THE DIRICHLET PROBLEM FOR SOME OVERDETERMINED SYSTEMS ON THE UNIT BALL IN $C^n$

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A characterization is given of those functions on $\partial B^n = \{|z| = 1\}$ which can be extended to be analytic, pluriharmonic, or $n$-harmonic in $B^n = \{|z| < 1\}$.

1. Introduction. If $f$ is a continuous function on $\partial B^n = \{z = (z_1, \cdots, z_n); |z| = 1\}$, then $f$ can be extended to a harmonic function $F$ in $B^n = \{z: |z| < 1\}$. That is, the Dirichlet problem is uniquely solvable. If we wish $F$, in addition, to be analytic, pluriharmonic, or $n$-harmonic, the extension is not always possible, and we must impose some restrictions on the function $f$. It is well-known that necessary and sufficient conditions for $f$ to have an analytic extension are that $f$ satisfy the tangential Cauchy-Riemann equation. In this paper we show that there are other systems that replace the tangential Cauchy-Riemann equations as consistency conditions. We also give the consistency conditions for a function to extend to be pluriharmonic or $n$-harmonic.

2. Pluriharmonic extension. Some important differential operators tangential to $\partial B^n$, $n \geq 2$ are:

\begin{align}
\mathcal{L}_{ij} &= \overline{\zeta}_i \frac{\partial}{\partial \zeta_j} - \overline{\zeta}_j \frac{\partial}{\partial \zeta_i} \\
\overline{\mathcal{L}}_{ij} &= \overline{\zeta}_i \frac{\partial}{\partial \zeta_j} - \overline{\zeta}_j \frac{\partial}{\partial \zeta_i}
\end{align}

where we take $1 \leq i, j \leq n$ and $\zeta = (\zeta_1, \cdots, \zeta_n) \in \partial B^n$. A simple computation shows that the real and imaginary parts of these operators are tangent to $\partial B^n$. These operators extend naturally into the interior of $B^n$. The following lemma shows the interplay between the action of the $\mathcal{L}_{ij}$ on $\partial B^n$ and in $B^n$.

**LEMMA 1.** Let $\mathcal{L}$ be one of the operators (1) or (2), and let $u \in C'(\partial B^n)$ be given. If $P(x, \zeta)$ is the Poisson kernel on $B^n$, we have:

\begin{align}
(\mathcal{L}_i u) \ast P(z) &= \mathcal{L}_i (u \ast P(z))
\end{align}

for $\zeta \in \partial B^n, z \in B^n$.

**Proof.** The operator $\mathcal{L}$ satisfies the hypotheses of Lemma 2, and thus the right hand side of (3) is harmonic (the left hand side
obviously is). Since (3) is valid for \(|z| = 1\), it must hold for all \(z \in B^*\).

**Lemma 2.** An operator \(\mathcal{D} = f(x, y)\frac{\partial}{\partial y} - g(x, y)\frac{\partial}{\partial x}\) preserves harmonic functions if and only if the pair \((f, g)\) satisfies the Cauchy-Riemann equations,

\[
\begin{align*}
    f_x &= g_y, \\
    f_y &= -g_x. 
\end{align*}
\]

**Proof.** It is a straightforward calculation that \((\mathcal{D}u)_x + (\mathcal{D}u)_y = 0\) for all harmonic \(u\) if and only if \(f_x = g_y\) and \(-f_y = g_x\).

