SETS GENERATED BY RECTANGLES

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For any family $F$ of sets, let $\mathcal{B}(F)$ denote the smallest $\sigma$-algebra containing $F$. Throughout this paper, $X$ denotes a set and $\mathcal{R}$ the family of sets of the form $A \times B$, for $A \subseteq X$ and $B \subseteq X$. It is of interest to find conditions under which the following holds:

(1) Each subset of $X \times X$ is a member of $\mathcal{B}(\mathcal{R})$

The interesting case is when

$$\omega_1 < \text{Card } X \leq c,$$

since results for other cases are known.

It is shown in Theorem 9 that (1) is equivalent to

There is a countable ordinal $\alpha$ such that

(2) each subset of $X \times X$ can be generated from $\mathcal{R}$ is $\alpha$ Baire process steps.

It is also shown that the two-dimensional statements (1) and (2) are equivalent to the one-dimensional statement

There is a countable ordinal $\alpha$ such that

(3) Card $H = \text{Card } X$, there is a countable family $G$ such that each member of $H$ can be generated from $G$ in $\alpha$ steps.

It is shown in Theorem 5 that the continuum hypothesis (CH) is equivalent to certain statements about rectangles of the form (1) and (2) with $\alpha = 2$.

Rao [7, 8] and Kunen [2] have shown that

**Theorem 1.** If Card $X \leq \omega_1$ (the first uncountable cardinal) then (1) is true and if Card $X > c$ then (1) is false.

The question of whether (1) is true (without the requirement Card $X \leq \omega_1$) was raised by Johnson [1] and earlier by Erdős, Ulam, and others (see [8], p. 197). The arguments in Kunen’s thesis actually showed that if Card $X \leq \omega_1$ then

(4) Each subset of $X \times X$ can be generated from $\mathcal{R}$ in 2 steps (i.e., each subset is a member of $\mathcal{R}_{\omega_1}$. See definitions in § 2.)

In Theorem 5 we generalize Theorem 1 and Kunen’s result (4),
and give a new characterization of CH by showing it to be equivalent to certain statements about rectangles of the form (1) and (4).

If CH is assumed the $\alpha$ appearing in statements (2) and (3) above is 2 (see Theorem 10). This raises the intriguing (but unanswered) question of whether $\alpha$ must always be 2 if (1) holds and CH is false.

It would also be interesting to know whether statements (1), (2), and (3) are equivalent to statement (5) below. Clearly (3) imples (5).

If $H$ is a family of subsets of $X$ with

\[ \text{Card } H = \text{Card } X, \]

then there is a countable family $G$ for which $H \subseteq \mathcal{B}(G)$.

The equivalence of (1) and (2) means for example, (assuming CH), that there is a countable family $G$ from which all real Borel sets (or analytic sets, or projective sets) can be generated in two steps (i.e., Borel sets $\subseteq G_\alpha$). This is remarkable in view of the well known result [4, 8] that if $G$ is a countable basis for the real topology, then the Borel sets cannot be generated from $G$ in less than $\omega_1$ steps.

As a generalization of this well known result we show in Theorem 12 that any countable family $G$ which is closed to complementation and which generates the Borel sets (i.e., Borel sets $\subseteq \mathcal{B}(G)$) must have order $\omega_1$. That is

\[ \mathcal{B}(G) \subseteq G_\alpha \]

for any countable ordinal $\alpha$. Thus, even though $G$ might generate the Borel sets in $\alpha$ steps (or 2 steps if CH is assumed), the process, nevertheless, continues to produce new members of $\mathcal{B}(G)$ until we reach $G_{\omega_1}$.

We would like to point out in conjunction with our characterization of CH that Kunen [2] has proved that if Martin's Axiom A holds (see [6]) and Card $X \leq c$ then (4) holds. He also proved that if $\omega_1 < \text{Card } X \leq c$ then (1) is independent of ZFC (Zermo-Frankel Axioms + the Axiom of Choice) together with the negation of CH.

