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Let G be a locally compact, noncompact, unimodular group. For $x \in G$, we denote by L_x, the left translation operator defined on $L^2(G)$ by $L_x f(y) = f(x^{-1}y)$. We let $\mathscr{L}_2(G)$ be the closure, in the weak operator topology, of the algebra generated by the operators $\{L_x : x \in G\}$. For $f \in L^p(G)$, $1 \leqq p \leqq 2$, we let L_f be the closed operator in $L^2(G)$, defined by $L_f g = f * g$, for $g \in L^1(G) \cap L^2(G)$. We prove, under a natural hypothesis on G, that for every $1 < p < 2$, there exists a projection $P \in \mathscr{L}_2(G)$, $P \neq 0$, with the property that if $f \in L^p(G)$, and $PL_f = L_f$, then $f = 0$. Thus P is a projection of uniqueness in the sense that the only element $f \in L^p$, such that the range of L_f is contained in the range of P is the zero element. Another way to express this result is the following: There exists a nontrivial closed subspace of $L^2(G)$, invariant under right translations and which contains no nonzero element of $L^p(G)$.

The additional hypothesis required on G is the following, which will be tacitly assumed in the sequel:

(H) *$\mathscr{L}_2(G)$ is not purely atomic, that is, it is not generated, as a von Neumann algebra, by its minimal projections.*

Hypothesis (H) is certainly satisfied if G is discrete. However, there exist noncompact groups for which (H) is not true. A discussion of hypothesis (H) will be given at the end of this paper.

The progenitor of the present work is a note by Y. Katznelson [11] in which it is proved that there exist nonnegligible subsets of the torus $\boldsymbol{T}$ which are sets of uniqueness for l^p when $p < 2$. Our result coincides with Katznelson's theorem if G is taken to be the group of integers $\boldsymbol{Z}$, with the discrete topology.

A. Figà-Talamanca and G. I. Gaudry have extended Katznelson's result to the case in which G is a locally compact noncompact Abelian group [6]. In fact for the theorem proved in the present paper, we use the techniques of [6] in the framework of I. E. Segal's noncommutative integration theory [18] as applied by R. Kunze [13] and W. Stinespring [19] to the canonical gage space of a locally compact unimodular group.

It is evident that hypothesis (H) plays the role of the hypothesis of nondiscreteness of the character group of G made in [6]. Whereas it is clear that discrete (Abelian) groups cannot have sets of uniqueness we cannot determine as yet if the hypothesis (H) in the general case is essential for the existence of projections of uniqueness. We note that compact (nonAbelian) groups do not have projections of uniqueness.

We mention also that I. Hirschman and Y. Katznelson [10] have improved Katznelson's result for l^p, be showing that if $p < q \leqq 2$, there exist sets which are of uniqueness for l^p, but not for l^q. This latter result was extended by A. M. Mantero [16] to the case of an arbitrary noncompact LCA group. Neither the techniques of [16] nor our techniques appear to be useful in order to extend Mantero's results to a noncommutative situation.

Section 2, which follows, contains the main result of this paper. In §3 we give some applications of the main result. In particular we introduce the spaces $\mathscr{L}_p(G)$ which consist of the bounded linear operators on $L^p(G)$ $(1 \leqq p \leqq 2)$ which are strong limits of linear combinations of left translations, and we prove that if $p \neq q$, the norms of $\mathscr{L}_p$ and $\mathscr{L}_q$ are different. We also discuss the noncommutative analogue of the "derived space" in the sense of [5] and [15, §4.7]. We conclude the paper with §4 which contains some remarks on hypothesis (H) and another related hypothesis. The authors are grateful to Bernard Russo for several interesting conversations. In particular to Russo are due the contents of §4.

Before entering in the heart of the matter we recall some facts, taken especially from [18] and [13], which will be used in the following section.

