THE NON ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES

Gokulananda Das and Ram N. Mohapatra
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The paper is devoted partly to the study of non-absolute Nörlund summability of Fourier series of \( \varphi(t) \) under the condition \( \varphi(t) \in AC[0, \pi] \) for suitable \( \varphi(t) \). The other aspect is to determine the order of variation of the Harmonic mean of the Fourier series whenever \( \varphi(t) \log k/t \in BV[0, \pi] \).

1. Let \( L \) denote the class of all real functions \( f \) with period \( 2\pi \) and integrable in the sense of Lebesgue over \( (-\pi, \pi) \) and let the Fourier series of \( f \in L \) be given by

\[
\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),
\]

assuming, as we may, the constant term to be zero.

We write

\[
\phi(t) = \frac{1}{2} \{f(x + t) + f(x - t)\}
\]

\[
g(n, t) = \int_{0}^{t} \cos nu \frac{\varphi(u)}{\chi(u)} du
\]

\[
h(n, t) = \int_{t}^{\pi} \cos nu \frac{\varphi(u)}{\chi(u)} du.
\]

Let \( \{p_n\} \) be a sequence of constants such that \( P_n = \sum_{n=0}^{\infty} p_n \neq 0 \) \((n \geq 0)\) and \( P_{-1} = p_{-1} = 0 \). For the definition of absolute Nörlund or \((N, p)\) method, see, for example, Pati [9]. When \( \sum_{n=0}^{\infty} a_n \) is absolutely \((N, p)\) summable, we shall write, for brevity, \( \sum_{n=0}^{\infty} a_n \in \{N, p\} \).

We define the sequence of constants \( \{c_n\} \) formally by \( (\sum_{n=0}^{\infty} p_n x^n)^{-1} = \sum_{n=0}^{\infty} c_n x^n \), \( c_{-1} = 0 \).

2. One of the objects of this paper is to study the non-absolute \((N, p)\) summability factors of Fourier series and generalize the following outstanding result of Pati in Theorems 1-2. Besides, the proof of Theorems 1-2 are short and simple and avoids the direct technique of Pati which is somewhat long and complicated.

If we write

\[
G = \left\{ f: f \in L, \varphi(t) \log k/t \in AC[0, \pi] \text{ and } \sum_{n=1}^{\infty} A_n(x) \in \left| N, \frac{1}{n + 1} \right| \right\}
\]

then Pati’s theorem is in the following form:

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Theorem P [9]. $G$ is nonempty.

Mohanty and Ray [8] subsequently constructed an example of $f \in G$.

We now establish

**Theorem 1.** Let $\lambda$ be a real differentiable function and $\{\varepsilon_n\}$ be a sequence satisfying the following conditions:

1. \[ \varphi(t)\lambda(t) \in \mathcal{AC}[0, \pi], \]
2. \[ \sum_{n=1}^{\infty} \left| \varepsilon_n \right| \frac{\left| g(n, \pi) \right|}{P_n} < \infty, \]
3. \[ \frac{\lambda^\prime(t)}{\lambda^\prime(\pi)} \to 0 \quad \text{as} \ t \to 0, \]
4. \[ \sum_{n=1}^{\infty} \frac{\left| \varepsilon_n \right| \lambda^\prime(\pi/n) \lambda^\prime(n/\pi)}{n^2 P_n} < \infty, \]
5. \[ \sum_{n=1}^{\infty} \left| \frac{\varepsilon_n}{nP_n} \right| < \infty, \]
6. \[ \varepsilon_n = 0(nP_n), \]
7. \[ \exists \text{a set } E: mE > 0 \text{ and } \exists \text{ a constant } \eta > 0 \text{ such that } \lambda(t)^{-1} > \eta \quad \forall t \in E. \]

Then

\[ \sum_{n=1}^{\infty} \left| \varepsilon_n \right| \frac{|A_n(t)|}{|P_n|} = \infty \quad (\forall t \in E), \]

if and only if

\[ \sum_{n=1}^{\infty} \frac{\left| \varepsilon_n \right|}{nP_n} = \infty. \]

Now, if we denote, $G^* = \{f: f \in L, \text{ conditions (1) through (7) and (9) hold and } \sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in |N, p| \}$ then we establish

**Theorem 2.** Let

\[ \sum_{n=0}^{\infty} |p_n| = O(|P_n|), \quad \sum_{n=0}^{\infty} |c_n| < \infty. \]

Then $G^*$ is nonempty.