2. Notation and definitions. If $G$ is any family of sets, let $G_0$ be the family $G$, and for each ordinal $\alpha$, $\alpha > 0$, let $G_\alpha$ be the family of all countable unions (intersections) of sets in $\bigcup_{\gamma < \alpha} G_{\gamma}$, if $\alpha$ is odd (even). Limit ordinals will be considered even. (Compare Kuratowski [3].) Thus we have

\[ G_0 = G, G_1 = G_\alpha, G_2 = G_{\alpha}, G_3 = G_{\alpha_2}, \ldots, G_{\alpha}, \ldots. \]

Also $G_{\alpha} \subseteq G_{\alpha+1}$ for each ordinal $\alpha$ and $G_{\alpha_1} = G_{\alpha_1+1}$, where $\omega_1$ is the first uncountable ordinal. If $\alpha > 0$, then the family $G_\alpha$ is closed under countable unions (intersections) if $\alpha$ is odd (even).
We define the order of $G$ to be the first ordinal $\alpha$, $\alpha > 0$, such that $G_{\alpha+1} = G_\alpha$.

For each $A \subseteq X$ (or $A \subseteq X \times X$), let $A'$ be the complement of $A$ with respect to $X$ (or $X \times X$), and for each family $G$ of subsets of $X$ (or $X \times X$) let $\mathcal{C}(G)$ be the family of complements of $G$. Note that if $\mathcal{C}(G) \subseteq G$, or even if $\mathcal{C}(G) \subseteq G_\omega$, then the family $G_\omega$ is the family $\mathcal{B}(G)$, the $\sigma$-algebra generated by $G$. Thus, since $$ (A \times B)' = A \times B' \cup A' \times X \in \mathcal{B} \,$$

it follows that $\mathcal{B}_\omega = \mathcal{B}(\mathcal{R})$.

If $G$ is a family of subsets of $X$, let $VG = \{A \times B : A \subseteq X, B \in G\}$, and let $HG = \{A \times B : A \in G, B \subseteq X\}$.

If $Z \subseteq X \times X$ and $x \in X$, let $Z_x$ denote the vertical section of $Z$ at $x$, $Z_x = \{y : (x, y) \in Z\}$.

3. Results. The following lemma is easily proved by transfinite induction.

**Lemma 2.** If $1 \leq \alpha < \omega_1$ and $A \in G_\alpha$, then there is a set $B$ in $G_\alpha$ such that $A \subseteq B$.

**Theorem 3.** If $G$ is a countable family of subsets of $X$, $Z \subseteq X \times X$, and $0 < \alpha < \omega_1$, then $Z \in (VG)_\alpha$ if and only if $Z_x \in G_\alpha$ for each $x \in$ domain $Z$.

**Proof.** By considering the natural projections of the sets involved on the second coordinate axis, it is easily seen that

if $Z \in (VG)_\alpha$, then $Z_x \in G_\alpha$ for each $x \in$ domain $Z$.

Now suppose that $Z_x \in G_\alpha$, for each $x \in$ domain $Z$, and let $G = \{\theta_1, \theta_2, \theta_3, \ldots\}$. We complete the proof by transfinite induction on $\alpha$.

**Case 1.** $\alpha = 1$.

For each $n$, let $A_n = \{x \in$ domain $Z : \theta_n \subseteq Z_x\}$, and let $Z_n = A_n \times \theta_n$. Then $Z_n \in VG$, for each $n$, and

$$Z = \bigcup_{n=1}^{\infty} Z_n \in (VG)_1 \,.$$

Now suppose $1 < \alpha < \omega_1$, and that the theorem holds for every $\gamma$, $0 < \gamma < \alpha$.

**Case 2.** $\alpha$ is even.
Let \( \{\gamma_n\}_{n=1}^\infty \) be a sequence of odd ordinals less than \( \alpha \) such that each odd ordinal less than \( \alpha \) appears infinitely often in \( \{\gamma_n\}_{n=1}^\infty \). For each \( x \in \text{domain } Z \), let

\[
D_1(x), D_2(x), D_3(x), \ldots
\]

be a sequence such that \( D_i(x) \in G_{r_i} \) for each \( i \), and

\[
Z_x = \bigcap_{i=1}^\infty D_i(x).
\]

This can be done in view of Lemma 2. For each \( i \), let

\[
Z^i = \bigcup_{x \in \text{domain } Z} \{x\} \times D_i(x).
\]

First note that \( Z = \bigcap_{i=1}^\infty Z^i \). Also each nonempty section \((Z^i)_x\) of \( Z^i \) is equal to \( D_i(x) \in G_{r_i} \). Hence, by the induction hypothesis, \( Z^i \in (VG)_{r_i} \), for each \( i \), and therefore

\[
Z = \bigcap_{i=1}^\infty Z^i \in (VG)_\alpha,
\]

by the definition of the family \((VG)_\alpha\).