DEFINITION 1.1. A regular gage space is a system $(\mathfrak{H}, \mathfrak{A}, m)$ composed of a Hilbert space $\mathfrak{H}$, a ring $\mathfrak{A}$ of everywhere defined bounded operators on $\mathfrak{H}$ and a nonnegative-valued function m on the projections in $\mathfrak{A}$ with the properties

(1) $m(P) > 0$ if $P \neq 0$ and $m(0) = 0$.

(2) m is completely additive.

(3) every projection is the l.u.b. of projections on which m is finite.

(4) if U is unitary and $U \in \mathfrak{A}$, then $m(U^*PU) = m(P)$.

We will assume from now on that the ring $\mathfrak{A}$ is a von Neumann algebra.

DEFINITION 1.2. A closed, not necessarily bounded operator T, on a Hilbert space $\mathfrak{H}$ is said to be measurable with respect to a von Neumann algebra $\mathfrak{A}$ of operators on $\mathfrak{H}$ if

(1) T commutes with every unitary operator in the commutation $\mathfrak{A}'$ of $\mathfrak{A}$.

(2) There exists an increasing sequence $K_1, K_2, \cdots$ of closed

linear subspaces in the domain of T, such that the restriction $T^{(i)}$ of T to K_i is bounded, for each i, $K_i^\perp$ is algebraically finite, $K_i^\perp \downarrow 0$ and the projections P_i on K_i belong to $\mathfrak{A}$ (see [18, p. 404] for a definition of algebraically finite).

Recall that given a gage space $(\mathfrak{H}, \mathfrak{A}, m) = \Gamma$, it is possible to extend the function m to be a linear functional on a class $L^1(\Gamma)$ of measurable operators (with respect to $\mathfrak{A}$). We have that $L^1(\Gamma)$ is a linear space on which m is a linear functional which is positive on positive elements. The space $L^1(\Gamma)$ becomes a Banach space with the norm $\|T\|_{L^1(\Gamma)} = m(|T|)$ where $U|T| = T$ is the canonical polar decomposition of T (it turns out that if T is measurable, then $U \in \mathfrak{A}$ and $|T|$ is measurable).

It is known [18] that if G is a unimodular group we can define *the canonical gage space* $\Gamma = (L^2(G), \mathscr{L}_2(G), m)$, where, as before, $\mathscr{L}_2$ is the von Neumann algebra generated by left translation by elements of G and m is defined as $m(P) = \|f\|_2^2$ when P is a projection and $P = L_f$ and as $m(P) = \infty$, otherwise.

In this context the operators L_f with $f \in L^p$, $1 \leqq p \leqq 2$ are measurable with respect to the ring $\mathscr{L}_2$ [13, p. 533]. We also have that $L^1(\Gamma)$ is isometrically isomorphic with the Fourier algebra $A(G)$ defined by P. Eymard [4]. This is a consequence of Eymard's result asserting that $A(G) = L^2(G) * L^2(G)$, and the fact that if $T \in L^1(\Gamma)$, then $T = L_f L_g$ for some $f, g \in L^2(G)$ and $\|T\|_{L^1(\Gamma)} = \|f\|_2 \|g\|_2$.

The isomorphism between $L^1(\Gamma)$ and $A(G)$ is given by $T \to T^\wedge(x) = m(L_x T)$ for $x \in G$, and $T \in L^1(\Gamma)$. This is a consequence of results of W. Stinespring on the inverse Fourier transform [19, § 9]. Finally we remark that if $L^2(\Gamma) = \{L_f : f \in L^2(G)\}$ the correspondence $f \to L_f$ defines a unitary transformation of $L^2(G)$ onto the Hilbert space $L^2(\Gamma)$, endowed with the inner product $(T, S) = m(TS^*)$, for $S, T \in L^2(\Gamma)$.

2. The main result.

We will start with a lemma which is a noncommutative version of Lemma 1.2 of [6].

