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**THE DIRICHLET PROBLEM FOR SOME OVERDETERMINED
SYSTEMS ON THE UNIT BALL IN C^n**

ERIC BEDFORD

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A characterization is given of those functions on $\partial B^n = \{|z| = 1\}$ which can be extended to be analytic, pluriharmonic, or n -harmonic in $B^n = \{|z| < 1\}$.

1. **Introduction.** If f is a continuous function on $\partial B^n = \{z = (z_1, \dots, z_n) : |z| = 1\}$, then f can be extended to a harmonic function F in $B^n = \{z : |z| < 1\}$. That is, the Dirichlet problem is uniquely solvable. If we wish F , in addition, to be analytic, pluriharmonic, or n -harmonic, the extension is not always possible, and we must impose some restrictions on the function f . It is well-known that necessary and sufficient conditions for f to have an analytic extension are that f satisfy the tangential Cauchy-Riemann equation. In this paper we show that there are other systems that replace the tangential Cauchy-Riemann equations as consistency conditions. We also give the consistency conditions for a function to extend to be pluriharmonic or n -harmonic.

2. **Pluriharmonic extension.** Some important differential operators tangential to ∂B^n , $n \geq 2$ are:

$$(1) \quad \mathcal{L}_{ij} = \bar{\zeta}_i \frac{\partial}{\partial \zeta_j} - \bar{\zeta}_j \frac{\partial}{\partial \zeta_i}$$

$$(2) \quad \bar{\mathcal{L}}_{ij} = \zeta_i \frac{\partial}{\partial \bar{\zeta}_j} - \zeta_j \frac{\partial}{\partial \bar{\zeta}_i}$$

where we take $1 \leq i, j \leq n$ and $\zeta = (\zeta_1, \dots, \zeta_n) \in \partial B^n$. A simple computation shows that the real and imaginary parts of these operators are tangent to ∂B^n . These operators extend naturally into the interior of B^n . The following lemma shows the interplay between the action of the \mathcal{L}_{ij} on ∂B^n and in B^n .

LEMMA 1. *Let \mathcal{L} be one of the operators (1) or (2), and let $u \in C^1(\partial B^n)$ be given. If $P(x, \zeta)$ is the Poisson kernel on B^n , we have:*

$$(3) \quad (\mathcal{L}_\zeta u) * P(z) = \mathcal{L}_z(u * P(z))$$

for $\zeta \in \partial B^n$, $z \in B^n$.

Proof. The operator \mathcal{L} satisfies the hypotheses of Lemma 2, and thus the right hand side of (3) is harmonic (the left hand side

obviously is). Since (3) is valid for $|z| = 1$, it must hold for all $z \in B^n$.

LEMMA 2. An operator $\mathcal{D} = f(x, y)\partial/\partial y - g(x, y)\partial/\partial x$ preserves harmonic functions if and only if the pair (f, g) satisfies the Cauchy-Riemann equations,

$$\begin{aligned} f_x &= g_y \\ f_y &= -g_x . \end{aligned}$$

Proof. It is a straightforward calculation that $(\mathcal{D}u)_{xx} + (\mathcal{D}u)_{yy} = 0$ for all harmonic u if and only if $f_x = g_y$ and $-f_y = g_x$.

COROLLARY 1. If $f \in L^1(\partial B^n)$, and $\mathcal{L}f = g$ in the weak sense, (i.e., $\int_{|\zeta|=1} f \mathcal{L}\varphi = -\int_{|\zeta|=1} g\varphi$ for all $\varphi \in C^\infty(\partial B^n)$), then

$$g * P(z) = \mathcal{L}_z(f * P(z)) .$$

Proof. Since the Poisson kernel on B^n is $P(\zeta, z) = 1 - |z|^2/|z - \zeta^{2n}|$, one can calculate that:

$$\mathcal{L}_z P(\zeta, z) = -\mathcal{L}_\zeta P(\zeta, z) .$$

Thus if dS is normalized surface area, we have:

$$\begin{aligned} \mathcal{L}_z(f * P(z)) &= \int_{|\zeta|=1} f(\zeta) \mathcal{L}_z P(\zeta, z) dS \\ &= -\int_{|\zeta|=1} f(\zeta) \mathcal{L}_\zeta P(\zeta, z) dS = \int_{|\zeta|=1} g(\zeta) P(\zeta, z) dS \\ &= g * P(z) . \end{aligned}$$

