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The purpose of this paper is to present a proof, under additional conditions, of the following conjecture: Let π be a set of primes, and let all π-subgroups of G be 2-closed. (If $2 \notin \pi$, this condition is satisfied.) If G is π-homogeneous, then G is π'-closed.

All groups considered here are *finite*. If π is a set of prime numbers, we say that the element x of a group G is a *π-element* if $|x|$ is divisible only by primes in π. In particular, one may speak of a *p-element*, p a prime. Similarly, a group G is called a *π-group* if $|G|$ is divisible only by primes in π. In addition, $\pi(G)$ will denote the set of primes dividing $|G|$. The set of primes not in π will be denoted by π'. A group G is termed *π-closed*, if the subset of G consisting of π-elements is a subgroup of G. We say that a group G is *π-homogeneous* if $N_G(H)/C_G(H)$ is a π-group for every nonidentity π-subgroup H of G.

It is well known that π'-closed groups are π-homogeneous. The converse, in general, does not hold. For instance, A_5 is not 5-closed, but it is $5'$-homogeneous.

For $\pi = \{p\}$, p a prime, the conjecture reduces to Frobenius' theorem ([11], Theorem 7.4.5).

The conjecture is closely connected to other well known problems in group theory. The proof of the conjecture would imply the solution of Baer's problem [3] (see also [5], p. 117), the answer to which is not known.

Baer's Problem. Let $\pi \subsetneqq \pi(G)$. Suppose that G is π and π'-homogeneous. Is G a direct product of a π-group and a π'-group?

In order to show the connection with Frobenius' problem, we need some additional notation. For any prime p, we denote by $|G|_p$ the highest power of the prime p that divides $|G|$. Define G to be *weakly π-closed* if for every subgroup U of G the number of π-elements of U is exactly $\prod_{p \in \pi} |U|_p$.

Baer proved that if G is weakly π-closed then G is π'-homogeneous ([2], Lemma 2). Therefore, in the case that $2 \in \pi$, the proof of the above conjecture would imply also a solution of Frobenius' problem ([2], p. 325).

Frobenius' Problem. Let G be a weakly π-closed group. Is G π-closed?

Our first result is that the conjecture holds if $2 \in \pi$.

THEOREM A. *Let π be a set of primes which includes 2. Assume that all π-subgroups of G are 2-closed. Then G is π'-closed if and only if G is π-homogeneous.* (Compare with [2], Satze A, A^*.)

In the next omnibus theorem, $2 \notin \pi$. The proofs of Theorems B and C, as well as the proof of Corollary B, rely on the recent classification of simple $3'$-groups by J. Thompson.

THEOREM B. *Let π be a set of odd primes. Then G is π'-closed if G is π-homogeneous and any one of the following conditions holds*:

(i) $3 \notin \pi(G)$.

(ii) *The π'-subgroups of G are solvable (hence if $N_G(H)$ is π'-closed for every nonidentity π-subgroup of G and the π'-subgroups of G are solvable, then G is π'-closed).*

(iii) *G has dihedral or abelian S_2-subgroups.*

(iv) *Every chain of subgroups has length at most 7.*

A similar result holds if every 3rd maximal subgroup is nilpotent, or if every 2nd maximal subgroup is $2'$-closed.

Theorem B (ii) together with Burnside's $p^\alpha q^\beta$ Theorem yields:

COROLLARY A. *If $|G|$ has exactly 4 prime divisors and π is a set of odd primes, then G is π'-closed if and only if G is π-homogeneous.*

The proof of part (ii) of Theorem B uses the following lemma, which follows from a theorem of Baer ([11], Theorem 3.8.2).

LEMMA 2.6. *If a group G is $2'$-homogeneous then G is 2-closed.*

We shall say that G is a D_π-group if all the maximal π-subgroups of G are conjugate S_π-subgroups of G.

We conjecture that if π is a set of primes, then D_π and π-homogeneity imply π'-closure. (The alternating group A_5, for example, is $5'$-homogeneous, but it is not a $D_{5'}$-group ([12], p. 143) and it is not $5'$-closed.) The following theorem proves this conjecture under additional conditions.