LEMMA 2.1. *Let $\mathscr{L}_2(G) = \mathfrak{A} \oplus \mathfrak{B}$, where $\mathfrak{B}$ is the von Neumann algebra generated by the minimal projections. Let P be a nonzero projection $P \in \mathfrak{A}$ with $m(P) < \infty$. (Such a projection exists by virtue of hypothesis* (H)*). Let $\varepsilon > 0$ be given and $1 < p < 2$. There exists a projection $P_\varepsilon \subset P$ such that*

(i) $m(P_\varepsilon) \geqq (1 - \varepsilon) m(P)$

(ii) *for each $T \in L^1(\Gamma)$ with $P_\varepsilon T = T$, the inequality*

$$\|T^\wedge\|_{L^\infty(G)} \leqq \varepsilon \|T^\wedge\|_{L^p(G)} ,$$

holds.

Proof. We notice first of all that $\mathfrak{A}$ contains no atoms; therefore for any projection $Q \in \mathfrak{A}$ and any positive number $\alpha < m(Q)$, there exists a projection $Q_\alpha \in \mathfrak{A}$, $Q_\alpha \subset Q$ such that $m(Q_\alpha) = \alpha$. Indeed it is not difficult to see that if $\{P_i\}_{i\in I}$ is a maximal chain of projections of $\mathfrak{A}$ contained in Q, and $Q = \operatorname{Sup}\{P_i: m(P_i) \leqq \alpha\}$ then $m(Q_\alpha) = \alpha$. Using this fact repeatedly with $\alpha = 1/2\ m(P)$, and assuming, without loss of generality, that $m(P) = 1$, we can define a sequence $\{\pi_n\}$ of partitions of P into 2^n orthogonal projections in the following fashion: The partition π_1 consists of two projections P_{11} and P_{12} with $m(P_{11}) = m(P_{12})$; the partition π_n is obtained by dividing each projection of the partition π_{n-1} into two projections of equal measure. We define then the Rademacher operators as $R_n = \sum_1^{2^n} (-1)^{k+1}P_{nk}$, where the partitions $\pi_n = \{P_{n,1}, \cdots, P_{n,2^n}\}$ are indexed in such a way that $P_{n-1,k} = P_{n,2k-1} + P_{n,2k}$ $(k=1, \cdots, 2^{n-1})$. We let then $W_0 = P$ and $W_n = R_{j_1}, \cdots, R_{j_s}$ where $n = 2^{j_1-1} + \cdots + 2^{j_s-1}$, $j_1 < j_2 \cdots < j_s$. The sequence $\{W_n\}_{n=0}^\infty$ is an orthonormal system in $L^2(\Gamma)$ which will be referred to as the Walsh system; its elements will be called the Walsh operators. We note that each P_{nk} belongs to linear span of $\{W_i\}$ and that all P_{nk} commute.

Let now N be a large positive integer; we shall define N operators $\Phi_1, \cdots, \Phi_n$, mutually orthogonal in $L^2(\Gamma)$ which are linear combinations of Walsh operators and satisfy the following conditions:

(1) $m(|\Phi_j|) \leqq 2$, $m(\Phi_j^2) \leqq 2^{k+1}$, where k is an integer such that $2^{-k} < \varepsilon/N < 2^{-k+1}$

(2) if Q_j is the projection on the subspace $\{u: \Phi_j u = u\}$, then $m(Q_j) \geqq (1 - \varepsilon/N)$

(3) $|\Phi_1^\wedge(x) + \cdots + \Phi_N^\wedge(x)| \leqq 4,\ x \in G$.