**Case 3.** \( \alpha \) is odd and greater than 1.

For each \( x \in \text{domain } Z \), let \( \{D_i(x)\}_{i=1}^\infty \) be a sequence of members of \( G_{\alpha-1} \) for which \( Z_x = \bigcup_{i=1}^\infty D_i(x) \), and let \( Z^i = \bigcup_{x \in \text{domain } Z} \{x\} \times D_i(x) \), for each \( i \).

Again it follows that \( Z^i \in G_{\alpha-1} \), for each \( i \), and

\[
Z = \bigcup_{i=1}^\infty Z^i \in (VG)_\alpha.
\]

**Corollary 4.** If \( Z \subseteq X \times X \) is the graph of a function then \( Z \in R_2 \subseteq B(\mathcal{R}) \).

**Proof.** Let \( G \) be a countable basis for the real topology and note that, for each \( x \in X \), \( Z_x \) is a singleton and hence \( Z_x \in G_x \). Thus by Theorem 3, \( Z \in (VG)_2 \subseteq R_2 \subseteq B(\mathcal{R}) \). Also see [7].

**Theorem 5.** Let \( X \) be the real numbers and let \( G \) be a countable base for the usual topology on \( X \). The following three statements are equivalent:

1. CH holds
2. if \( Z \subseteq X \times X \), then \( Z = A \cap B \), where \( A \in (VG)_2 \) and \( B \in (HG)_2 \)
3. if \( E \subseteq X \times X \), then \( E = C \cup D \), where \( C \in B(VG) \) and \( D \in B(HG) \).
Proof. First, assume CH and suppose \( Z \subseteq X \times X \). As is well known [7], the complement of \( Z \) is the union of two sets \( H \) and \( K \) such that each vertical section of \( H \) is countable and each horizontal section of \( K \) is countable.

Let \( A \) be the complement of \( H \) and let \( B \) be the complement of \( K \). Then each vertical section of \( A \) is a \( G_2 \) set and by Theorem 3, \( A \in (V_G)_2 \). Similarly, \( B \in (H_G)_2 \). Of course, \( Z = A \cap B \).

Since \( A \in (V_G)_2 \subseteq \mathcal{R}_2 \) and \( B \in (H_G)_2 \subseteq \mathcal{R}_2 \) and \( \mathcal{R}_2 \) is closed under finite intersections, \( Z \in \mathcal{R}_2 \). Thus, if CH holds, then the order of \( \mathcal{R} \) is \( \leq 2 \). Since the graph of the identity function, \( f(x) = x \), is not in \( \mathcal{R}_2 \), it follows that the order of \( \mathcal{R} \) is 2.

Now, suppose statement 2 holds and \( E \subseteq X \times X \). Then, the complement of \( E \) can be expressed as the intersection of sets \( A \) and \( B \) with \( A \in (V_G)_2 \) and \( B \in (H_G)_2 \). It follows that \( A' \in (V_G)_2 \subseteq \mathcal{B}(V_G) \) and \( B' \in (H_G)_2 \subseteq \mathcal{B}(H_G) \). Thus, \( E \) is the union of two sets \( C \) and \( D \), where \( C \in \mathcal{B}(V_G) \) and \( D \in \mathcal{B}(H_G) \).

Finally, assume statement 3 holds. Let \( T \) be a totally imperfect subset of \( X \) of cardinality \( c \). The existence of such a set can be proven without assuming CH [3, p. 514]. Let \( E = T \times T \) and let \( E = C \cup D \), with \( C \in \mathcal{B}(V_G) \) and \( D \in \mathcal{B}(H_G) \). Then each vertical section of \( C \) is a subset of \( T \) which is a Borel set. Since an uncountable Borel set contains a perfect set and \( T \) contains no perfect set, we have that each vertical section of \( C \) is countable. Similarly, each horizontal section of \( D \) is countable. But, as is well known [10] this implies CH.