THEOREM C. *If G is a D_π-group and π-homogeneous, then G is π'-closed if one of the following conditions holds:*
(i) $3 \notin \pi(G)$.
(ii) *The proper subgroups of G are π'-closed.*

Theorems A, B, and C imply the following corollary about groups all of whose proper subgroups are π'-closed.

COROLLARY B. *Let π be a set of primes. Let G be a finite group such that every proper subgroup of G is π'-closed, and assume that any one of the following conditions holds:*
(i) $2 \in \pi$ *and the π-subgroups of G are 2-closed.*
(ii) $2 \notin \pi$ *and* $3 \notin \pi(G)$.
(iii) $2 \notin \pi$ *and the π'-subgroups of G are solvable.*
(iv) $2 \notin \pi$ *and G has dihedral or abelian S_2-subgroups.*
(v) $2 \notin \pi$ *and every chain of subgroups has length at most 7.*
(vi) *G is a D_π-group.*
Then G is one of the following:
(a) *G is π'-closed, or*
(b) $\pi = \{p\}$, *p a prime, every proper subgroup of G is nilpotent,* $|G| = p^a q^b$, *q a prime, the S_q-subgroup of G are cyclic and G is p-closed.*
(Compare this corollary with ([14], Chap. (iv), Satz 5.4.)

EXAMPLE. Let $\pi = \{2, 3\}$. Every proper subgroup of the alternating group A_5 is π'-closed. But A_5 is neither π'-closed nor solvable.

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2. *Proofs.* We incorporate a portion of the proofs of Theorems A and B into independent lemmas.

LEMMA 2.1. *Let G be either* PSL $(2, r^t)$ *or* $S_Z(q)$. *Let π be a subset of $\pi(G)$ consisting of odd primes and assume $|\pi| \geqq 2$. Then G is not π-homogeneous. Moreover, if P is an S_p-subgroup of* PSL $(2, r^t)$ *where $p \in \pi$ and $p \neq r$, or P is an S_p-subgroup of $S_Z(q)$ where $p \in \pi$ then* $2/|N_G(P)/C_G(P)|$.

Proof. If P is an S_p-subgroup of PSL $(2, r^t)$, where $p \in \pi$ and $p \neq r$, then it is well known that $2/|N_G(P)/C_G(P)|$. Therefore, PSL $(2, r^t)$ is not π-homogeneous.

It follows by Theorem 4, Proposition 16, and Theorem 9 of [17] that in $S_Z(q)$, $2/|N_G(H)/C_G(H)|$ for every nonidentity subgroup H of $S_Z(q)$ of odd order.

The following four basic results concerning π-homogeneous groups were proved in [1].

LEMMA 2.2 ([1], Lemma 2.3). *Subgroups, direct products, and epimorphic images of π-homogeneous groups are π-homogeneous.*

LEMMA 2.3 ([1], Lemma 2.4). *If K is a normal subgroup of the π'-homogeneous group G, and if K and G/K are π-closed, then G is π-closed.*

LEMMA 2.4 ([1], Theorem 2.5). *The group G is π-closed if, and only if, G is π-separable and π'-homogeneous.*

LEMMA 2.5 ([1], Lemma 2.1). *π-closed groups are π'-homogeneous.*

We now obtain at once

LEMMA 2.6. *If a group G is $2'$-homogeneous then G is 2-closed.*

Proof. Let G be a minimal counterexample. Lemmas 2.2 and 2.3 imply that G is a nonabelian simple group. Let K be the conjugate class of an involution u of G; obviously $|K| > 1$. Then by Theorem 3.8.2 of [11] there exists $v \in K$, $v \neq u$, such that uv is not a 2-element. If $|uv| = 2^k m$, $m > 1$ odd, set $t = (uv)^{2^k}$; then $|t| = m > 1$ is odd. Now $t^u = t^{-1}$; therefore, $N_G(\langle t \rangle)/C_G(\langle t \rangle)$ is not a $2'$-group. Hence G is not $2'$-homogeneous, a contradiction.