To construct these operators we let $\Phi_1 = P$ and we suppose that $\Phi_1, \cdots, \Phi_n$ have heen constructed satisfying (1), (2), and

$$(3') \qquad |\Phi_1^\wedge(x) + \cdots + \Phi_n^\wedge(x)| \leqq \left(2 + \frac{2n}{N}\right), \qquad x \in G .$$

Since $\Phi_j \in L^1(\Gamma)$ and $\Phi_j^\wedge \in A(G) \subseteqq C_0(G)$, there exists a compact set $K \subseteqq G$ such that $|\Phi_j^\wedge(x)| \leqq 1/N$ for $1 \leqq j \leqq n$ and $x \notin K$. We notice that if $\mathfrak{H}$ is the Hilbert space generated by $\{W_i\}$, and T is the projection of $L^2(\Gamma)$ onto $\mathfrak{H}$, $\|T(L_xP)\|_2^2$ is a continuous function of x.

On the other hand $\sum |m(L_x W_i)|^2 = \|T(L_xP)\|_2^2$, since $\{W_i\}$ is a complete orthonormal system in $\mathfrak{H}$. Therefore, by Dini's theorem the convergence of the series

$$\sum |m(L_x W_i)|^2$$

is uniform on K. In conclusion we can choose a positive integer ν

such that $\sum_{i>\nu} | m(L_x W_i) |^2$ is uniformly and arbitrarily small on K. We can also assume that ν is greater than the indices of the Walsh operators appearing in the expansions of $\Phi_1, \cdots, \Phi_n$. This assumption implies that $m(\Phi_j W_h) = 0$ if $1 \leqq j \leqq n$ and $h > \nu$. For any such choice of ν, we can choose a partition π_m of P such that $2^m > \nu$. For typographical convenience we shall denote the projections of the partitions as $P_{ij} = P(i, j)$. Then if $P(m, j)$, $j = 1, \cdots, 2^m$ are the projections of π_m, $P(m, j) = P(m + k, (j - 1)2^k + 1) + P(m + k, (j - 1)2^k + 2) + \cdots + P(m + k, j2^k)$, where $j = 1, \cdots, 2^m$, and k is the fixed integer satisfying $2^{-k} < \varepsilon/N < 2^{-k+1}$. Define now

$$\Phi_{n+1} = P - 2^k \sum_{j=1}^{2m} P(m + k, (j - 1)2^k + 1) ,$$

then

$$\Phi_{n+1} P(m, j) = P(m, j) - 2^k P(m + k, (j - 1)2^k + 1) ,$$

and

$$\Phi_{n+1} = \Phi_{n+1} P = \sum_1^{2m}{}_j \, \Phi_{n+1} P(m, j) .$$

Since $m(\Phi_{n+1} P(m, j)) = 0$ and for $i \leqq \nu$, $W_i P(m, j) = \pm P(m, j)$, it follows that

$$m(\Phi_{n+1} W_i) = \sum_1^2{}_j \pm m(\Phi_{n+1} P(m, j)) = 0 .$$

In particular Φ_{n+1} is orthogonal to all Φ_j, when $1 \leqq j \leqq n$, and

$$\Phi_{n+1} = \sum_{j>\nu} a_j W_j , \tag{4}$$

where the a_j are real numbers and the sum is finite. To check that condition (2) holds we notice that

$$Q_{n+1} = P - \sum_{j=1}^{2m} P(m + k, (j - 1)2^k + 1) ,$$

and therefore,

$$m(Q_{n+1}) = 1 - 2^m(2^{-m-k}) = 1 - 2^{-k} \geqq \left(1 - \frac{\varepsilon}{N}\right) ,$$

by definition of k. If $x \notin K$, $| \Phi_j^\wedge(x) | < 1/N$ $(1 \leqq j \leqq n)$; therefore,

$$| \Phi_1^\wedge(x) + \cdots + \Phi_{n+1}^\wedge(x) | \leqq \frac{n}{N} + | \Phi_{n+1}^\wedge(x) | \leqq 2 + \frac{n}{N} .$$