**Corollary 1.** If \(f \in L^1(\partial B^*)\), and \(\mathcal{L}f = g\) in the weak sense, (i.e., \(\int_{|z|=1} f\mathcal{L}\phi = -\int_{|z|=1} g\phi\) for all \(\phi \in C^\infty(\partial B^*)\), then

\[
g * P(z) = \mathcal{L}^*(f * P(z)).
\]

**Proof.** Since the Poisson kernel on \(B^*\) is \(P(\zeta, z) = 1 - |z|^2/|z - \zeta|^2\), one can calculate that:

\[
\mathcal{L}^*P(\zeta, z) = \frac{1}{|\zeta|^2 - |z|^2}.
\]

Thus if \(dS\) is normalized surface area, we have:

\[
\mathcal{L}^*(f * P(z)) = \int_{|z|=1} f(\zeta)\mathcal{L}^*P(\zeta, z)dS = -\int_{|z|=1} g(\zeta)P(\zeta, z)dS = g * P(z).
\]

**Definition.** If \(\alpha\) and \(\beta\) are multi-indices, then \(z^\alpha \overline{z}^\beta = \prod_{j=1}^n z_j^\alpha \overline{z}_j^\beta\) has type \((p, q)\) if \(|\alpha| = p\) and \(|\beta| = q\). If \(h(z, \overline{z})\) is a sum of monomials of type \((p, q)\), then \(h\) is of type \((p, q)\).

Observe that if \(h\) is of type \((p, q)\), then \(\mathcal{L}_i h\) is either zero or of type \((p + 1, q - 1)\). Similarly, \(\mathcal{L}_i^* h\) is either of type \((p - 1, q + 1)\) or zero.

By \(L\) we will denote the matrix of operators \(L = (\mathcal{L}_i^*)\).

If \(K = (K_{ij})\) and \(M = (M_{ij})\) are two matrices of operators, then \(KM\) will denote the tensor product of the two matrices:

\[
KM(u) = K \otimes M(u) = (K_{ij}M_{ij}u).
\]

**Lemma 3.** Let \(F \in C^1(\overline{B}^*)\) satisfy \(\Delta F = 0\). If \(\mathcal{L}F(z) = 0\) for all \(z \in B^*\), then \(F\) is analytic.

**Proof.** The system \(\mathcal{L}F = 0\) is precisely the tangential Cauchy-Riemann equations.
Riemann equations (see [1], [2]). Thus if $f$ is the restriction of $F$ to $\partial B^n$, then $f$ has a holomorphic extension to $B^n$, which must coincide with $F$, since $F$ is harmonic.

**Remark.** The lemma may also be proved directly without mention of the tangential Cauchy-Riemann equations.

**Theorem 1.** If $u \in C^\infty(\partial B^n)$, then

$$
\bar{L}LL(u) = 0
$$

if and only if $u$ extends to a pluriharmonic function $U$ on $B^n$.

**Proof.** If $u$ extends to a pluriharmonic $U$, then we write $U(z, \bar{z}) = f(z) + g(\bar{z})$ where $f$ and $g$ are analytic. An entry of the matrix $\bar{L}LLU$ looks like:

$$
\bar{L}(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} U) = \bar{L}(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} f - \bar{z}_i f_z)
$$

$$
= \bar{L}(z_i \frac{\partial^2}{\partial z_j \partial \bar{z}_k} f_z - z_k \frac{\partial^2}{\partial z_j \partial \bar{z}_i} f_z)
$$

$$
- z_k \frac{\partial^2}{\partial z_i \partial \bar{z}_j} f_z + z_j \frac{\partial^2}{\partial z_i \partial \bar{z}_k} f_z)
$$

$$
= \bar{L} \text{ (analytic)} = 0 .
$$

To prove the converse, we show that the harmonic extension $U$ of $u$ is pluriharmonic. Since $U$ is harmonic, we may write, as before:

$$
U(z, \bar{z}) = \sum_{p+q \leq 0} F_{p,q} .
$$

By Lemma 1, we have:

$$
\bar{L}LL(\sum F_{p,q}) = \sum_{p+q \leq 0} \bar{L}LLF_{p,q} = 0 .
$$

Recall that $\bar{L}LL$ takes a polynomial of type $(p, q)$ into one of type $(p + 1, q - 1)$ or zero. Thus $\bar{L}LLF_{p,q} = 0$ for each $p, q \geq 0$.

