$\pi$-HOMOGENEITY AND $\pi'$-CLOSURE OF FINITE GROUPS

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The purpose of this paper is to present a proof, under additional conditions, of the following conjecture: Let \( \pi \) be a set of primes, and let all \( \pi \)-subgroups of \( G \) be 2-closed. (If \( 2 \in \pi \), this condition is satisfied.) If \( G \) is \( \pi \)-homogeneous, then \( G \) is \( \pi' \)-closed.

All groups considered here are finite. If \( \pi \) is a set of prime numbers, we say that the element \( x \) of a group \( G \) is a \( \pi \)-element if \( |x| \) is divisible only by primes in \( \pi \). In particular, one may speak of a \( p \)-element, \( p \) a prime. Similarly, a group \( G \) is called a \( \pi \)-group if \( |G| \) is divisible only by primes in \( \pi \). In addition, \( \pi(G) \) will denote the set of primes dividing \( |G| \). The set of primes not in \( \pi \) will be denoted by \( \pi' \). A group \( G \) is termed \( \pi \)-closed, if the subset of \( \pi \)-elements is a subgroup of \( G \). We say that a group \( G \) is \( \pi \)-homogeneous if \( N_G(H)/C_G(H) \) is a \( \pi \)-group for every nonidentity \( \pi \)-subgroup \( H \) of \( G \).

It is well known that \( \pi' \)-closed groups are \( \pi \)-homogeneous. The converse, in general, does not hold. For instance, \( A_5 \) is not 5-closed, but it is 5'-homogeneous.

For \( \pi = \{p\}, p \) a prime, the conjecture reduces to Frobenius' theorem ([11], Theorem 7.4.5).

The conjecture is closely connected to other well known problems in group theory. The proof of the conjecture would imply the solution of Baer's problem [3] (see also [5], p. 117), the answer to which is not known.

Baer's Problem. Let \( \pi \subseteq \pi(G) \). Suppose that \( G \) is \( \pi \) and \( \pi' \)-homogeneous. Is \( G \) a direct product of a \( \pi \)-group and a \( \pi' \)-group?

In order to show the connection with Frobenius' problem, we need some additional notation. For any prime \( p \), we denote by \( |G|_p \) the highest power of the prime \( p \) that divides \( |G| \). Define \( G \) to be weakly \( \pi \)-closed if for every subgroup \( U \) of \( G \) the number of \( \pi \)-elements of \( U \) is exactly \( \prod_{p \in \pi} |U|_p \).

Baer proved that if \( G \) is weakly \( \pi \)-closed then \( G \) is \( \pi' \)-homogeneous ([2], Lemma 2). Therefore, in the case that \( 2 \in \pi \), the proof of the above conjecture would imply also a solution of Frobenius' problem ([2], p. 325).
Frobenius’ Problem. Let $G$ be a weakly $\pi$-closed group. Is $G$ $\pi$-closed?

Our first result is that the conjecture holds if $2 \in \pi$.

**Theorem A.** Let $\pi$ be a set of primes which includes 2. Assume that all $\pi$-subgroups of $G$ are 2-closed. Then $G$ is $\pi'$-closed if and only if $G$ is $\pi$-homogeneous. (Compare with [2], Satze A, A*.)

In the next omnibus theorem, $2 \in \pi$. The proofs of Theorems B and C, as well as the proof of Corollary B, rely on the recent classification of simple $3'$-groups by J. Thompson.

**Theorem B.** Let $\pi$ be a set of odd primes. Then $G$ is $\pi'$-closed if $G$ is $\pi$-homogeneous and any one of the following conditions holds:

(i) $3 \in \pi(G)$.

(ii) The $\pi'$-subgroups of $G$ are solvable (hence if $N_{\pi}(H)$ is $\pi'$-closed for every nonidentity $\pi$-subgroup of $G$ and the $\pi'$-subgroups of $G$ are solvable, then $G$ is $\pi'$-closed).

(iii) $G$ has dihedral or abelian $S_2$-subgroups.

(iv) Every chain of subgroups has length at most 7.

A similar result holds if every 3rd maximal subgroup is nilpotent, or if every 2nd maximal subgroup is 2'-closed.

