above problems. (For a proof see [8].)

**Lemma 1.** Let $A, B \in B(H)$ and let $A'$ belong to the commutant $\{A\}'$ of $A$. Then

(a) both $A' B(\Delta_A)$ and $B(\Delta_A) A'$ are contained in $B(\Delta_A)$.

(b) if $B(\Delta_A) \subset B(\Delta_A)$, then both $\Delta_A' (B) B(H)$ and $B(H) \Delta_A' (B)$ are contained in $B(\Delta_A)$.

(c) $B B(\Delta_A) \subset B(\Delta_A)$ if and only if $\Delta_A(B) B(H) \subset B(\Delta_A)$.

(d) $B(\Delta_A) B \subset B(\Delta_A)$ if and only if $B(H) \Delta_A(B) \subset B(\Delta_A)$.

From (b) of Lemma 1 it follows that if $B(\Delta_A)$ does not contain left- or right-ideals, then a necessary condition for $B(\Delta_B) \subset B(\Delta_A)$ is that $B \in \{A\}'$. In fact, more is true:

**Lemma 2.** Let $A \in B(H)$. If $B(\Delta_A)$ contains either no nonzero left-ideals or no nonzero right-ideals, then $\Delta_A(\mathcal{I}) \subset B(\Delta_A)$ implies
Proof. Assume that \( \mathcal{R}(A_A) \) contains no nonzero left-ideals (the argument for the other assumption is similar). Let \( P \) be a finite rank projection. If \( A' \in \{A\}' \), then
\[
A'_a(B)P = A'_a(PX) - A'_d(A'PX)
\]
is in \( \mathcal{R}(A_A) \) for all \( X \in \mathcal{B}(\mathcal{H}) \). Therefore, \( A'_a(B)P \mathcal{B}(\mathcal{H}) \subset \mathcal{R}(A_A) \) and hence \( A'_a(B)P = 0 \). However, this is true for any such \( P \) and hence \( A'_a(B) = 0 \).

For the sake of completeness we include a somewhat simpler proof of a theorem of Stampfli [6]. In the proof, \( \sigma_l(A) \) denotes the left essential spectrum of \( A \) and is defined to be the set of those \( \lambda \) for which the coset of the Calkin algebra \( \mathcal{B}(\mathcal{H})/\mathcal{K} \) (where \( \mathcal{K} \) is the ideal of compact operators) containing \( A - \lambda \) fails to have a left inverse. The right essential spectrum \( \sigma_r(A) \) is defined in the obvious way.

**Theorem 1.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then \( \mathcal{R}(A_A) \) contains no nonzero two-sided ideals of \( \mathcal{B}(\mathcal{H}) \).