This completes the proof of Theorem 5.

The following two lemmas are well known.

**Lemma 6.** If \( F \) is a family of sets, \( \alpha \) is a countable ordinal, and \( A \in F_\alpha \), then there is a countable subfamily \( J \) of \( F \) for which \( A \in J_\alpha \).

**Lemma 7.** If \( F \) is a family of sets, \( \mathcal{C}(F) \subseteq F \), and \( A \in \mathcal{B}(F) \) then there is a countable subfamily \( J \) of \( F \) and a countable ordinal \( \alpha \) for which \( A \in J_\alpha \).

**Theorem 8.** (a) The following two statements are equivalent:

(i) For each subset \( Z \) of \( X \times X \) there is a countable ordinal \( \alpha \) such that \( Z \in \mathcal{R}_\alpha \).

(ii) If \( H \) is a family of subsets of \( X \) and \( \text{Card } H = \text{Card } X \), then there is a countable family \( G \) of subsets of \( X \) and a countable ordinal \( \alpha \) for which \( H \subseteq G_\alpha \).

(b) If \( \alpha \) is a countable ordinal, the following two statements are equivalent:

(i) Each subset of \( X \times X \) is a member of \( \mathcal{R}_\alpha \).
(ii) If $H$ is a family of subsets of $X$ and $\text{Card} \ H = \text{Card} \ X$ then there is a countable family $G$ of subsets of $X$ for which $H \subseteq G\alpha$.

Proof. The proof of part (b) is similar to that of part (a) which is given below.

First suppose (i) holds, and suppose that $H$ satisfies the hypotheses of (ii). Define the subset $Z \subseteq X \times X$ by letting each member of $H$ be a vertical section of $Z$. More precisely, let $f$ be a 1-1 function from $X$ to $H$ and let

$$Z = \bigcup_{x \in X} \{x\} \times f(x).$$

By (i) there is a countable ordinal $\alpha$ such that $Z \in \mathcal{R}_\alpha$ and hence by Lemma 6, there is a countable subfamily $J$ of $\mathcal{R}$ for which $Z \in J\alpha$. Let

$$G = \{B: A \times B \in J\},$$

note that $Z \in (VG)_\alpha$, and use Theorem 3 to conclude that $H \subseteq G\alpha$.

Now suppose (ii) holds, and that $Z \subseteq X \times X$. Let $H$ be the family of vertical sections of $Z$, and use (ii) to secure a countable family $G$ and a countable ordinal $\alpha$ for which $H \subseteq G\alpha$. Thus $Z \in G\alpha$ for each $x \in \text{domain} \ Z$ and by Theorem 3

$$Z \in (VG)\alpha \subseteq \mathcal{R}_\alpha.$$

Theorem 9. The following four statements are equivalent:

(i) Each subset of $X \times X$ is a member of $\mathcal{R}(\mathcal{R})$.

(ii) If $H$ is a family of subsets of $X$ and $\text{Card} \ H = \text{Card} \ X$ then there is a countable family $G$ and a countable ordinal $\alpha$ for which $H \subseteq G\alpha$.

(iii) There is a countable ordinal $\alpha$ such that, for each family $H$ of subsets of $X$ with $\text{Card} \ H = \text{Card} \ X$, there is a countable family $G$ for which $H \subseteq G\alpha$.

(iv) There is a countable ordinal $\alpha \geq 2$ such that each subset of $X \times X$ is a member of $\mathcal{R}_\alpha$.

Proof. Statements (i) and (ii) are equivalent by Lemma 7 and Theorem 8a. Clearly (iii) implies (ii) and (iv) implies (i). Also by Theorem 8b it follows that (iii) implies (iv). $\alpha$ cannot be equal to 1 in (iv) because by (i) the identity function $f(x) = x$ is not in $\mathcal{R}_1$.

We complete the proof by showing that (ii) implies (iii). Since this result is immediate if $X$ is countable we will assume that $\text{Card} \ X \geq \omega_1$.

Suppose that (ii) holds and that (iii) does not. Then for each $\alpha < \omega_1$, there is a family $H(\alpha)$ of subsets of $X$ for which $\text{Card} \ H(\alpha) = \omega_1$.
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Card $X$ and

(1) for each countable $G$, $H(\alpha) \subsetneq G_{\alpha}$. 