In §3, we discuss some special cases of interest of Theorem 2.
Since Theorem 2 implies that the total variation of the \((N, p)\) mean of the series \(\sum_{n=1}^{\infty} \varepsilon_n A_n(x)\) is unbounded, the natural question now is to determine the order of the variation. And this is achieved in Theorem 3 in §4.

3. We need the following lemmas for the proof of Theorem 1.

**Lemma 1.** (2) Suppose that \(\{f_n(x)\}\) is measurable in \((a, b)\) where \(b - a \leq \infty\), for \(n = 1, 2, \cdots\). Then a necessary and sufficient condition that, for every function \(\psi(x)\) integrable in the sense of Lebesgue over \((a, b)\), the functions \(f_n(x)\psi(x)\) should be integrable \(L\) over \((a, b)\) and

\[
\sum_{n=1}^{\infty} \left| \int_{a}^{b} \psi(x) f_n(x) dx \right| \leq K
\]

is that

\[
\sum_{n=1}^{\infty} |f_n(x)| \leq K
\]

where \(K\) is an absolute constant for almost every \(x\) in \((a, b)\).

**Lemma 2.** Let condition (3) hold. Then

\[
h(n, t) = \frac{\sin n t}{n \lambda(t)} + O\left(\frac{1}{n^2}\right) \frac{X'(\pi/n)}{X^2(\pi/n)}.
\]

**Proof.** We have, by integration by parts, and second mean-value theorem,

\[
h(n, t) = \left(\int_{\pi/n}^{t} \frac{\cos nu}{\lambda(u)} du \right)
\]

\[
= \frac{\sin nt}{n \lambda(t)} + \frac{1}{n} \left(\int_{\pi/n}^{t} \frac{\lambda'(u)}{\lambda^2(u)} \right) \sin nu du
\]

\[
= \frac{\sin nt}{n \lambda(t)} + O\left(\frac{1}{n}\right) \frac{|X'(\pi/n)|}{X^2(\pi/n)} \left(\int_{\pi/n}^{t} - \int_{\pi/n}^{\zeta}\right) \sin nu du
\]

\[
= \frac{\sin nt}{n \lambda(t)} + O\left(\frac{1}{n^2}\right) \frac{|X'(\pi/n)|}{X^2(\pi/n)},
\]

where \(\pi/n \leq \zeta \leq \pi, \pi/n \leq \zeta \leq t\).

This completes the proof.

**Proof of Theorem 1.** We have, by integration by parts,

\[
A_n(x) = \frac{2}{\pi} \int_{0}^{\pi} \phi(t) \cos ntdt = F(0) g(n, \pi) + \int_{0}^{\pi} F'(t) h(n, t) dt,
\]

where \(F(t) = \phi(t) \lambda(t)\). Hence by condition (2) the statement (8) is
equivalent to proving that:

\[(11) \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} \int_0^\infty F''(t) h(n, t) \, dt = \infty \quad (\forall t \in E).\]

Since, by hypothesis (1)

\[\int_0^\infty |F''(t)| \, dt < \infty,
\]

by Lemma 1, the statement (11) is equivalent to proving that there exists a set $E: mE > 0$ and

\[(12) \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |h(n, t)| = \infty \quad (\forall t \in E).
\]

Whenever conditions (3) and (4) hold, by virtue of Lemma 2, the statement (12) is easily seen to be equivalent to proving that

\[(13) M(t) = \frac{1}{|X(t)|} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} |\sin nt| = \infty \quad (\forall t \in E).
\]

Now, since

\[|\sin nt| \geq \sin^2 nt = \frac{1}{2} (1 - \cos 2nt),\]

we have

\[M(t) \geq \frac{1}{2|X(t)|} \left( \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} - \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} \cos 2nt \right).
\]

Using conditions (5) and (6) and using Dedekind’s theorem we observe that the series

\[\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} \cos 2nt\]

is convergent for $0 < t < \pi$. Hence (13) is equivalent to showing that

\[(14) \frac{1}{|X(t)|} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} = \infty \quad (\forall t \in E).
\]

Now the result follows from (14) by using the conditions (7) and (8).