Proof. Replace \( A \) by \( A - \lambda \) where \( \lambda \in \sigma_l(A) \cap \sigma_r(A) \) if necessary in order to assume that there exist orthonormal sequences \( \{f_n\} \) and \( \{g_n\} \) such that \( \sum ||Af_n||^{1/2} < \infty \) and \( \sum ||A^*g_n||^{1/2} < \infty \). (See [6].) Then for all \( X \in \mathcal{B}(\mathcal{H}) \),
\[
\sum |((AX -XA)f_n, g_n)|^{1/2} \leq \sum ||X||^{1/2}(||A^*g_n||^{1/2} + ||Af_n||^{1/2}) < \infty.
\]
If \( \mathcal{R}(A_A) \) contains a two-sided ideal, then it contains all finite rank operators. In particular, if \( f \otimes g \) denotes the rank one operator \( f \otimes g(x) = (x, g)f \), then \( (f \otimes f)X \in \mathcal{R}(A_A) \) for all \( f \in \mathcal{H} \) and \( X \in \mathcal{B}(\mathcal{H}) \). Hence
\[
\sum |((f \otimes f)Xf_n, g_n)|^{1/2} < \infty.
\]
Since
\[
\sum |((f \otimes f)Xf_n, g_n)|^{1/2} = \sum |(Xf_n, (f \otimes f)g_n)|^{1/2} = \sum |(Xf_n, f)(g_n, f)|^{1/2},
\]
then
\[
\sum |(Xf_n, f)(g_n, f)|^{1/2} < \infty
\]
for all \( f \in \mathcal{H} \) and \( X \in \mathcal{B}(\mathcal{H}) \). However, if we choose \( X \) such that \( Xf_n = g_n \) and \( f \) such that \( \{(g_n, f)\} \) is not summable, we have a contradiction.
2. Let \( \mathcal{S} \) denote the set of Hilbert-Schmidt operators on \( \mathcal{H} \). Equipped with the trace inner product \( (A, B) = \text{tr}(AB^*) \), \( \mathcal{S} \) is a Hilbert space [5]. If \( A \in \mathcal{B}(\mathcal{H}) \), then the restriction of \( \Delta_A \) to \( \mathcal{S} \) is a bounded operator on \( \mathcal{S} \) with adjoint \( (\Delta_A|_{\mathcal{S}})^* = \Delta_{A^*}|_{\mathcal{S}} \). Hence \( \mathcal{S} = \mathcal{B}(\Delta_A|_{\mathcal{S}})^\perp \oplus ([A^*]' \cap \mathcal{S}) \) where the double bar indicates closure with respect to the topology on \( \mathcal{S} \).

**Theorem 2.** Let \( A \in \mathcal{S} \). Then \( \mathcal{R}(\Delta_A)^\perp = \mathcal{R}(\Delta_A|_{\mathcal{S}})^\perp \).

**Proof.** It follows from the above remarks that \( \mathcal{R}(\Delta_A)^\perp \subset \mathcal{R}(\Delta_A|_{\mathcal{S}})^\perp = [A^*]' \cap \mathcal{S} \). It remains to show the reverse inclusion. Let \( T \in [A^*]' \cap \mathcal{S} \). Then for \( \lambda \in \mathbb{C} \),

\[
\lambda T \Rightarrow \text{tr} (\lambda T^* \Delta_A(X)) = \text{tr} (T^*AX) - \text{tr} (X^*TA) = \text{tr} (AT^*X) - \text{tr} (T^*XA) = 0.
\]

Therefore \( T \in \mathcal{R}(\Delta_A)^\perp \).

**Corollary.** Let \( A \in \mathcal{S} \). Then \( \mathcal{R}(\Delta_A)^\perp = \mathcal{R}(\Delta_A) \cup ([A^*]' \cap \mathcal{S}) = \mathcal{S} \).

**Theorem 3.** If \( A \in \mathcal{S} \), then \( \mathcal{R}(\Delta_A) \) does not contain any nonzero left- or right-ideals.

In the proof of Theorem 3 we will make use of the following result.

**Lemma 3.** Let \( A \in \mathcal{S} \). If \( (f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \), then \( Af = 0 \).

**Proof.** Since \( \mathcal{R}(\Delta_A) \perp [A^*]' \cap \mathcal{S} \), then \( 0 = \text{tr} (A(f \otimes f)X) = \text{tr} (Af \otimes X^*f) = (Af, X^*f) \) for all \( X \in \mathcal{B}(\mathcal{H}) \). Hence \( Af = 0 \).

**Proof of Theorem 3.** Suppose that \( (f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \). Then \( f \otimes f = \Delta_A(X) \) for some \( X \in \mathcal{B}(\mathcal{H}) \) and by Lemma 3, \( f = (f \otimes f)f = AXf - XAf = AXf \). Since \( (f \otimes f)\mathcal{B}(\mathcal{H}) = \Delta_A(X)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \), then by Lemma 1, \( X\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A) \). Therefore, \( (Xf \otimes Xf)\mathcal{B}(\mathcal{H}) \subset X(f \otimes f)\mathcal{B}(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \) and by Lemma 3, \( Xf \in \ker(A) \). Hence \( f = AXf = 0 \). The remainder follows by taking adjoints.