DEFINITION. If α and β are multi-indices, then $z^\alpha \bar{z}^\beta = \prod_{j=1}^n z_j^{\alpha_j} \bar{z}_j^{\beta_j}$ has type (p, q) if $|\alpha| = p$ and $|\beta| = q$. If $h(z, \bar{z})$ is a sum of monomials of type (p, q) , then h is of type (p, q) .

Observe that if h is of type (p, q) , then $\overline{\mathcal{L}_{ij}}h$ is either zero or of type $(p+1, q-1)$. Similarly, $\mathcal{L}_{ij}h$ is either of type $(p-1, q+1)$ or zero.

By L we will denote the matrix of operators $L = (\mathcal{L}_{ij})$.

If $K = (K_{rs})$ and $M = (M_{ij})$ are two matrices of operators, then KM will denote the tensor product of the two matrices:

$$KM(u) = K \otimes M(u) = (K_{rs} M_{ij} u) .$$

LEMMA 3. Let $F \in C^1(\bar{B}^n)$ satisfy $\Delta F = 0$. If $\bar{L}F(z) = 0$ for all $z \in B^n$, then F is analytic.

Proof. The system $\bar{L}F = 0$ is precisely the tangential Cauchy-

Riemann equations (see [1], [2]). Thus if f is the restriction of F to ∂B^n , then f has a holomorphic extension to B^n , which must coincide with F , since F is harmonic.

REMARK. The lemma may also be proved directly without mention of the tangential Cauchy-Riemann equations.

THEOREM 1. *If $u \in C^3(\partial B^n)$, then*

$$(4) \quad \bar{L}\bar{L}L(u) = 0$$

if and only if u extends to a pluriharmonic function U on B^n .

Proof. If u extends to a pluriharmonic U , then we write $U(z, \bar{z}) = f(z) + g(\bar{z})$ where f and g are analytic. An entry of the matrix $\bar{L}\bar{L}LU$ looks like:

$$\begin{aligned} \bar{L}(\bar{\mathcal{L}}_{ij}\bar{\mathcal{L}}_{kl}U) &= \bar{L}\bar{\mathcal{L}}_{ij}(\bar{z}_k f_{z_l} - \bar{z}_l f_{z_k}) \\ &= \bar{L}\left(z_i\left(\frac{\partial \bar{z}_k}{\partial \bar{z}_j}\right)f_{z_l} - z_i\left(\frac{\partial \bar{z}_l}{\partial \bar{z}_j}\right)f_{z_k} \right. \\ &\quad \left. - z_j\left(\frac{\partial \bar{z}_k}{\partial \bar{z}_i}\right)f_{z_l} + z_j\left(\frac{\partial \bar{z}_l}{\partial \bar{z}_i}\right)f_{z_k}\right) \\ &= \bar{L}(\text{analytic}) = 0. \end{aligned}$$

To prove the converse, we show that the harmonic extension U of u is pluriharmonic. Since U is harmonic, we may write, as before:

$$U(z, \bar{z}) = \sum_{p,q \geq 0} F_{p,q}.$$

By Lemma 1, we have:

$$\bar{L}\bar{L}L(\sum F_{p,q}) = \sum_{p,q \geq 0} \bar{L}\bar{L}LF_{p,q} = 0.$$

Recall that $\bar{L}\bar{L}L$ takes a polynomial of type (p, q) into one of type $(p+1, q-1)$ or zero. Thus $\bar{L}\bar{L}LF_{p,q} = 0$ for each $p, q \geq 0$.