Theorem B (ii) together with Burnside's $p^aq^b$ Theorem yields:

**Corollary A.** If $|G|$ has exactly 4 prime divisors and $\pi$ is a set of odd primes, then $G$ is $\pi'$-closed if and only if $G$ is $\pi$-homogeneous.

The proof of part (ii) of Theorem B uses the following lemma, which follows from a theorem of Baer ([11], Theorem 3.8.2).

**Lemma 2.6.** If a group $G$ is $2'$-homogeneous then $G$ is 2-closed.

We shall say that $G$ is a $D_\pi$-group if all the maximal $\pi$-subgroups of $G$ are conjugate $S_\pi$-subgroups of $G$.

We conjecture that if $\pi$ is a set of primes, then $D_\pi$ and $\pi$-homogeneity imply $\pi'$-closure. (The alternating group $A_5$, for example, is $5'$-homogeneous, but it is not a $D_5$-group ([12], p. 143) and it is not 5'-closed.) The following theorem proves this conjecture under additional conditions.
THEOREM C. If $G$ is a $D_\pi$-group and $\pi$-homogeneous, then $G$ is $\pi'$-closed if one of the following conditions holds:

(i) $3 \in \pi(G)$.

(ii) The proper subgroups of $G$ are $\pi'$-closed.

Theorems A, B, and C imply the following corollary about groups all of whose proper subgroups are $\pi'$-closed.

COROLLARY B. Let $\pi$ be a set of primes. Let $G$ be a finite group such that every proper subgroup of $G$ is $\pi'$-closed, and assume that any one of the following conditions holds:

(i) $2 \in \pi$ and the $\pi$-subgroups of $G$ are 2-closed.

(ii) $2 \in \pi$ and $3 \in \pi(G)$.

(iii) $2 \in \pi$ and the $\pi'$-subgroups of $G$ are solvable.

(iv) $2 \in \pi$ and $G$ has dihedral or abelian $S_r$-subgroups.

(v) $2 \in \pi$ and every chain of subgroups has length at most 7.

(vi) $G$ is a $D_\pi$-group.

Then $G$ is one of the following:

(a) $G$ is $\pi'$-closed, or

(b) $\pi = \{p\}$, $p$ a prime, every proper subgroup of $G$ is nilpotent, $|G| = p^aq^b$, $q$ a prime, the $S_r$-subgroup of $G$ are cyclic and $G$ is $p$-closed.

(Compare this corollary with ([14], Chap. (iv), Satz 5.4.)

EXAMPLE. Let $\pi = \{2, 3\}$. Every proper subgroup of the alternating group $A_5$ is $\pi'$-closed. But $A_5$ is neither $\pi'$-closed nor solvable.

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2. Proofs. We incorporate a portion of the proofs of Theorems A and B into independent lemmas.

LEMMA 2.1. Let $G$ be either $\text{PSL}(2, r')$ or $S_z(q)$. Let $\pi$ be a subset of $\pi(G)$ consisting of odd primes and assume $|\pi| \geq 2$. Then $G$ is not $\pi$-homogeneous. Moreover, if $P$ is an $S_r$-subgroup of $\text{PSL}(2, r')$ where $p \in \pi$ and $p \neq r$, or $P$ is an $S_r$-subgroup of $S_z(q)$ where $p \in \pi$ then $2| |N_0(P)/C_0(P)|$.

Proof. If $P$ is an $S_r$-subgroup of $\text{PSL}(2, r')$, where $p \in \pi$ and $p \neq r$, then it is well known that $2| |N_0(P)/C_0(P)|$. Therefore, $\text{PSL}(2, r')$ is not $\pi$-homogeneous.
It follows by Theorem 4, Proposition 16, and Theorem 9 of [17] that in $S_z(q)$, $2/|N_G(H)/C_G(H)|$ for every nonidentity subgroup $H$ of $S_z(q)$ of odd order.

The following four basic results concerning $\pi$-homogeneous groups were proved in [1].

**Lemma 2.2** ([1], Lemma 2.3). Subgroups, direct products, and epimorphic images of $\pi$-homogeneous groups are $\pi$-homogeneous.