**Corollary 1.** Let \( A \in \mathcal{S} \) and \( B \in \mathcal{B}(\mathcal{H}) \). Then \( B\mathcal{R}(\Delta_A) \subset \mathcal{R}(\Delta_A) \) if and only if \( B \in [A]' \).

**Proof.** This follows from Lemma 1 and the theorem.

**Corollary 2.** Let \( A \in \mathcal{S} \). If \( \Delta_B(\mathcal{F}) \subset \mathcal{R}(\Delta_A) \) then \( B \in [A]' \).

**Proof.** This follows from Lemma 2 and the theorem.
We now turn our attention to diagonal operators. When expressing a diagonal operator as the sum \( A = \sum \alpha_n P_n \), unless otherwise stated we shall assume that \( P_n \) is the rank one projection onto the subspace spanned by \( e_n \), where \( \{e_n\} \) is an orthonormal basis. (However, we do not require that the \( \alpha_n \)'s be distinct.) Each operator \( X \) has a matrix \( (x_{ij}) \) with respect to this fixed basis.

The principle result of this section is that the range of a derivation generated by a diagonal operator contains no nonzero left- or right-ideals. The theorem is slightly more general.

**Theorem 4.** Let \( A \in \mathcal{B}(\mathcal{H}) \) have the property that there exist reducing subspaces \( \mathcal{M}_n \) of \( A \), each finite dimensional, such that \( \mathcal{H} = \sum \oplus \mathcal{M}_n \). Then \( \mathcal{R}(\Delta_A) \) contains no nonzero positive operators.

*Proof.* Let \( P = \Delta_A(X) \) where \( P \) is positive. If \( P_n \) is the orthogonal projection onto \( \mathcal{M}_n \), then \( P_n P |_{\mathcal{M}_n} = A_n X_n - X_n A_n \) where \( A_n = A |_{\mathcal{M}_n} \) and \( X_n \) is the compression of \( X \) to \( \mathcal{M}_n \). Since \( \mathcal{M}_n \) is finite dimensional, then \( \text{tr} (P_n P |_{\mathcal{M}_n}) = 0 \). Hence \( P_n P |_{\mathcal{M}_n} \) being a positive operator with zero trace, must be 0. Therefore, \( P_n P P_n = 0 \) (on \( \mathcal{H} \)). Hence \( P^1/2 P_n = 0 \) and \( P^{1/2} = 0 \).

**Corollary 1.** If \( A \) satisfies the hypothesis of the theorem and if either \( B \in \mathcal{R}(\Delta_A) \) or \( \mathcal{R}(\Delta_A) B \) is contained in \( \mathcal{R}(\Delta_A) \), then \( B \in \{A\}' \).

**Corollary 2.** If \( A \) satisfies the hypothesis of the theorem and \( \Delta_B(\mathcal{H}) \subset \mathcal{R}(\Delta_A) \), then \( B \in \{A\}'' \).

**Corollary 3.** Let \( A \) be normal with finite spectrum. Then for \( B \in \mathcal{B}(\mathcal{H}) \), \( \mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A) \) if and only if \( B \in \{A\}'' \).

*Proof.* If \( B \in \{A\}'' \) then \( B \) is a polynomial of \( A \) and hence \( \mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A) \). (See [1, p. 79] .) The converse follows from Corollary 2.

**Lemma 4.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) where \( A = \sum \alpha_i P_i \). Then \( \mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A) \) if and only if \( B = \sum \beta_i P_i \) for some set of scalars \( \beta_0, \beta_1, \ldots \) and for every operator \( X = (x_{ij}) \in \mathcal{B}(\mathcal{H}) \) there exists an operator \( Y = (y_{ij}) \in \mathcal{B}(\mathcal{H}) \) such that \( (\alpha_i - \alpha_j) x_{ij} = (\beta_i - \beta_j) y_{ij} \) for all \( i, j \).