By Lemma 3, the entries of the matrix $\bar{L}\bar{L}LF_{p,q}$ are analytic. But on the other hand, they must be of type (p, q) or zero. Thus if $q \geq 1$, we conclude that $\bar{L}\bar{L}LF_{p,q} = 0$.

Again by Lemma 3, the entries of $LF_{p,q}$ are analytic if $q \geq 1$. But since they will be type $(p-1, q+1)$ or zero, we conclude that $LF_{p,q} = 0$ for $q \geq 1$. This means that $\bar{F}_{p,q} = 0$ is analytic if $q \geq 1$. Thus if $p, q \geq 1$, then $F_{p,q} = 0$.

Thus we may write

$$U(z, \bar{z}) = \sum_{j \geq 1} (F_{j,0} + F_{0,j}) + F_{0,0}.$$

Hence U is pluriharmonic.

REMARK. It was observed by L. Nirenberg that there is no second order operator \mathcal{D} which gives the consistency conditions for pluriharmonic functions ∂B^n .

COROLLARY 2. *Let $m \geq 2$ and $u \in C^\infty(\partial B^n)$ be given. Then u can be extended to U pluriharmonic in B^n if and only if (5) or (6) holds:*

$$(5) \quad \bar{L}^2(L^2\bar{L}^2)^m Lu = 0$$

$$(6) \quad (L^2\bar{L}^2)^m Lu = 0.$$

Proof. If u can be extended, then the above equations are clearly valid.

We prove the other implication by induction. Line (5) holds for $m = 0$ (Theorem 1). We assume that (6) is valid for $m = k$ and show that (5) also holds for $m = k$. The other part, showing that (5) is valid for $m = k$ implies (6) valid for $m = k + 1$ is identical. If U is the harmonic extension of u , Lemma 1 applied to (5) yields:

$$\bar{L}^2 L^2 (\bar{L}^2 L^2)^{k-1} \bar{L}(\bar{L} L U) = 0.$$

Conjugating, we get:

$$(L^2 \bar{L}^2)^k L(LL\bar{U}) = 0.$$

Thus the entries of $L\bar{L}\bar{U}$ are pluriharmonic. Thus if we write $U = \sum F_{p,q}$, we have $\bar{L}L F_{p,q} = 0$ for $p, q \geq 1$, since $\bar{L}L$ preserves type. Thus $L F_{p,q}$ is analytic for $p, q \geq 1$. Hence $F_{p,q} = 0$ for $p, q \geq 1$. Hence $F_{p,q} = 0$ for $p, q \geq 1$.

3. Cauchy-Riemann equations.

LEMMA 4. *If $f \in C^2(\bar{B}^n)$, then $\bar{\mathcal{L}}_{ij} f = 0$ if and only if*

$$\mathcal{L}_{ij} \bar{\mathcal{L}}_{ij} f = 0.$$

Proof. If $\bar{\mathcal{L}} f = 0$, then clearly $\mathcal{L}_{ij} \bar{\mathcal{L}}_{ij} f = 0$. To prove the converse, we fix all variables except z_i and z_j and restrict f to

$$C_r = \{|z_i|^2 + |z_j|^2 = r^2\}.$$

Let dS_r be the normalized surface area, and integrate by parts:

$$\int_{C_r} \bar{\mathcal{L}}_{ij} f (\overline{\mathcal{L}_{ij} f}) dS_r = - \int_{C_r} f (\overline{\mathcal{L}_{ij} \bar{\mathcal{L}}_{ij} f}) = 0.$$

Thus $\bar{\mathcal{L}}_{ij} f = 0$ on C_r . Since this must hold for all r , $\bar{\mathcal{L}}_{ij} f = 0$.