On the other hand, if $x \in K$, we have that

$$|\Phi_{n+1}^{\wedge}(x)| \leqq \sum_{j \geqq \nu} |a_j m(L_x W_j)| \leqq \Big(\sum_{j \geqq \nu} |a_j|^2\Big)^{1/2} \Big(\sum_{j \geqq \nu} |m(L_x W_j)|^2\Big)^{1/2}$$
$$\leqq \|\Phi_{n+1}\|_{L^2(\Gamma)} \Big(\sum_{j \geqq \nu} |m(L_x W_j)|^2\Big)^{1/2} .$$

Now, as we saw, by choosing ν large enough, we can make the right-hand side of this equation arbitrarily small on K. We suppose then that ν has been chosen in such a way that

$$|\Phi_{n+1}^{\wedge}(x)| \leqq \frac{2}{N} .$$

This proves that there exist operators $\Phi_1, \cdots, \Phi_N$, satisfying (1), (2), and (3).

To complete the proof of the lemma, we write

$$F = (\Phi_1 + \cdots + \Phi_N)/N .$$

By construction the operators Φ_j are orthogonal, and it follows that

$$\|F\|_{L^2(\Gamma)} = \frac{1}{N} \big(\sum \|\Phi_j\|_{L^2(\Gamma)}^2\big)^{1/2} \Big(\frac{2^{k+1}}{N}\Big)^{1/2} .$$

Since $\varepsilon/N < 2^{-k+1}$, $2^{-k}N > \varepsilon/2$, therefore

$$\|F\|_{L^2(\Gamma)} \leqq C ,$$

where the constant C is independent of N. On the other hand,

$$\|F^{\wedge}\|_{\infty} \leqq \|\sum \Phi_j^{\wedge}\|_{\infty}/N \leqq \rho/N ,$$

where the constant ρ is independent of N, by (3). Finally if $2 < q < \infty$ and $1/p + 1/q = 1$, the relation

$$\|F^{\wedge}\|_{L^q(G)} \leqq \|F^{\wedge}\|_{L^2(G)}^{2/q} \|F^{\wedge}\|_{L^\infty(G)}^{1-2/q} = 0(N^{-1+2/q}) ,$$

holds. If N is large enough, this inequality implies that

$$\|F^{\wedge}\|_{L^q(G)} < \varepsilon .$$

Write now $Y_\varepsilon = \{u \in L^2(G): Fu = u\}$ and let P_ε be the projection of $L^2(G)$ onto Y_ε. Since

$$Y_\varepsilon \supseteqq \bigcap_{j=1}^{N} \{u: \Phi_j u = u\} ,$$

and Q_j are the projections onto $\{u: \Phi_j u = u\}$, satisfying $m(Q_j) \geqq (1 - \varepsilon/N)$, we have that $m(P_\varepsilon) \geqq 1 - \varepsilon$. Let now $T \in L^1(\Gamma)$ and suppose that $T^{\wedge} \in L^p(G)$ and that $P_\varepsilon T = T$. Then $FT = FP_\varepsilon T = P_\varepsilon T = T$, since $FP_\varepsilon = P_\varepsilon$. Furthermore $T \in L^q(\Gamma)$, by [13, Theorem 6 and Corollary 7.4] (see [13] for a definition of $L^q(\Gamma)$). Therefore $T \in L^2(\Gamma)$. We have then,

$$| T^\wedge(x) | = | m(L_x T) | = | m(L_x F T) |$$
$$= \left| \int_G (L_x F)^\wedge(y) T^\wedge(y) dy \right| \leqq \| (L_x F)^\wedge \|_{L^q(G)} \| T^\wedge \|_{L^q(G)} \leqq \varepsilon \| T^\wedge \|_{L^q(G)} .$$

The application of the Parseval formula [13, Lemma 7.2] is licit because both T and $L_x F$ bolong to $L^2(\Gamma)$. We have thus proved that

$$\| T^\wedge \|_\infty \leqq \varepsilon \| T^\wedge \|_p .$$

We are now ready to prove the main theorem.