Proof of Theorem A. If G is π'-closed, then without any assumption on π G is π-homogeneous by Lemma 2.5. Therefore, we will prove here that, under the assumptions of Theorem A, if G is π-homogeneous then G is π'-closed. Let $\pi_1 = \pi \cap \pi(G)$. If $2 \notin \pi(G)$ then Lemma 2.4 and [8] imply that G is π'-closed. If $\pi_1 = \{2\}$ this is Frobenius' theorem. Let G be a minimal counterexample. Then G has the following properties:

(a) G is π_1-homogeneous, $2 \in \pi_1$ and $|\pi_1| \geqq 2$.

(b) The π_1-subgroups of G are 2-closed.

(c) G is not π_1'-closed.

For the remainder of the proof we shall denote π_1 by π. Lemma 2.2 implies that subgroups and epimorphic images of G are π-homogeneous. Clearly π-subgroups of subgroups of G are 2-closed. Therefore we also have:

(d) Proper subgroups of G are π'-closed (hence solvable, by [8]). We want to prove

(e) G is simple.

Suppose not, and let N be a minimal normal subgroup of G. Since by (d) N is solvable, N is a p-group. If $p \in \pi$ and K/N is a π-subgroup of G/N, then K is a π-subgroup of G. Therefore, the π-subgroups of G/N are 2-closed. G/N is π'-closed, by induction. By Lemma 2.3, G is π'-closed, a contradiction. Assume now that $p \notin \pi$. If K/N is a π-subgroup of G/N, then by the Schur-Zassenhaus theorem $K = K_\pi N$ where K_π is an S_π-subgroup of K. Therefore, K/N has a normal S_2-subgroup. By induction G/N, and hence G, are π'-closed, a contradiction. Hence G is simple.

Moreover, by (d) G is a minimal simple group. By [21] G is one of the following:

(1) $\mathrm{PSL}_2(2^p)$ where p is any prime.

(2) $\mathrm{PSL}_2(3^p)$ where $p > 2$ is any prime.

(3) $\mathrm{PSL}_2(p)$ where p is any prime with $p > 3$, and $p \equiv 2$ or 3 (mod 5).

(4) $S_Z(2^p)$ where p is any odd prime.

(5) $\mathrm{PSL}_3(3)$.

If G is a group of type (1) or (4), then for $q \in \pi$, q odd ($|\pi| \geqq 2$), there exist Q, a q-subgroup of G, and a 2-element u of G, such that $u \in N_G(Q)$ but $u \notin C_G(Q)$, by Lemma 2.1. Now $T = \langle u \rangle Q$ is a non 2-closed π-group, a contradiction.

If G is $\mathrm{PSL}_2(r^t)$ of type (2) or (3) and π contains a prime $u \neq r$, 2, then again Lemma 2.1 yields a contradiction. Hence $\pi = \{2, r\}$. Let R be an S_r-subgroup of G. It is well known that $C_G(R) = R$ and that $|N_G(R)| = 1/2(r^t - 1)|R|$. Since G is π-homogeneous we obtain that $1/2(r^t - 1) = 2^\alpha$ and therefore $N_G(R)$ is a π-subgroup of G. By assumption $N_G(R)$ is 2-closed, a contradiction.

If G is $\mathrm{PSL}_3(3)$, then $\pi(G) = \{2, 3, 13\}$. If $\pi = \{2, 13\}$ then ([14], Satz 7.3, p. 187) implies that $3/|N_G(P)/C_G(P)|$, where P is an S_{13}-subgroup of G. Hence G is not π-homogeneous, a contradiction. If G is isomorphic to $\mathrm{PSL}_3(3)$ and $\pi = \{2, 3\}$, then a study of the character table of $\mathrm{PSL}_3(3)$ implies the existence of a subgroup K of order 54 in $\mathrm{PSL}_3(3)$ which is not 2-closed, in contradiction to (b). The proof of Theorem A is now complete.

Before beginning the proof of Theorem B we need several definitions.

A chain of subgroups of G is a set of subgroups of G linearly ordered by inclusion:

$$G = G_0 \supset G_1 \supset \cdots \supset G_k \supset \cdots \supset 1 .$$

The length of a chain is the number of its distinct terms, minus 1.