*Proof.* This follows from Corollary 2 and the fact that \( \Delta_A(X)_{ij} = (\alpha_i - \alpha_j) x_{ij} \) if \( X = (x_{ij}) \).

**Theorem 5.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be diagonal. If for \( B \in \mathcal{B}(\mathcal{H}), \mathcal{R}(\Delta_B) \subset \mathcal{R}(\Delta_A) \), then \( B = f(A) \) for some function \( f \) which is Lipschitz on the spectrum of \( A \).
Proof. Let $A = \sum \alpha_i P_i$. If $\mathcal{A}(\Delta_\beta) \subset \mathcal{A}(\Delta_A)$, then by Corollary 2, $B = \sum \beta_i P_i$ for some sequence of scalars $\{\beta_i\}$ and for any $X = (x_{ij}) \in \mathcal{B}(\mathcal{A})$, there exists a $Y = (y_{ij}) \in \mathcal{B}(\mathcal{A})$ such that $y_{ij} = ((\beta_i - \beta_j)(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$. It follows that $((\beta_i - \beta_j)(\alpha_i - \alpha_j))$ is bounded by some positive number $M$. Define $f$ such that $f(\alpha_i) = \beta_i$. Then $f$ is a Lipschitz function defined on a dense subset of $\sigma(A)$ onto a dense subset of $\sigma(B)$. Therefore, we can extend $f$ to be Lipschitz on $\sigma(A)$ onto $\sigma(B)$.

It was shown in [7] that if $B$ is an analytic function of $A$, then $\mathcal{A}(\Delta_\beta) \subset \mathcal{A}(\Delta_A)$. To have range inclusion it is neither necessary that $B$ be an analytic function of $A$ nor sufficient that $B$ be a continuous function of $A$ as seen in the next two examples.

**Example 1.** Let $A = \sum \alpha_n P_n$ where $\dim P_n = 1$, $\alpha_0 = 0$, and

$$
\alpha_n = \begin{cases} 
  i/n & \text{for } n \text{ even} \\
  1/n & \text{for } n \text{ odd}.
\end{cases}
$$

Let $B = \sum \beta_n P_n$ where $\beta_0 = 0$ and $\beta_n = -i/n^2$ for $n \geq 1$. A direct computation shows that if $n < m$, then $|(\beta_n - \beta_m)/(\alpha_n - \alpha_m)| \leq 2/n$. Now, for any $X = (x_{ij}) \in \mathcal{B}(\mathcal{A})$, consider the matrix $Y = (y_{ij})$ where $y_{ij} = ((\beta_i - \beta_j)(\alpha_i - \alpha_j))x_{ij}$ whenever $\alpha_i \neq \alpha_j$ and zero otherwise. Then

$$
\sum_{i,j} |y_{ij}|^2 = \sum_{n=0}^\infty \sum_{j=n}^\infty |y_{nj}|^2 + \sum_{m=0}^\infty \sum_{i=m}^\infty |y_{im}|^2.
$$

For $m > 0$,

$$
\sum_{i=m}^\infty |y_{im}|^2 \leq 4/m^2 \sum_{i=m}^\infty |x_{im}|^2 \leq 4/m^2 \|X\|^2
$$

and for $n > 0$,

$$
\sum_{j=n}^\infty |y_{nj}|^2 \leq 4/n^2 \|X\|^2.
$$

Hence

$$
\sum_{i,j} |y_{ij}|^2 \leq \|X\|^2 + \sum_{m=1}^\infty 4/n^2 \|X\|^2 + \|X\|^2 + \sum_{m=1}^\infty 4/m^2 \|X\|^2.
$$

Therefore, $Y \in \mathcal{B}(\mathcal{A})$ and by Lemma 4, $\mathcal{A}(\Delta_\beta) \subset \mathcal{A}(\Delta_A)$. Now, assume $f$ is an analytic function on $\sigma(A)$ such that for even $n$, $f(i/n) = -i/n^2$. Then $f(z) = z^2 i$. Hence for odd $n$, $f(1/n) = i/n^2 \neq -i/n^2$ and $B \neq f(A)$.