THEOREM 2.2. *With the notation of Lemma 2.1, let $1 < p < 2$ and let $P \in \mathfrak{A}$, $m(P) > 0$. Then there exists $Q \in \mathfrak{A}$, $Q \subset P$ such that if $T \in L^1(\Gamma)$, $QT = T$ and $T^\wedge \in L^p$, it follows that $T = 0$. Furthermore, Q can be chosen in such a way that $m(Q)$ is arbitrarily close to $m(P)$.*

Proof. Again we can assume without loss of generality that $m(P) = 1$. We let $\sum \varepsilon_n \leqq \eta < 1$ and we define P_{ε_n} as in Lemma 2.1. Let $Q = \bigcap_{n=1}^\infty P_{\varepsilon_n}$. Then $m(P - Q) \leqq \eta$ and therefore $m(Q) \geqq 1 - \eta$. On the other hand, if $T \in L^1(\Gamma)$ and $QT = T$, then $P_{\varepsilon_n} T = T$ and therefore $\| T^\wedge \|_\infty \leqq \varepsilon_n \| T^\wedge \|_{L^p(G)}$ for every n. This inequality can hold only if $\| T^\wedge \|_{L^p(G)} = \infty$ or if $T = 0$. The theorem is thus proved.

3. Applications. We denote by $\mathscr{L}_p$, $1 \leqq p \leqq 2$ the algebra of operators on $L^p(G)$ which are limits in the weak operator topology of linear combinations of left translations. We remark that $\mathscr{L}_1$ is isomorphic to $M(G)$, the algebra of regular bounded measures on G; the isomorphism is defined by the map $\mu \to L_\mu$, for $\mu \in M(G)$, where $L_\mu f = \mu * f$. This implies that $\mathscr{L}_1 \subseteqq \mathscr{L}_p$ for every p. The spaces $\mathscr{L}_p$, are endowed with a locally convex Hausdorff topology, with respect to which the unit sphere is compact. Indeed for $p = 1$ this topology is the weak* topology, for $p > 1$ it is the weak operator topology, for which the unit sphere is compact by virtue of the reflexivity of $L^p(G)$.

In this section we establish first of all a result which states that the norms of $\mathscr{L}_p$ and $\mathscr{L}_q$ are inequivalent on $\mathscr{L}_1$ (which is contained in both), when $p \neq q$. The result, as before is proved only for groups satisfying (H).

THEOREM 3.1. *Let $1 \leqq p < q \leqq 2$, then the norms $\| \cdot \|_{\mathscr{L}_2} + \| \cdot \|_{\mathscr{L}_q}$ and $\| \cdot \|_{\mathscr{L}_2} + \| \cdot \|_{\mathscr{L}_p}$ defined on the space $\mathscr{L}_1$ are not equivalent.*

Proof. If the norms were equivalent, then for some $K > 0$

$$\| \cdot \|_{\mathscr{L}_p} + \| \cdot \|_{\mathscr{L}_2} \leqq K(\| \cdot \|_{\mathscr{L}_q} + \| \cdot \|_{\mathscr{L}_2}) . \tag{5}$$

But the density theorem of Kaplanski [2, p. 46] and the definition of $\mathscr{L}_2$, imply that there exists a generalized sequence $\{T_\alpha\}$ of elements of $\mathscr{L}_1$, (indeed $T_\alpha = L_{f_\alpha}$, with $f_\alpha \in L^1(G)$), such that $\lim_\alpha T_\alpha = Q$, in the weak operator topology, where Q is the projection whose existence is established in Theorem 2.2, and $\| T_\alpha \|_{\mathscr{L}_2} \leqq 1$. Since $Q \notin \mathscr{L}_p$, no subnet of $\{T_\alpha\}$ can converge in the weak operator topology of $\mathscr{L}_p$. It follows that T_α cannot be bounded in the norm of $\mathscr{L}_p$. Therefore, we can construct a sequence $\{T_n\} \subseteqq \{T_\alpha\}$ such that $\| T_n \|_{\mathscr{L}_p} \to \infty$.