A subgroup G_k of G is kth maximal if it is the kth term in some chain of proper subgroups, each of which is maximal in its predecessor and k is the smallest such integer.

Proof of Theorem B. Let G be a minimal counterexample.

Proof of (i). Lemmas 2.2 and 2.3 imply that G is simple. By Thompson's classification of simple $3'$-groups G isomorphic to $S_Z(q)$. Therefore, Lemma 2.1 implies that G is not π-homogeneous, a contradiction.

Proof of (ii). G has the following properties:
(a) G is π-homogeneous, $2 \notin \pi$ and $|\pi \cap \pi(G)| \geqq 2$.
(b) The π'-subgroups of G are solvable.
(c) G is not π'-closed.

Lemma 2.2 implies that subgroups and epimorphic images of G are π-homogeneous. Clearly subgroups of G have solvable π'-subgroups. Therefore we also have:

(d) Proper subgroups of G are π-closed (hence solvable, by [8]). We want to prove:

(e) G is simple.

Suppose not, and let N be a minimal normal subgroup of G. Since by (d) N is solvable, N is a p-group. If $p \in \pi'$ and K/N is a π'-subgroup of G/N, then K is a π'-subgroup, so that K is solvable, by hypothesis. Thus K/N is solvable. If $p \in \pi$ and K/N is a π'-subgroup of G/N, then by the Schur-Zassenhaus theorem $K = NK_{\pi'}$ where $K_{\pi'}$ is an $S_{\pi'}$-subgroup of K. By assumption K/N is solvable. Therefore, G/N has solvable π'-subgroups. By induction G/N, and hence G (by Lemma 2.3), are π'-closed, a contradiction. Hence G is simple. Moreover, by (d) G is a minimal simple group. By [21] G is of one of the 5 types mentioned in the proof of Theorem A.

Lemma 2.1 implies that G is not of type (1), (2), (3) or (4). Frobenius' theorem and Lemma 2.6 imply that G is not $\mathrm{PSL}_3(3)$, since $|\mathrm{PSL}_3(3)|$ has only 3 prime divisors, a contradiction.

Now, if $N = N_G(H)$ is π'-closed, for any π-subgroup $H \neq 1$ of G, then $N/C_G(H)$ is a π-group. Hence by the preceding paragraph G is π'-closed.

We now obtain at once

Proof of Corollary A. If $|G|$ has only 4 prime divisors; then Frobenius' theorem, Lemma 2.6, and Theorem B (ii), together with Burnside's $p^\alpha q^\beta$ theorem, yield that G is π'-closed.

We return to the proof of Theorem B.

Proof of (iii). Let G have a dihedral S_2-subgroup. If there exists $1 \neq N \lhd G$, then the S_2-subgroups of N are of one of the following types: dihedral, cyclic or trivial. In the first case N is π'-closed by induction, in the second case N is $2'$-closed and in the third N is solvable by [8]. Lemma 2.4 then implies that in every case N is π'-closed. Similarly G/N is also π'-closed. Therefore, Lemma 2.3 implies that G is π'-closed, a contradiction. Hence G is simple. By Theorem 16.3 of [11] G is isomorphic to either PSL$(2, q)$, q odd, $q > 3$, or to A_7. Lemma 2.1 implies that G is isomorphic to A_7. But $|A_7|$ has only 4 prime divisors, therefore, Corollary A implies that G is π'-closed, a contradiction.

Let G have abelian S_2-subgroups. Clearly G is simple. Walter [18, 19] proved that one of the following holds:

(1) G is isomorphic to $L_2(q)$, $q > 3$, $q \equiv 3, 5 \pmod 8$ or $q = 2^n$;

(2) G is isomorphic to $J(11)$; or

(3) G is of Ree type.