**Lemma 2.3** ([1], Lemma 2.4). If $K$ is a normal subgroup of the $\pi'$-homogeneous group $G$, and if $K$ and $G/K$ are $\pi$-closed, then $G$ is $\pi$-closed.

**Lemma 2.4** ([1], Theorem 2.5). The group $G$ is $\pi$-closed if, and only if, $G$ is $\pi$-separable and $\pi'$-homogeneous.

**Lemma 2.5** ([1], Lemma 2.1). $\pi$-closed groups are $\pi'$-homogeneous.

We now obtain at once

**Lemma 2.6.** If a group $G$ is $2'$-homogeneous then $G$ is $2$-closed.

**Proof.** Let $G$ be a minimal counterexample. Lemmas 2.2 and 2.3 imply that $G$ is a nonabelian simple group. Let $K$ be the conjugate class of an involution $u$ of $G$; obviously $|K| > 1$. Then by Theorem 3.8.2 of [11] there exists $v \in K$, $v \neq u$, such that $uv$ is not a $2$-element. If $|uv| = 2^km$, $m > 1$ odd, set $t = (uv)^{2^k}$; then $|t| = m > 1$ is odd. Now $t^*=t^{-1}$; therefore, $N_G(\langle t \rangle)/C_G(\langle t \rangle)$ is not a $2$-group. Hence $G$ is not $2'$-homogeneous, a contradiction.

**Proof of Theorem A.** If $G$ is $\pi'$-closed, then without any assumption on $\pi$ $G$ is $\pi$-homogeneous by Lemma 2.5. Therefore, we will prove here that, under the assumptions of Theorem A, if $G$ is $\pi$-homogeneous then $G$ is $\pi'$-closed. Let $\pi = \pi \cap \pi(G)$. If $2 \in \pi(G)$ then Lemma 2.4 and [8] imply that $G$ is $\pi'$-closed. If $\pi = \{2\}$ this is Frobenius' theorem. Let $G$ be a minimal counterexample. Then $G$ has the following properties:

(a) $G$ is $\pi_i$-homogeneous, $2 \in \pi_i$ and $|\pi_i| \geq 2$.

(b) The $\pi_i$-subgroups of $G$ are $2$-closed.

(c) $G$ is not $\pi'_i$-closed.

For the remainder of the proof we shall denote $\pi_i$ by $\pi$. Lemma 2.2 implies that subgroups and epimorphic images of $G$ are $\pi$-homogeneous. Clearly $\pi$-subgroups of subgroups of $G$ are $2$-closed. Therefore we also have:
(d) Proper subgroups of $G$ are $\pi'$-closed (hence solvable, by [8]). We want to prove
(e) $G$ is simple.

Suppose not, and let $N$ be a minimal normal subgroup of $G$. Since by (d) $N$ is solvable, $N$ is a $p$-group. If $p \in \pi$ and $K/N$ is a $\pi$-subgroup of $G/N$, then $K$ is a $\pi$-subgroup of $G$. Therefore, the $\pi$-subgroups of $G/N$ are 2-closed. $G/N$ is $\pi'$-closed, by induction. By Lemma 2.3, $G$ is $\pi'$-closed, a contradiction. Assume now that $p \notin \pi$. If $K/N$ is a $\pi$-subgroup of $G/N$, then by the Schur-Zassenhaus theorem $K = K_{r}N$ where $K_{r}$ is an $S_{r}$-subgroup of $K$. Therefore, $K/N$ has a normal $S_{r}$-subgroup. By induction $G/N$, and hence $G$, are $\pi'$-closed, a contradiction. Hence $G$ is simple.

Moreover, by (d) $G$ is a minimal simple group. By [21] $G$ is one of the following:

1. $\text{PSL}_{2}(2^{\alpha})$ where $p$ is any prime.
2. $\text{PSL}_{3}(3^{\alpha})$ where $p > 2$ is any prime.
3. $\text{PSL}_{4}(p)$ where $p$ is any prime with $p > 3$, and $p \equiv 2$ or 3 (mod 5).
4. $S_{2}(2^{\alpha})$ where $p$ is any odd prime.
5. $\text{PSL}_{3}(3)$.