Applying (5) one gets

$$\| T_n \|_{\mathscr{L}_p} + \| T_n \|_{\mathscr{L}_2} \leqq K(\| T_n \|_{\mathscr{L}_q} + \| T_n \|_{\mathscr{L}_2}) .$$

On the other hand, the Riesz convexity theorem implies that

$$\| T_n \|_{\mathscr{L}_q} \leqq \| T_n \|_{\mathscr{L}_p}^t \| T_n \|_{\mathscr{L}_2}^{1-t} ,$$

for some $0 < t < 1$.

Hence,

$$\| T_n \|_{\mathscr{L}_p} + \| T_n \|_{\mathscr{L}_2} \leqq K(\| T_n \|_{\mathscr{L}_p}^t \| T_n \|_{\mathscr{L}_2}^{1-t} + \| T_n \|_{\mathscr{L}_2}) ,$$

which implies

$$\| T_n \|_{\mathscr{L}_p}^{1-t} + \| T_n \|_{\mathscr{L}_2} \| T_n \|_{\mathscr{L}_p}^{-t} \leqq K(\| T_n \|_{\mathscr{L}_2}^{1-t} + \| T_n \|_{\mathscr{L}_2} \| T_n \|_{\mathscr{L}_p}^{-t}) ,$$

which is absurd because $\| T_n \|_{\mathscr{L}_p} \to \infty$, and $\| T_n \|_{\mathscr{L}_2} \leqq 1$.

When G is amenable we have the norm-decreasing inclusion $\mathscr{L}_p \subseteqq \mathscr{L}_q$ for $1 \leqq p \leqq q \leqq 2$. This result is due to C. Herz for the general case [9, Theorem C]. For the case of a commutative group the inclusion is simply a consequence of the Riesz convexity theorem. For the compact case a continuous, but not norm-decreasing inclusion, had been established in [7, pp. 511–512].

Applying Theorem 3.1 to the case of an amenable group satisfying (H), we obtain:

COROLLARY 3.2. *Let G be an amenable group satisfying* (H), *then the inclusion $\mathscr{L}_p \subseteqq \mathscr{L}_q$, for $1 \leqq p < q \leqq 2$ is proper.*

This corollary was established in [6] for noncompact commutative groups.

Our second application concerns the "derived space" of $L^p(G)$. The following definition is based on that which was given in [5] for the commutative case (see also [15]).

DEFINITION 3.2. For $f \in L^p$, $1 \leqq p \leqq 2$, define

$$\|f\|_0 = \sup \{\| \mathrm{L}_h f \|_p : \| L_h \|_{\mathscr{L}_2} \leqq 1, h \in L^1(G)\} .$$

We call the derived space of L^p and we denote by $(L^p)_0$, the linear subspace of L^p consisting of the elements for which $\|f\|_0 < \infty$.

We notice that, as it was proved in [6], $(L^p)_0 = 0$, when G is a noncompact Abelian group. We cannot prove or disprove the same result even in the case when G satisfies property (H). However, the result is true with essentially the same proof, when a stronger condition holds, that is when $\mathscr{L}_2(G)$ has *no minimal projections.*

THEOREM 3.3. *If G is a noncompact group and $\mathscr{L}_2(G)$ has no minimal projections, then $(L^p)_0 = 0$.*

Proof. Let $f \in (L^p)_0$ and $f \neq 0$. There exists a projection $P \in \mathscr{L}_2(G)$, such that $PL_f \neq 0$ and $m(P) < \infty$. Since P can be approximated in the weak* topology of $\mathscr{L}_2(G)$, by elements of the type L_h, $h \in L^1(G)$ and $\| L_h \|_{\mathscr{L}_2} \leqq 1$, and since $L^p(G)$ has a weakly compact unit sphere, it is not difficult to see that, by definition of $(L^p)_0$, $PL_f = L_g$, with $g \in (L^p)_0$. It follows that $PL_g = L_g$. This means that if ε is sufficiently small and $Q \subset P$ is the projection constructed in Theorem 2.2, and such that $m(Q) > (1 - \varepsilon)m(P)$, then $QL_g \neq 0$. By the same reasoning $QL_g = L_h = QL_h$, with $h \in (L^p)_0$, and $h \neq 0$. But this is impossible on the basis of Theorem 2.2.