Lemma 2.1 eliminates the first possibility. Now $J(11)$ is of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. If P is an S_p-subgroup of $J(11)$ for $p = 3, 5, 7, 11, 19$, then $2/|N(P)/C(P)|$ by [15]. Hence $J(11)$ is not π-homogeneous, so that G must be of Ree type. Then G is of order $q^3(q-1)(q+1)(q^2-q+1)$ where $q = 3^{2k+1}$, $k \geqq 1$. If $3 \in \pi$ and P is an S_3-subgroup of G, then $N(P) = PW$, where W is cyclic of order $q-1$. Now if J is the involution of W, then $J \notin C(P)$. Hence if $3 \in \pi$ then G is not π-homogeneous. We know also [20] that G possesses Abelian Hall subgroups M^+ and M^- of orders $q+1+3m$ and $q+1-3m$, where $m = 3^k$ and $q^2 - q + 1 = (q+1+3m)(q+1-3m)$. If t is a prime such that either $t/|M^+|$ or $t/|M^-|$ and T is an S_t-subgroup of $M^\pm$, then $N(T) \supseteqq N(M^\pm) = M^\pm W^\pm$, where $W^\pm$ are cyclic of order 6. But $C(T) = M^\pm$. Hence if $t \in \pi$ then G is not π-homogeneous. Now by the definition of G [20] there exist cyclic subgroups $R^\pm$ of order $1/2(q \pm 1)$. The normalizer $N_G(R_0)$ of any subgroup $R_0 \neq 1$ of $R^\pm$ is contained in $\langle J \rangle \times L_2(q)$, where J is an involution of G. If R_0 is of odd order then $R_0 \subseteqq L_2(q)$ and $2/|N_G(R_0)/C_G(R_0)|$. Therefore, if π contains of primes dividing either $q+1$ or $q-1$, then G is not π-homogeneous. Since $|G| = q^3(q-1)(q+1)(q^2-q+1)$ where $q = 3^{2k+1}$, $k \geqq 1$, (iii) follows.

Proof of (iv). Lemmas 2.2 and 2.3 imply that G is simple. Gagen's theorem [9] and Harada's theorem [13] imply that G is isomorphic to one of the following groups: $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_3(5)$, A_7, M_{11}, $J(11)$, or PSL$(2, q)$, for certain values of q. The last possibility is eliminated by Lemma 2.1. In the proof of (iii) we found that $J(11)$ is not π-homogeneous. Since the remaining groups have orders with at most 4 prime divisors, they are π'-closed, by Corollary A and

Lemma 2.6.

Proof of Theorem C. Let G be a minimal counterexample. In both cases Lemmas 2.2, 2.3, and ([14], Chap. (iv), Hilf. 7.2, p. 444) imply that G is simple. Therefore, if (i) $3 \notin \pi(G)$ then, assuming Thompson's classification of simple 3'-groups, G is isomorphic to $S_Z(q)$. If in addition $2 \notin \pi$ then Theorem B implies that G is π'-closed, a contradiction. If $2 \in \pi$ then Theorem 9 of [17] implies that G is not a D_π-group, again a contradiction. In case (ii) Theorem 3.1 of [7] implies that G is π'-closed. This contradiction completes the proof of Theorem C.

It is well known that if every proper subgroup of G is p'-closed but G is not p'-closed, then every proper subgroup of G is nilpotent, $|G| = p^\alpha q^\beta$, q a prime, and the S_q-subgroups of G are cyclic (see [14], Chap. (iv), Satz 5.4, p. 434).

Theorems A, B, and C imply the same conclusion under additional conditions for groups every proper subgroup of which is π'-closed.

Proof of Corollary B. Let G be a minimal counterexample. If G is not π'-closed, then Theorems A, B, and C imply that there exist S, a π-subgroup of G, and x, a π'-element of G, such that $x \in N_G(S)$ but $x \notin C_G(S)$. Therefore, Theorem 6.2.2 of [11] implies that there exists a prime p in π and P, an S_p-subgroup of S, such that $x \in N_G(P)$ but $x \notin C_G(P)$. Set $T = P\langle x\rangle$. If $T \subset G$, then by hypothesis $T = P \times \langle x\rangle$ and $x \in C_G(P)$, a contradiction. If $T = G = P\langle x\rangle$, then every proper subgroup of G is by hypothesis p'-closed, but G itself is not p'-closed. Hence ([14], Chap. (iv), Satz 5.4, p. 434) implies (b).

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