**Example 2.** Let $A = \sum \alpha_n P_n$ where $P_n$ is rank one for all $n$, $\alpha_0 = 0$, and $\alpha_n = 1/n^2$ for $n > 0$ and let $B = \sum \beta_n P_n$ where $\beta_0 = 0$
and \( \beta_n = 1/n \) for \( n > 0 \). Then \( B \) is a continuous function of \( A \), in fact \( B = f(A) \) where \( f(z) = z^{1/2} \). Let \( X = (x_{ij}) \in \mathcal{B}(\mathcal{H}) \) where
\[
x_{nj} = \begin{cases} 1/n & \text{for } n > 0 \text{ and } j = 0 \\
0 & \text{otherwise} \end{cases}
\]
If \( \Delta_B(X) = \Delta_A(Y) \) where \( Y = (y_{ij}) \), then
\[
y_{n0} = x_{n0}(\beta_n - \beta_0)/(\alpha_n - \alpha_0) = (1/n)(1/n)/(1/n^2) = 1
\]
for all \( n \). Hence \( Y \in \mathcal{B}(\mathcal{H}) \) and \( \mathcal{B}(\Delta_B) \not\subset \mathcal{B}(\Delta_A) \).

Other derivations whose ranges do not contain any nonzero one-sided ideals are those generated by unitary and self-adjoint operators. (See [9].)

It was shown in [7] that the range of a derivation generated by a nonunitary isometry \textit{does} contain nonzero left-ideals. Other operators which possess this property are some of the weighted shifts.

4. Another question concerning the range of a derivation and, in this case, a two-sided ideal \( \mathcal{I} \) of \( \mathcal{B}(\mathcal{H}) \) is whether \( \mathcal{B}(\Delta_A) = \Delta_A(\mathcal{I}) \).

**Theorem 6.** Let \( A \in \mathcal{B}(\mathcal{H}) \) and let \( \mathcal{I} \) be a proper two-sided ideal of \( \mathcal{B}(\mathcal{H}) \). Consider the following conditions:

- (a) \( \{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H}) \).
- (b) \( \mathcal{B}(\Delta_A) = \Delta_A(\mathcal{I}) \).
- (c) \( \mathcal{B}(\Delta_A) \subset \mathcal{I} \).
- (d) \( A = T - \lambda \) for some \( T \in \mathcal{I} \) and \( \lambda \in \mathbb{C} \).

(a) is equivalent to (b), (c) is equivalent to (d), and (b) implies (c).

**Proof.** That (a) is equivalent to (b) is a consequence of the fact that \( X = T + A' \) for some \( T \in \mathcal{I} \) and \( A' \in \{A\}' \) if and only if \( \Delta_A(X) \in \Delta_A(\mathcal{I}) \). That (c) is equivalent to (d) is a consequence of a theorem of Calkin [2] where he shows that the center of \( \mathcal{B}(\mathcal{H})/\mathcal{I} \) consists of scalars. It is immediate that (b) implies (c).

**Remark.** An example to show that (c) does not imply (b) for the case when \( \mathcal{I} \) is the ideal of compact operators can be obtained by letting \( A \) be the adjoint of the weighted shift with weights \( \{2, 1, 1/2, 1/3, \cdots\} \) and showing that each element of \( \{A\}' \) is the translate of a Hilbert-Schmidt operator. (See [8].)