By Lemma 3, the entries of the matrix $\bar{L}LF_{p,q}$ are analytic. But on the other hand, they must be of type $(p, q)$ or zero. Thus if $q \geq 1$, we conclude that $\bar{L}LF_{p,q} = 0$.

Again by Lemma 3, the entries of $LF_{p,q}$ are analytic if $q \geq 1$. But since they will be type $(p - 1, q + 1)$ or zero, we conclude that $LF_{p,q} = 0$ for $q \geq 1$. This means that $\bar{F}_{p,q} = 0$ is analytic if $q \geq 1$.

Thus if $p, q \geq 1$, then $F_{p,q} = 0$.

Thus we may write

$$
U(z, \bar{z}) = \sum_{j \geq 1} (F_{j,0} + F_{0,j}) + F_{0,0} .
$$

Hence $U$ is pluriharmonic.
REMARK. It was observed by L. Nirenberg that there is no second order operator $\mathcal{D}$ which gives the consistency conditions for pluriharmonic functions $\partial B^\ast$.

**Corollary 2.** Let $m \geq 2$ and $u \in C^\infty(\partial B^\ast)$ be given. Then $u$ can be extended to $U$ pluriharmonic in $B^\ast$ if and only if (5) or (6) holds:

(5) \[ \bar{L}^4(L^4\bar{L}^4)^mLu = 0 \]

(6) \[ (L^4\bar{L}^4)^mLu = 0 . \]

**Proof.** If $u$ can be extended, then the above equations are clearly valid.

We prove the other implication by induction. Line (5) holds for $m = 0$ (Theorem 1). We assume that (6) is valid for $m = k$ and show that (5) also holds for $m = k$. The other part, showing that (5) is valid for $m = k$ implies (6) valid for $m = k + 1$ is identical. If $U$ is the harmonic extension of $u$, Lemma 1 applied to (5) yields:

\[ \bar{L}^4L^4(\bar{L}^4L^4)^{k-1}L(\bar{L}LU) = 0 . \]

Conjugating, we get:

\[ (L^4\bar{L}^4)^kL(\bar{L}LU) = 0 . \]

Thus the entries of $LLU$ are pluriharmonic. Thus if we write $U = \sum F_{p,q}$, we have $LLF_{p,q} = 0$ for $p, q \geq 1$, since $LL$ preserves type. Thus $LF_{p,q}$ is analytic for $p, q \geq 1$. Hence $F_{p,q} = 0$ for $p, q \geq 1$. Hence $F_{p,q} = 0$ for $p, q \geq 1$.


**Lemma 4.** If $f \in C^\infty(\bar{B}^\ast)$, then $\bar{\mathcal{L}}_i f = 0$ if and only if $\mathcal{L}_i \bar{\mathcal{L}}_i f = 0$.

**Proof.** If $\bar{L}f = 0$, then clearly $\mathcal{L}_i \bar{\mathcal{L}}_i f = 0$. To prove the converse, we fix all variables except $z_i$ and $\bar{z}_i$ and restrict $f$ to $C_r = \{|z_i|^2 + |z_j|^2 = r^2\}$.

Let $dS_r$ be the normalized surface area, and integrate by parts:

\[ \int_{C_r} \bar{\mathcal{L}}_i f(\bar{\mathcal{L}}_i f)dS_r = -\int_{C_r} f(\bar{\mathcal{L}}_i \bar{\mathcal{L}}_i f) = 0 . \]

Thus $\bar{\mathcal{L}}_i f = 0$ on $C_r$. Since this must hold for all $r$, $\bar{\mathcal{L}}_i f = 0$. 

3* Cauchy-Riemann equations*
Remark. If $\Omega = \{p = 0\}$ is a smooth domain, $\nabla p \neq 0$ on $\partial \Omega$, then we set $\overline{\mathcal{L}}_{ij} = \rho_j(\partial/\partial \overline{z}_j) - \rho_j(\partial/\partial z_i)$. The proof above shows that for $f \in C^\prime(\partial \Omega)$, $\overline{\mathcal{L}}_{ij}f = 0$ on $\partial \Omega$ if and only if $\overline{\mathcal{L}}_{ij} \overline{\mathcal{L}}_{ij}f = 0$ on $\partial \Omega$.