REMARK. If $\Omega = \{\rho = 0\}$ is a smooth domain, $\text{grad } \rho \neq 0$ on $\partial\Omega$, then we set $\overline{\mathcal{L}}_{ij} = \rho_{z_i}(\partial/\partial\bar{z}_j) - \rho_{z_j}(\partial/\partial\bar{z}_i)$. The proof above shows that for $f \in C^2(\partial\Omega)$, $\overline{\mathcal{L}}_{ij}f = 0$ on $\partial\Omega$ if and only if $\mathcal{L}_{ij}\overline{\mathcal{L}}_{ij}f = 0$ on $\partial\Omega$.

THEOREM 2. Let $m \geq 1$ and $u \in C^m(\partial B^n)$ be given. Then u can be extended to an analytic function on B^n if and only if:

$$(7) \quad \overline{\mathcal{L}}_{ij}(\mathcal{L}_{ij}\overline{\mathcal{L}}_{ij})^{(m-1)/2}u(\zeta) = 0 \quad (m \text{ odd})$$

$$(8) \quad \mathcal{L}(\mathcal{L}_{ij}\overline{\mathcal{L}}_{ij})^{m/2}u(\zeta) = 0 \quad (m \text{ even})$$

for all $\zeta \in \partial B^n$ and $1 \leq i, j \leq n$.

Proof. In Lemma 4 we have shown that $\text{Range}(\mathcal{L}_{ij}) \cap \text{Null}(\overline{\mathcal{L}}_{ij}) = 0$. Similarly, $\text{Range}(\overline{\mathcal{L}}_{ij}) \cap \text{Null}(\mathcal{L}_{ij}) = 0$. Thus equations (7) and (8) will hold if and only if $\overline{\mathcal{L}}_{ij}u = 0$. Since $\overline{L}u$ is the tangential Cauchy-Riemann system, (7) and (8) will hold if and only if u can be extended to an analytic function.

REMARK. The above theorem remains valid for $f \in C^\infty(\partial\Omega)$, as in the remark following Lemma 4.

4. N -Harmonic functions.

DEFINITION. Let Γ be the set of subsets of $\{1, 2, \dots, n\}$. For $\gamma \in \Gamma$, we say that u is γ -regular if $\partial u/\partial\bar{z}_k = 0$ when $k \in \gamma$ and $\partial u/\partial z_k = 0$ when $k \notin \gamma$. We define a new operator $T = (\mathcal{L}_{ij}\overline{\mathcal{L}}_{ij})$. For $\gamma \in \Gamma$, we define T^γ (resp. L^γ) to be T (resp. L) with the variables z_k and \bar{z}_k interchanged whenever $k \notin \gamma$.

The function z_1 , for instance, is γ -regular for many γ , but $z_1\bar{z}_1$ is not γ -regular for any γ . Note that every γ -regular function is n -harmonic.

LEMMA 5. If f is harmonic on B^n , then $T^\gamma f = 0$ if and only if f is γ -regular.

Proof. We have established in Lemma 4 that $Tg = 0$ if and only if g is analytic. Consider the real linear map $\gamma: C^n \rightarrow C^n$

$$\gamma(x_1, y_1, \dots, x_n, y_n) = (\zeta_1, \dots, \zeta_n)$$

where

$$\begin{aligned} \zeta_k &= x_k + iy_k & \text{if } k \in \gamma \\ \zeta_k &= x_k - iy_k & \text{if } k \notin \gamma. \end{aligned}$$

Any γ -regular function f can be obtained from some analytic g by composition:

$$f = g \circ \gamma .$$

Hence $T^r f = Tg = 0$ if and only if f is γ -regular.

THEOREM 3. *A function $u \in C^\infty(\partial B^n)$ can be extended to a function U which is n -harmonic in B^n if and only if:*

$$(9) \quad \left(\prod_{\gamma \in \Gamma} T^\gamma \right) u = 0 .$$

(Since the T^r 's do not commute, the product (9) is taken in an arbitrary but fixed order.)