If we require only that the closures be equal, we have the following:

**Theorem 7.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be compact and let \( \mathcal{I} \) be the ideal of finite rank operators. Then \( \mathcal{B}(\Delta_A)^{-} = \Delta_A(\mathcal{I})^{-} \).
Proof. Let \( f \in \mathcal{B}(\mathcal{H})^* \). Then \( f = f_0 + f_\tau \) for some trace-class operator \( T \) where \( f_\tau(X) = \text{tr}(XT) \) and where \( f_0 \) annihilates the compact operators. (See Dixmier [3].) If \( f \) annihilates \( \Delta(A) \) then \( f_\tau(\Delta(A)(F)) = f(\Delta(A)(F)) = 0 \) for all \( F \in \mathcal{I} \). However,

\[
f_\tau(\Delta(A)(F)) = \text{tr}((AF - FA)T) = \text{tr}(AFT - FAT) = \text{tr}(FTA - FAT) = \text{tr}(FA(\tau - T))
\]

for all \( F \in \mathcal{I} \). Since \( \mathcal{I} \) is dense in the trace-class operators, then \( \Delta(A)(-T) = 0 \) and \( T \in \{A\}' \). Hence \( f_\tau \) annihilates the range of \( \Delta(A) \) and since \( A \) is compact, \( f(\Delta(A)(X)) = f_\tau(\Delta(A)(X)) = 0 \) for all \( X \in \mathcal{B}(\mathcal{H}) \).

If \( A \) is normal then Theorem 6 can be improved;

**Theorem 8.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be normal and let \( \mathcal{I} \) be a proper two-sided ideal of \( \mathcal{B}(\mathcal{H}) \). The following are equivalent:

\[
\begin{align*}
(a) & \quad \{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H}), \\
(b) & \quad \mathcal{B}(\Delta(A)) = \Delta(A)(\mathcal{I}).
\end{align*}
\]

(c) \( \mathcal{B}(\Delta(A)) \subset \mathcal{I} \) and \( \sigma(A) \) is finite.

(d) \( A = T - \lambda \) for some \( T \in \mathcal{I} \), some \( \lambda \in \mathcal{E} \) and \( \sigma(A) \) is finite.

**Proof.** That (a) is equivalent to (b) and (c) is equivalent to (d) follows from Theorem 6. If \( A \) is normal with finite spectrum, then by a theorem of Anderson [1, p. 96] \( \mathcal{B}(\Delta(A)) + \{A\}' = \mathcal{B}(\mathcal{H}) \). Hence, if \( A = T - \lambda \) for some \( T \in \mathcal{I} \) and \( \lambda \in \mathcal{C} \) then \( \mathcal{B}(\Delta(A)) \subset \mathcal{I} \) and (d) implies (a). To show that (a) implies (d), assume that \( \sigma(A) \) is infinite and that \( \{A\}' + \mathcal{I} = \mathcal{B}(H) \). Then by Theorem 6, \( A - \lambda \in \mathcal{I} \) for some \( \lambda \in \mathcal{E} \). Since \( \mathcal{I} \) is contained in the ideal of compact operators, we can assume that \( A \) is compact. Let \( A = A_1 \oplus A_2 \) on \( \mathcal{M} \oplus \mathcal{M}^\perp \) where \( A_1 \) is an infinite dimensional diagonal operator with distinct eigenvalues and let \( P \) be the orthogonal projection onto \( \mathcal{M} \). Hence, if \( X \in \{A\}' \), then \( PXP \) is diagonal. However, if we let \( U \) be the unilateral shift on \( \mathcal{M} \), then \( \{A\}' + \mathcal{I} = \mathcal{B}(\mathcal{H}) \) implies that \( U = D + K \) for some diagonal operator \( D \) and some compact operator \( K \). This is clearly a contradiction (let \( \{e_n\} \) be an orthonormal basis for \( \mathcal{M} \) by which \( U \) is the shift, then \( ((D - U)e_n, e_{n+1}) = 1 \) for all \( n \)).

**References**


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