Theorem 2. Let $m \geq 1$ and $u \in C^m(\partial B^n)$ be given. Then $u$ can be extended to an analytic function on $B^n$ if and only if:

(7) $\overline{\mathcal{L}}_{ij}(\mathcal{L}_{ij} \overline{\mathcal{L}}_{ij})^{n-1/2} u(\zeta) = 0$ (m odd)

(8) $\overline{\mathcal{L}}(\mathcal{L}_{ij} \overline{\mathcal{L}}_{ij})^{n-1/2} u(\zeta) = 0$ (m even)

for all $\zeta \in \partial B^n$ and $1 \leq i, j \leq n$.

Proof. In Lemma 4 we have shown that $\text{Range}(\mathcal{L}_{ij}) \cap \text{Null}(\overline{\mathcal{L}}_{ij}) = 0$. Similarly, $\text{Range}(\overline{\mathcal{L}}_{ij}) \cap \text{Null}(\mathcal{L}_{ij}) = 0$. Thus equations (7) and (8) will hold if and only if $\overline{\mathcal{L}}_{ij} u = 0$. Since $\bar{L}u$ is the tangential Cauchy-Riemann system, (7) and (8) will hold if and only if $u$ can be extended to an analytic function.

Remark. The above theorem remains valid for $f \in C^\infty(\partial \Omega)$, as in the remark following Lemma 4.


Definition. Let $\Gamma$ be the set of subsets of $\{1, 2, \cdots, n\}$. For $\gamma \in \Gamma$, we say that $u$ is $\gamma$-regular if $\partial u/\partial z_k = 0$ when $k \in \gamma$ and $\partial u/\partial \overline{z}_k = 0$ when $k \in \gamma$. We define a new operator $T = (\mathcal{L}_{ij} \overline{\mathcal{L}}_{ij})$. For $\gamma \in \Gamma$, we define $T^\gamma$ (resp. $L^\gamma$) to be $T$ (resp. $L$) with the variables $z_k$ and $\overline{z}_k$ interchanged whenever $k \in \gamma$.

The function $z_i$, for instance, is $\gamma$-regular for many $\gamma$, but $z_i \overline{z}_i$ is not $\gamma$-regular for any $\gamma$. Note that every $\gamma$-regular function is $n$-harmonic.

Lemma 5. If $f$ is harmonic on $B^n$, then $T^\gamma f = 0$ if and only if $f$ is $\gamma$-regular.

Proof. We have established in Lemma 4 that $T^\gamma g = 0$ if and only if $g$ is analytic. Consider the real linear map $\gamma : C^* \rightarrow C^n$

$\gamma(x_1, y_1, \cdots, x_n, y_n) = (\zeta_1, \cdots, \zeta_n)$

where

$\zeta_k = x_k + iy_k$ if $k \in \gamma$

$\zeta_k = x_k - iy_k$ if $k \notin \gamma$. 

Any γ-regular function \( f \) can be obtained from some analytic \( g \) by composition:

\[
f = g \circ \gamma.
\]

Hence \( T^\gamma f = Tg = 0 \) if and only if \( f \) is γ-regular.

**Theorem 3.** A function \( u \in C^\infty(\partial B^n) \) can be extended to a function \( U \) which is \( n \)-harmonic in \( B^n \) if and only if:

\[
(\prod_{i \in I} T^i)u = 0.
\]

(Since the \( T^i \)'s do not commute, the product (9) is taken in an arbitrary but fixed order.)