**Proof of Theorem 2.** Das [4], in particular, proved that whenever condition (10) holds, then

\[\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in N, p \implies \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |A_n(x)| < \infty.
\]

Now the result follows from Theorem 1.
4. In this section we apply Theorem 2 to some special cases. If we take \( \chi(t) = \log k/t, \ E = \{t: k/e \leq t < \pi\} \) we get

**COROLLARY 1.** Let \( \{\varepsilon_n\} \) satisfy the conditions:

i) \( \varepsilon_n = O(\log n) \),

ii) \( \sum_{n=1}^{\infty} |\varepsilon_n|/n \log^3(n + 1) < \infty \),

iii) \( \sum_{n=1}^{\infty} |\Delta \varepsilon_n|/n \log(n + 1) < \infty \),

iv) \( \sum_{n=1}^{\infty} |\varepsilon_n|/n \log(n + 1) = \infty \).

Then

\[
\varphi(t) \log k/t \in AC[0, \pi] \implies \sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in N, \frac{1}{n + 1}.
\]

**Proof.** Since the Fourier series of the even periodic function \( (\log k/t)^{-1} \) is absolutely convergent (see Mohanty [7]) we get that

\[
\sum_{n=1}^{\infty} \left| \int_0^x \frac{\cos nu}{\log k/u} \, du \right| < \infty .
\]

It may be observed that if we take \( \varepsilon_n = 1, \ p_n = 1/(n + 1) \) in Corollary 1, then we get Theorem P.

**COROLLARY 2.** Let \( \varphi(t) \in BV[0, \pi] \) and let conditions (5), (6), and (9) hold. Then \( \sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in N, p \).

Take \( \chi(t) = 1, \ E = [0, \pi] \) in Theorem 2. In this case \( g(n, \pi) = 0 \).

**REMARK.** Corollary 2 in the case \( \varepsilon_n = 1 \) gives that

\[
\varphi(t) \in BV[0, \pi] \implies \sum_{n=1}^{\infty} A_n(x) \notin N, \frac{1}{n + 1} .
\]

This interalia establishes the result that \( \varphi(t) \in BV[0, \pi] \) is not sufficient to guarantee the absolute convergence of the series \( \sum_{n=1}^{\infty} A_n(x) \). See Bosanquet (1) who showed this by taking an example.

5. Throughout this section we consider the case \( p_n = 1/(n + 1) \) only. We write \( t_n \) and \( \tau_n \) respectively for the \( (N, 1/(n + 1)) \) means of the sequences \( \{\sum_{n=1}^{N} \varepsilon_n A_n(x)\} \) and \( \{n\varepsilon_n A_n(x)\} \). It follows from a result of Das [4] Theorem 6 on general infinite series that

\[
\sum_{n=1}^{m} \frac{\left|\tau_n\right|}{n} = O(1) \quad \text{if and only if} \quad \sum_{n=1}^{m} \left| t_n - t_{n-1} \right| = O(1) .
\]

Proceeding as in the proof of above result we in fact get that for any positive nondecreasing sequence \( \{\lambda_n\} \)
\[ (17) \quad \frac{\sum_{n=1}^{m} |\tau_n|}{n} = O(\lambda_m) \] if and only if \[ \sum_{n=1}^{m} |t_n - t_{n-1}| = O(\lambda_m). \]

Since Theorem P implies that the variation of \{t_n\} is of unbounded order, we are immediately confronted with the problem of determining the order of variation of \{t_n\}. Because of relation (17) this problem simplifies to determining the order of \( \sum_{n=1}^{m} |\tau_n|/n \) and this is achieved in