If $G$ is a group of type (1) or (4), then for $q \in \pi$, $q$ odd ($|\pi| \geq 2$), there exist $Q$, a $q$-subgroup of $G$, and a 2-element $u$ of $G$, such that $u \in N_{G}(Q)$ but $u \notin C_{G}(Q)$, by Lemma 2.1. Now $T = \langle u \rangle Q$ is a non 2-closed $\pi$-group, a contradiction.

If $G$ is $\text{PSL}_{2}(r^{\alpha})$ of type (2) or (3) and $\pi$ contains a prime $u \neq r, 2$, then again Lemma 2.1 yields a contradiction. Hence $\pi = \{2, r\}$. Let $R$ be an $S_{r}$-subgroup of $G$. It is well known that $C_{G}(R) = R$ and that $|N_{G}(R)| = 1/2(r^{\alpha} - 1)|R|$. Since $G$ is $\pi$-homogeneous we obtain that $1/2(r^{\alpha} - 1) = 2^{\alpha}$ and therefore $N_{G}(R)$ is a $\pi$-subgroup of $G$. By assumption $N_{G}(R)$ is 2-closed, a contradiction.

If $G$ is $\text{PSL}_{3}(3)$, then $\pi(G) = \{2, 3, 13\}$. If $\pi = \{2, 13\}$ then ([14], Satz 7.3, p. 187) implies that $3/|N_{G}(P)/C_{G}(P)|$, where $P$ is an $S_{13}$-subgroup of $G$. Hence $G$ is not $\pi$-homogeneous, a contradiction. If $G$ is isomorphic to $\text{PSL}_{4}(3)$ and $\pi = \{2, 3\}$, then a study of the character table of $\text{PSL}_{4}(3)$ implies the existence of a subgroup $K$ of order 54 in $\text{PSL}_{4}(3)$ which is not 2-closed, in contradiction to (b). The proof of Theorem A is now complete.

Before beginning the proof of Theorem B we need several definitions.

A chain of subgroups of $G$ is a set of subgroups of $G$ linearly ordered by inclusion:

$$G = G_{0} \supset G_{1} \supset \cdots \supset G_{k} \supset \cdots \supset 1.$$ 

The length of a chain is the number of its distinct terms, minus 1.
A subgroup $G_k$ of $G$ is $k$th maximal if it is the $k$th term in some chain of proper subgroups, each of which is maximal in its predecessor and $k$ is the smallest such integer.

**Proof of Theorem B.** Let $G$ be a minimal counterexample.

**Proof of (i).** Lemmas 2.2 and 2.3 imply that $G$ is simple. By Thompson’s classification of simple $3'$-groups $G$ isomorphic to $S_3(q)$. Therefore, Lemma 2.1 implies that $G$ is not $\pi$-homogeneous, a contradiction.

**Proof of (ii).** $G$ has the following properties:
(a) $G$ is $\pi$-homogeneous, $2 \in \pi$ and $|\pi \cap \pi(G)| \geq 2$.
(b) The $\pi'$-subgroups of $G$ are solvable.
(c) $G$ is not $\pi'$-closed.

Lemma 2.2 implies that subgroups and epimorphic images of $G$ are $\pi$-homogeneous. Clearly subgroups of $G$ have solvable $\pi'$-subgroups. Therefore we also have:
(d) Proper subgroups of $G$ are $\pi$-closed (hence solvable, by [8]).
We want to prove:
(e) $G$ is simple.

Suppose not, and let $N$ be a minimal normal subgroup of $G$. Since by (d) $N$ is solvable, $N$ is a $p$-group. If $p \in \pi'$ and $K/N$ is a $\pi'$-subgroup of $G/N$, then $K$ is a $\pi'$-subgroup, so that $K$ is solvable, by hypothesis. Thus $K/N$ is solvable. If $p \in \pi$ and $K/N$ is a $\pi'$-subgroup of $G/N$, then by the Schur-Zassenhaus theorem $K = NK'$, where $K'$ is an $S_3$-subgroup of $K$. By assumption $K/N$ is solvable. Therefore, $G/N$ has solvable $\pi'$-subgroups. By induction $G/N$, and hence $G$ (by Lemma 2.3), are $\pi'$-closed, a contradiction. Hence $G$ is simple. Moreover, by (d) $G$ is a minimal simple group. By [21] $G$ is of one of the 5 types mentioned in the proof of Theorem A.