Let $H' = \bigcup_{\alpha<\omega_1} H(\alpha)$. Thus $\text{Card } H' = \text{Card } X$ and hence by (ii) there is a countable family $G'$ and a countable ordinal $\alpha'$ for which $H' \subseteq G'_{\alpha'}$. But then $H(\alpha') \subseteq H' \subseteq G'_{\alpha'}$, in contradiction of (1).

Therefore (ii) implies (iii).

In part (ii) above the family $G$ can be chosen so that $G_{\omega_1}$ is closed to complementation (i.e., is a $\sigma$-algebra).

In view of condition (ii) of Theorem 9, it is interesting to note that R. Mansfield has shown that if $G$ is a countable family of Lebesgue measurable sets, then $B(G)$ does not contain all analytic sets [5].

As was mentioned in the introduction it would be interesting to know whether the formula "$H \subseteq G_\alpha$" in Theorem 9 could be replaced by $H \subseteq \mathcal{B}(G)$. We do not know the answer to this question.

**Theorem 10.** If CH holds, Card $X = c$, $H$ is a family of subsets of $X$, and Card $H = c$, then there is a countable family $G$ for which $H \subseteq G_\omega$.

**Proof.** By Theorem 5 each subset $Z$ of $X \times X$ is a member of $\mathcal{R}_2$. The desired conclusion now follows from Theorem 8b.

4. Generating Borel sets. Let $R$ be the set of reals, and let $H$ be the family of all Borel subsets of $R$. This family has cardinality $c$. Suppose $G$ is a countable family of subsets of $R$ such that $H \subseteq G_{\omega_1}$ and $G_{\omega_1}$ is closed to complementation. The next two theorems show that, even if the family $G$ generates all the Borel sets at an early stage, the order of $G$ is $\omega_1$. This is a generalization of the well known result [4, 9] that if $G$ is a countable basis for the real topology then $G$ has order $\omega_1$. Our proof which is a usual "diagonal" type argument, parallels somewhat Lebesgue’s proof of that result [3, p. 368].

Let $G = \{V_1, V_2, V_3, \ldots\}$, let $N$ be the set of irrational numbers between 0 and 1 and let $K$ be the family $\{\theta_1, \theta_2, \theta_3, \ldots\}$ of all intersections of the members of $G$ with $N$.

$$\theta_i = V_i \cap N.$$  

It will be shown that the order of $K$ is $\omega_1$. It then follows that the order of $G$ is $\omega_1$.

For each $z \in N$, let $(z_1, z_2, z_3, \ldots)$ be the sequence of integers appearing in the continued fraction expansion of $z$. This defines a
reversible transformation from $N$ onto the set of all sequences of positive integers. Let
\[ z^1 = (z_1, z_3, z_5, \ldots) \quad \text{(odd indices)} \]
\[ z^2 = (z_2, z_6, z_{10}, \ldots) \]
\[ z^3 = (z_4, z_{12}, z_{20}, \ldots) \]
\[ \vdots \]
\[ z^n = (z_{2n-1}, z_{3n-1}, z_{5n-1}, \ldots) \quad \text{(n)} \]
\[ \vdots \]

This defines a homeomorphism between $N$ and $N^\omega$ (see Kuratowski [3], p. 369). Also note that if $f$ is a continuous function from $N$ into $N$, then the functions $f_n$ from $N$ into the space of positive integers are continuous, where

\[ f(z) = (f_1(z), f_2(z), f_3(z), \ldots), \quad \text{or} \quad (f_n(z) = f(z)_n). \]

Recall that $K = \{\theta, \theta_2, \theta_3, \ldots\}$. The family $K_\alpha$ which appears in Theorem 11 is defined in §2.

**Theorem 11.** For each countable ordinal $\alpha, \alpha > 0$, there is a function $U_\alpha$ from $N$ onto $K_\alpha$ such that if $f$ is a continuous function from $N$ into $N$, then the set

\[ A_f = \{z : z \in U_\alpha(f(z))\} \]

is in $G(\alpha)$.

**Proof.** Let $U_\alpha(z) = \bigcup_{n=1}^{\infty} \theta_{f_\alpha(z)}$, for each $z \in N$. Clearly $U_\alpha$ maps $N$ onto $K_\alpha$.