REMARK. For $p = 1$ the analogue of Theorem 3.3 has been established by S. Helgason for several special cases [8] and by Sakai for *every* noncompact group [17].

4. **The hypothesis (H).** There exist noncompact groups which fail to satisfy (H). This is shown in [1, §4]. This is the only example known at present to the authors of a noncompact group for which $\mathscr{L}_2(G)$ is purely atomic. Many groups have some minimal projections and still satisfy (H), e.g., SL $(2, \boldsymbol{R})$ [14]. At the other extreme the following classes of groups have no minimal projections and thus Theorem 3.3 applies to them:

(a) noncompact complex connected semi-simple Lie groups,

(b) noncompact real simply connected nilpotent Lie groups,

(c) noncompact (unimodular) SIN groups (i.e., there is a basis of neighborhoods of the identity invariant under all inner automorphisms).

(d) the Euclidean group.

For references to the proofs of (a) and (b) see [3, 14.6.3]. Basically these facts follow from the correspondence between minimal projections in $\mathscr{L}_2(G)$ (equivalently in $\mathscr{L}_2(G)'$), and irreducible subrepresentations of the regular representation of G (square integrable representations) [3, §14].

The proposition which follows is known [20, p. 70], but the proof is new and provides a proof of (c). The proof of (d) follows from the Plancherel formula for the Euclidean group [12], which shows that the Plancherel measure has no atoms and hence [3, 18.8.5], the group has no square integrable representations.

PROPOSITION 4.1. *If the left regular representation of a locally compact unimodular group G contains a finite-dimensional subrepresentation π, then G is compact.*

Proof. Let E be the minimal projection in $\mathscr{L}_2'$(= the commutator of $\mathscr{L}_2$) corresponding to π and let F be the central support of π in $\mathscr{L}_2 \cap \mathscr{L}_2'$. Then $\mathscr{L}_2 \cong \mathscr{L}_F \oplus \mathscr{L}_{1-F}$, where $\mathscr{L}_F$ is isomorphic to the ring of all bounded operators on a finite-dimensional Hilbert space. From Stinespring [17, §10] we know that $x \rightarrow L_x$ is a homeomorphism of G onto $\{L_x: x \in G\}$ with the weak operator topology and that $\{L_x: x \in G\} \cup \{0\}$ is weakly closed in the unit sphere of $\mathscr{L}_2$. In our case $\{L_x: x \in G\}$ is closed in $\{L_x: x \in G\} \cup \{0\}$, since if $L_{x_\alpha} \rightarrow T$, weakly, we have $FL_{x_\alpha} \rightarrow FT$, in norm, because $\mathscr{L}_F$ is finite-dimensional, so $T \neq 0$. Thus $\{L_x: x \in G\}$ is compact and so is G.

COROLLARY 4.2. *A noncompact unimodular* SIN *group G has no minimal projections in $\mathscr{L}_2(G)$.*

Proof. As in the above proof we have $\mathscr{L}_2 = \mathscr{L}_F \oplus \mathscr{L}_{1-F}$, where $\mathscr{L}_F$ is isomorphic to the ring of all bounded operators on some Hilbert space, not a priori finite-dimensional. But as G is SIN, $\mathscr{L}_2$ and hence $\mathscr{L}_F$ must be a finite von Neumann algebra, [3, 13.10.5] and [2, p. 97]. Thus the subrepresentation determined by F is finite-dimensional.

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