Lemma 2.1 implies that $G$ is not of type (1), (2), (3) or (4). Frobenius’ theorem and Lemma 2.6 imply that $G$ is not $\text{PSL}_3(3)$, since $|\text{PSL}_3(3)|$ has only 3 prime divisors, a contradiction.

Now, if $N = N_{\alpha}(H)$ is $\pi'$-closed, for any $\pi$-subgroup $H \neq 1$ of $G$, then $N/C_{\alpha}(H)$ is a $\pi$-group. Hence by the preceding paragraph $G$ is $\pi'$-closed.

We now obtain at once

**Proof of Corollary A.** If $|G|$ has only 4 prime divisors; then Frobenius’ theorem, Lemma 2.6, and Theorem B (ii), together with Burnside’s $p^aq^b$ theorem, yield that $G$ is $\pi'$-closed.

We return to the proof of Theorem B.
Proof of (iii). Let \( G \) have a dihedral \( S_2 \)-subgroup. If there exists \( 1 \neq N \triangleleft G \), then the \( S_2 \)-subgroups of \( N \) are of one of the following types: dihedral, cyclic or trivial. In the first case \( N \) is \( \pi' \)-closed by induction, in the second case \( N \) is \( 2' \)-closed and in the third \( N \) is solvable by [8]. Lemma 2.4 then implies that in every case \( N \) is \( \pi' \)-closed. Similarly \( G/N \) is also \( \pi' \)-closed. Therefore, Lemma 2.3 implies that \( G \) is \( \pi' \)-closed, a contradiction. Hence \( G \) is simple. By Theorem 16.3 of [11] \( G \) is isomorphic to either \( \text{PSL}(2, q) \), \( q \) odd, \( q > 3 \), or to \( A_7 \). Lemma 2.1 implies that \( G \) is isomorphic to \( A_7 \). But \( |A_7| \) has only 4 prime divisors, therefore, Corollary A implies that \( G \) is \( \pi' \)-closed, a contradiction.

Let \( G \) have abelian \( S_2 \)-subgroups. Clearly \( G \) is simple. Walter [18, 19] proved that one of the following holds:

1. \( G \) is isomorphic to \( L_3(q) \), \( q > 3 \), \( q \equiv 3, 5 \) (mod 8) or \( q = 2^n \);
2. \( G \) is isomorphic to \( J(11) \); or
3. \( G \) is of Ree type.

Lemma 2.1 eliminates the first possibility. Now \( J(11) \) is of order \( 2^1 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \). If \( P \) is an \( S_p \)-subgroup of \( J(11) \) for \( p = 3, 5, 7, 11, 19 \), then \( 2|N(P)/C(P)| \) by [15]. Hence \( J(11) \) is not \( \pi \)-homogeneous, so that \( G \) must be of Ree type. Then \( G \) is of order \( q^3(q - 1)(q + 1) \) \( (q^2 - q + 1) \) where \( q = 3^{2k+1} \), \( k \geq 1 \). If \( 3 \in \pi \) and \( P \) is an \( S_2 \)-subgroup of \( G \), then \( N(P) = PW \), where \( W \) is cyclic of order \( q - 1 \). Now if \( J \) is the involution of \( W \), then \( J \not\in C(P) \). Hence if \( 3 \in \pi \) then \( G \) is not \( \pi \)-homogeneous. We know also [20] that \( G \) possesses Abelian Hall subgroups \( M^+ \) and \( M^- \) of orders \( q + 1 + 3m \) and \( q + 1 - 3m \), where \( m = 3^t \) and \( q^2 - q + 1 = (q + 1 + 3m)(q + 1 - 3m) \). If \( t \) is a prime such that either \( t|M^+ \) or \( t|M^- \) and \( T \) is an \( S_2 \)-subgroup of \( M^\pm \), then \( N(T) \supseteq N(M^\pm) = M^\pm W^\pm \), where \( W^\pm \) are cyclic of order 6. But \( C(T) = M^\pm \). Hence if \( t \in \pi \) then \( G \) is not \( \pi \)-homogeneous. Now by the definition of \( G \) [20] there exist cyclic subgroups \( R^\pm \) of order \( 1/2(q \pm 1) \). The normalizer \( N_o(R_0) \) of any subgroup \( R_0 \neq 1 \) of \( R^\pm \) is contained in \( \langle J \rangle \times L_3(q) \), where \( J \) is an involution of \( G \). If \( R_0 \) is of odd order then \( R_0 \subseteq L_3(q) \) and \( 2|N_o(R_0)/C_o(R_0)| \). Therefore, if \( \pi \) contains of primes dividing either \( q + 1 \) or \( q - 1 \), then \( G \) is not \( \pi \)-homogeneous. Since \( |G| = q^3(q - 1)(q + 1)(q^2 - q + 1) \) where \( q = 3^{2k+1} \), \( k \geq 1 \), (iii) follows.