**Theorem 3.** If \( g(t) \equiv \varphi(t) \log k/t \in BV[0, \pi] \). Then
\[ \frac{\sum_{n=1}^{m} |\tau_n|}{n} = O(\log \log m). \]

**Proof.** We have
\[ \tau_n = \frac{2}{\pi P_n} \sum_{\nu=1}^{n} p_{n-v} \int_0^\pi \varphi(t) \cos \nu \right) dt. \]

Since
\[ \int_0^\pi \varphi(t) \cos \nu \right) dt = g(0) \int_0^\pi \cos \nu \right) \log k/t \right) dt + \int_0^\pi g(t) \int_0^\pi \frac{\cos \nu \right) u \right)} du, \]
we get
\[ \sum_{n=1}^{m} \frac{|\tau_n|}{n} \leq \frac{2}{\pi} |g(0)| \sum_{n=1}^{m} \frac{1}{nP_n} \left| \int_0^\pi \frac{dt}{\log k/t} \left( \sum_{\nu=1}^{n} p_{n-v} \cos \nu \right) \right| \]
\[ + \frac{2}{\pi} \int_0^\pi |dg(t)| \sum_{n=1}^{m} \frac{1}{nP_n} \left| \int_0^\pi \frac{dt}{\log k/t} \left( \sum_{\nu=1}^{n} p_{n-v} \cos \nu \right) \right| \]
\[ = \frac{2}{\pi} \left| g(0) \right| \left( G_m + H_m \right). \]

Since the series \( \sum_{n=1}^{\infty} \int_0^\pi \cos n \right) \log k/u \right) du \) is absolutely convergent (see (15)) and therefore it is absolutely \((N, 1/(n + 1))\) summable, we get that \( G_m = O(1) \) by using relation (16).

Since \( \int_0^\pi |dg(t)| < \infty \), using Lemma 2 with \( \log k/t \) in place of \( \lambda(t) \) we get that
\[ H_m = O(1) \sum_{n=1}^{m} \frac{1}{n \log (n + 1)} \left| \sum_{\nu=1}^{n} \frac{\sin \nu \right) t}{n - \nu + 1} \right| \]
\[ + O(1) \sum_{n=1}^{m} \frac{1}{n \log (n + 1)} \sum_{\nu=1}^{n} (n - \nu + 1) \frac{1}{\log^2 (n + 1)} = H_m^{(1)} + H_m^{(2)}. \]

By a result of McFadden ([6], Lemma 5.10) we get
\[ \sum_{\nu=1}^{n} \frac{\sin \nu \right) t}{n - \nu + 1} = O(\log \tau), (\tau = [k/t]) \]
and consequently
\[ H_m^{(1)} = O(1) \frac{\log \tau}{\log h/t} \sum_{n=1}^{m} \frac{1}{n \log (n + 1)} = O(\log \log m). \]

On change of order of summation in \( H_m^{(1)} \) and by use of the fact that
\[ \sum_{n=v}^{m} \frac{1}{(n - v + 1)n \log (n + 1)} = O\left(\frac{1}{v + 1}\right), \]
we get
\[ H_m^{(2)} = O(1) \sum_{v=1}^{m} \frac{1}{v \log^2 (v + 1)} = O(1) \quad (m \to \infty); \]
and this completes the proof.

**Remarks.** In view of Corollary 1, one is naturally led to determine suitable sequences \( \{\varepsilon_n\} \) such that \( g(t) \in BV[0, \pi] \Rightarrow \sum \varepsilon_n \xi_n(x) \in |N, 1/(n + 1)| \). But in view of Theorem 3 it is enough to determine the sequence of factors \( \{\varepsilon_n\} \) such that \( \sum_{n=1}^{\infty} \varepsilon_n \xi_n(x) \in |N, 1/(n + 1)| \) whenever \( \sum_{n=1}^{m} \tau_n |/n = O(\log \log m) \). Such a result is contained in the more general result of Das [5].

**References**


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