Proof. We shall show that the harmonic extension U of u is n -harmonic if and only if (9) holds. The function U is n -harmonic if and only if we may write:

$$U = \sum_{\gamma \in \Gamma} w^\gamma \text{ where } w^\gamma \text{ is } \gamma\text{-regular .}$$

The "if" is clear since each w^γ is n -harmonic. The "only if" follows because we may use the Cauchy integral formula in z_1 to write:

$$u(z, \bar{z}) = f(z_1, w) + g(\bar{z}_1, w) \quad w = (z_2, \bar{z}_2, \dots, z_n, \bar{z}_n)$$

where f and g are n -harmonic. If we continue and split each part in a similar fashion we obtain the desired representation.

Now we show that if f is γ -regular, then so is Tf . We compute:

$$(10) \quad \begin{aligned} \mathcal{L}_{ij} \overline{\mathcal{L}_{ij}} f &= z_i \bar{z}_i f_{z_j \bar{z}_j} - z_i \bar{z}_j f_{z_i \bar{z}_j} \\ &- z_j \bar{z}_i f_{z_j \bar{z}_i} + z_j \bar{z}_j f_{z_i \bar{z}_i} - \bar{z}_j f_{\bar{z}_j} - \bar{z}_i f_{\bar{z}_i} . \end{aligned}$$

In expression (10), f will be multiplied by the variable ξ only if $f_\xi \neq 0$. Thus if f is γ -regular so is Tf .

If we perform the analogous computation for T^σ , we can use the same argument to show that if f is γ -regular then so is $T^\sigma f$.

Now if U is n -harmonic, then $U = \sum_{\sigma \in \Gamma} u^\sigma$; and

$$\begin{aligned} \prod_{\gamma \in \Gamma} T^\gamma u^\sigma &= \prod_{\Gamma_1} T^\gamma T^\sigma \prod_{\Gamma_2} T^\gamma u^\sigma \\ &= 0 . \end{aligned}$$

This is because $\prod T^\gamma u^\sigma$ is σ -regular and will be annihilated by T^σ .

To prove the converse we establish the following result:

LEMMA 6. *Let v, v_1, \dots, v_k be harmonic. If v_j is γ_j -regular and*

$$(11) \quad T^r v = v_1 + \dots + v_k ,$$

then we may write $v = u + u_1 + \cdots + u_k$ where u_j is γ_j -regular, and u is γ -regular.

Proof of lemma. Let $u_0 = u_1 + \cdots + u_k$ be the sum of all monomials of v that are γ_j -regular for some $j = 1, 2, \dots, k$. Thus u_0 is harmonic and so is $v - u_0$. We now claim that $T^r(v - u_0)$ is zero.

By the construction of u_0 , every monomial $z^\alpha \bar{z}^\beta$ of $v - u_0$ is not γ_j -regular for any $j = 1, 2, \dots, k$. From an inspection of (10), one can see that if $T^r(v - u_0)$ is nonzero, then it will be a sum of monomials, none of which is γ_j -regular for any $j = 1, 2, \dots, k$.

On the other hand, from (11) and the construction of u_0 , it is clear that $T^r(v) - T^r u_0$ is a sum of γ_j -regular functions. Hence $T^r(v - u_0)$ must vanish. By Lemma 5, we conclude that $v - u_0 = u$ is γ -regular, concluding the proof of this lemma.

Proof of theorem. We iterate Lemma 6 several times and find that if (8) is valid, then

$$U = \sum_{r \in \Gamma} u^r, \quad \text{as desired.}$$

COROLLARY 3. *A function $u \in C^\infty(\partial B^n)$ can be extended to a function $U = \sum_{j=1}^k u_j$, where u_j is γ_j -regular if and only if*

$$\left(\prod_{j=1}^k T^{r_j} \right) u = 0.$$

Proof. This follows easily from Lemma 6.

REMARK. All of the above results remain valid if the boundary differential operators are interpreted in the weak sense of Corollary 1.

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