Let $f$ be a continuous function from $N$ onto $N$. We have

\[ A_f = \{z : z \in U_\alpha(f(z))\} = \bigcup_{n=1}^{\infty} \theta_{f_\alpha(z)} \]

For each $n$,

\[ \{z : z \in \theta_{f_\alpha(z)}\} = \bigcup_{i=1}^{\infty} \{J_{n_i} \cap \theta_i\} \]

where $J_{n_i} = \{z : f_\alpha(z) = i\}$. Since each $f_\alpha$ is continuous it follows that each $J_{n_i}$ is open and therefore the set $A_f$ belongs to $G_{\omega_1}$.

Suppose $1 < \alpha < \omega_1$ and suppose that the function $U_\gamma$ has been defined for each ordinal $\gamma$ with $1 \leq \gamma < \alpha$. (Induction hypothesis.)

If $\alpha$ is odd, let
$U_a(z) = \bigcup_{n=1}^{\infty} U_{a-1}(z^n)$, for $z \in N$.

Clearly $U_a$ maps $N$ onto $K_a$.

If $\alpha$ is even, let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of odd ordinals less than $\alpha$ such that each odd ordinal less than $\alpha$ appears infinitely often in $\{\gamma_n\}_{n=1}^{\infty}$, and let

$U_a(z) = \bigcap_{n=1}^{\infty} U_{\gamma_n}(z^n)$, for $z \in N$.

If $A \in K_a$ ($\alpha$ is still even), then

$A = \bigcap_{n=1}^{\infty} D_n$,

where $D_n \in K_{\gamma_n}$, for each $n$. For each $n$, let $y_n$ be a point of $N$ such that

$D_n = U_{\gamma_n}(y_n)$.

And let $z$ be the point mapped by the transformation described by (*) to the point $(y_1, y_2, y_3, \ldots)$ of $N^{\omega_1}$. Thus

$U_a(z) = A$

and $U_\alpha$ maps $N$ onto $K_\alpha$.

This completes the definition of the functions $U_a$. Now let $f$ be a continuous function from $N$ into $N$. It will be shown that if $\alpha$ is even the set

$A_f = \{z: z \in U_a(f(z))\}$

is in $G_{\omega_1}$. The argument for the case $\alpha$ is odd is similar.

We have

$A_f = \left\{z: z \in \bigcap_{n=1}^{\infty} U_{\gamma_n}((f(z))^n)\right\}$

$= \bigcap_{n=1}^{\infty} \left\{z: z \in U_{\gamma_n}((f(z))^n)\right\}$.

But, for each $n$, the function $z \mapsto (f(z))^n$, being the composition of two continuous functions, is a continuous function from $N$ to $N$.

Thus by the induction hypothesis, the sets $\{z: z \in U_{\gamma_n}((f(z))^n)\}$ are in the family $G_{\omega_1}$. Therefore $A_f \in G_{\omega_1}$.

**Theorem 12.** If $G$ is a countable family of subsets of real numbers with $\mathcal{G}(G) \subseteq G$, and each Borel set is a member of $\mathcal{B}(G)$ then $G$ has order $\omega_1$. 
Proof. Let $\alpha$ be any countable ordinal, and let

$$I_\alpha = \{z : z \not\in U_\alpha(z)\}.$$ 

Suppose $I_\alpha \in K_\alpha$, and let $U_\alpha(z) = I_\alpha$. If $z \in I_\alpha$ then $z \in U_\alpha(z)$. But this contradicts the definition of $I_\alpha$. If $z \not\in I_\alpha$, then $z \not\in U_\alpha(z) = I_\alpha$, $z \in I_\alpha$. This contradiction shows that $I_\alpha \not\in K_\alpha$.

Since $G(G) = G_{\omega_1}$ (because $\mathcal{E}(G) \subseteq G$), and $I_\alpha = \{z : z \in U_\alpha(z)\} \in G_{\omega_1}$, by Theorem 11, it follows that $I_\alpha \in G_{\omega_1} - G_\alpha$. Thus $G_\alpha \neq G_{\omega_1}$, and hence $G$ has order $\omega_1$ [3, p. 371].

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