**Proof.** We shall show that the harmonic extension \( U \) of \( u \) is \( n \)-harmonic if and only if (9) holds. The function \( U \) is \( n \)-harmonic if and only if we may write:

\[
U = \sum_{i \in I} w \quad \text{where} \quad w \text{ is \( \gamma \)-regular}.
\]

The "if" is clear since each \( w \) is \( n \)-harmonic. The "only if" follows because we may use the Cauchy integral formula in \( z_i \) to write:

\[
u(z, \bar{z}) = f(z, w) + g(\bar{z}, w) \quad w = (z_\gamma, \bar{z}_\gamma, \cdots, z_n, \bar{z}_n)
\]

where \( f \) and \( g \) are \( n \)-harmonic. If we continue and split each part in a similar fashion we obtain the desired representation.

Now we show that if \( f \) is \( \gamma \)-regular, then so is \( Tf \). We compute:

\[
(10) \quad \mathcal{L}_i \mathcal{Z}_i f = z_i \bar{z}_i f_{i\bar{i}} - \bar{z}_i \bar{z}_i f_{i\bar{i}}
\]

In expression (10), \( f \) will be multiplied by the variable \( \xi \) only if \( f_\xi \neq 0 \). Thus if \( f \) is \( \gamma \)-regular so is \( Tf \).

If we perform the analogous computation for \( T^\gamma \), we can use the same argument to show that if \( f \) is \( \gamma \)-regular then so is \( T^\gamma f \).

Now if \( U \) is \( n \)-harmonic, then \( U = \sum_{\sigma \in \mathcal{S}} u^\sigma \); and

\[
(11) \quad \prod_{i \in I} T^i w^\sigma = \prod_{i_1} T^{i_1} T^\sigma \prod_{i_2} T^{i_2} w^\sigma = 0.
\]

This is because \( \prod T^i w^\sigma \) is \( \sigma \)-regular and will be annihilated by \( T^\sigma \).

To prove the converse we establish the following result:

**Lemma 6.** Let \( v, v_1, \cdots, v_k \) be harmonic. If \( v_j \) is \( \gamma_j \)-regular and

\[
(11) \quad T^j v = v_1 + \cdots + v_k
\]
then we may write \( v = u + u_1 + \cdots + u_k \) where \( u_j \) is \( \gamma_j \)-regular, and \( u \) is \( \gamma \)-regular.

**Proof of lemma.** Let \( u_0 = u_1 + \cdots + u_k \) be the sum of all monomials of \( v \) that are \( \gamma_j \)-regular for some \( j = 1, 2, \ldots, k \). Thus \( u_0 \) is harmonic and so is \( v - u_0 \). We now claim that \( T^r(v - u_0) \) is zero.

By the construction of \( u_0 \), every monomial \( z^a \bar{z}^b \) of \( v - u_0 \) is not \( \gamma_j \)-regular for any \( j = 1, 2, \ldots, k \). From an inspection of (10), one can see that if \( T^r(v - u_0) \) is nonzero, then it will be a sum of monomials, none of which is \( \gamma_j \)-regular for any \( j = 1, 2, \ldots, k \).

On the other hand, from (11) and the construction of \( u_0 \), it is clear that \( T^r(v) - T^r u_0 \) is a sum of \( \gamma_j \)-regular functions. Hence \( T^r(v - u_0) \) must vanish. By Lemma 5, we conclude that \( v - u_0 = u \) is \( \gamma \)-regular, concluding the proof of this lemma.

**Proof of theorem.** We iterate Lemma 6 several times and find that if (8) is valid, then

\[
U = \sum_{j \in I} w^j, \quad \text{as desired}.
\]

**COROLLARY 3.** A function \( u \in C^\infty(\partial B^n) \) can be extended to a function \( U = \sum_{j=1}^k u_j \), where \( u_j \) is \( \gamma_j \)-regular if and only if

\[
\left( \prod_{j=1}^k T^{\gamma_j} \right) u = 0.
\]

**Proof.** This follows easily from Lemma 6.

**Remark.** All of the above results remain valid if the boundary differential operators are interpreted in the weak sense of Corollary 1.

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**References**


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