Proof of (iv). Lemmas 2.2 and 2.3 imply that \( G \) is simple. Gagen's theorem [9] and Harada's theorem [13] imply that \( G \) is isomorphic to one of the following groups: \( \text{PSU}_3(3) \), \( \text{PSU}_3(5) \), \( A_7 \), \( M_{11} \), \( J(11) \), or \( \text{PSL}(2, q) \), for certain values of \( q \). The last possibility is eliminated by Lemma 2.1. In the proof of (iii) we found that \( J(11) \) is not \( \pi \)-homogeneous. Since the remaining groups have orders with at most 4 prime divisors, they are \( \pi' \)-closed, by Corollary A and
Lemma 2.6.

Proof of Theorem C. Let $G$ be a minimal counterexample. In both cases Lemmas 2.2, 2.3, and ([14], Chap. (iv), Hlf. 7.2, p. 444) imply that $G$ is simple. Therefore, if (i) $3 \in \pi(G)$ then, assuming Thompson's classification of simple $3'$-groups, $G$ is isomorphic to $S_{\pi}(q)$. If in addition $2 \in \pi$ then Theorem B implies that $G$ is $\pi'$-closed, a contradiction. If $2 \in \pi$ then Theorem 9 of [17] implies that $G$ is not a $D_{\pi}$-group, again a contradiction. In case (ii) Theorem 3.1 of [7] implies that $G$ is $\pi'$-closed. This contradiction completes the proof of Theorem C.

It is well known that if every proper subgroup of $G$ is $p'$-closed but $G$ is not $p'$-closed, then every proper subgroup of $G$ is nilpotent, $|G| = p^{a}q^{b}$, $q$ a prime, and the $S_{\pi}$-subgroups of $G$ are cyclic (see [14], Chap. (iv), Satz 5.4, p. 434).

Theorems A, B, and C imply the same conclusion under additional conditions for groups every proper subgroup of which is $\pi'$-closed.

Proof of Corollary B. Let $G$ be a minimal counterexample. If $G$ is not $\pi'$-closed, then Theorems A, B, and C imply that there exist $S$, a $\pi$-subgroup of $G$, and $x$, a $\pi'$-element of $G$, such that $x \in N_{G}(S)$ but $x \notin C_{G}(S)$. Therefore, Theorem 6.2.2 of [11] implies that there exists a prime $p$ in $\pi$ and $P$, an $S_{p}$-subgroup of $S$, such that $x \in N_{G}(P)$ but $x \notin C_{G}(P)$. Set $T = P \langle x \rangle$. If $T \subset G$, then by hypothesis $T = P \times \langle x \rangle$ and $x \notin C_{G}(P)$, a contradiction. If $T = G = P \langle x \rangle$, then every proper subgroup of $G$ is by hypothesis $p'$-closed, but $G$ itself is not $p'$-closed. Hence ([14], Chap. (iv), Satz 5.4, p. 434) implies (b).

References


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