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## **THE NON ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES**

GOKULANANDA DAS AND RAM N. MOHAPATRA

## THE NON ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES

G. DAS AND R. N. MOHAPATRA

The paper is devoted partly to the study of non-absolute Nörlund summability of Fourier series of  $\varphi(t)$  under the condition  $\varphi(t)\chi(t) \in AC[0, \pi]$  for suitable  $\chi(t)$ . The other aspect is to determine the order of variation of the Harmonic mean of the Fourier series whenever  $\varphi(t) \log k/t \in BV[0, \pi]$ .

1. Let  $L$  denote the class of all real functions  $f$  with period  $2\pi$  and integrable in the sense of Lebesgue over  $(-\pi, \pi)$  and let the Fourier series of  $f \in L$  be given by

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),$$

assuming, as we may, the constant term to be zero.

We write

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\}$$

$$g(n, t) = \int_0^t \frac{\cos nu}{\chi(u)} du$$

$$h(n, t) = \int_t^\pi \frac{\cos nu}{\chi(u)} du.$$

Let  $\{p_n\}$  be a sequence of constants such that  $P_n = \sum_{v=0}^n p_v \neq 0$  ( $n \geq 0$ ) and  $P_{-1} = p_{-1} = 0$ . For the definition of absolute Nörlund or  $(N, p)$  method, see, for example, Pati [9]. When  $\sum_{n=0}^{\infty} a_n$  is absolutely  $(N, p)$  summable, we shall write, for brevity,  $\sum_{n=0}^{\infty} a_n \in |N, p|$ .

We define the sequence of constants  $\{c_n\}$  formally by  $(\sum_{n=0}^{\infty} p_n x^n)^{-1} = \sum_{n=0}^{\infty} c_n x^n, c_{-1} = 0$ .

2. One of the objects of this paper is to study the non-absolute  $(N, p)$  summability factors of Fourier series and generalize the following outstanding result of Pati in Theorems 1-2. Besides, the proof of Theorems 1-2 are short and simple and avoids the direct technique of Pati which is somewhat long and complicated.

If we write

$$G = \left\{ f: f \in L, \varphi(t) \log k/t \in AC[0, \pi] \text{ and } \sum_{n=1}^{\infty} A_n(x) \in \left| N, \frac{1}{n+1} \right| \right\}$$

then Pati's theorem is in the following form:

THEOREM P [9].  $G$  is nonempty.

Mohanty and Ray [8] subsequently constructed an example of  $f \in G$ .

We now establish

THEOREM 1. Let  $\chi$  be a real differentiable function and  $\{\varepsilon_n\}$  be a sequence satisfying the following conditions:

$$(1) \quad \phi(t)\chi(t) \in AC[0, \pi],$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |g(n, \pi)| < \infty,$$

$$(3) \quad \frac{|\chi'(t)|}{\chi^2(t)} \nearrow \text{ as } t \searrow 0,$$

$$(4) \quad \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n^2 |P_n|} \frac{|\chi'(\pi/n)|}{\chi^2(\pi/n)} < \infty,$$

$$(5) \quad \sum_{n=1}^{\infty} \left| \Delta \left( \frac{\varepsilon_n}{nP_n} \right) \right| < \infty,$$

$$(6) \quad \varepsilon_n = O(nP_n),$$

$$(7) \quad \exists \text{ a set } E: mE > 0 \text{ and } \exists \text{ a constant } \eta > 0 \text{ such that } \chi(t)^{-1} > \eta \quad \forall t \in E.$$

Then

$$(8) \quad \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |A_n(t)| = \infty \quad (\forall t \in E),$$

if and only if

$$(9) \quad \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} = \infty.$$

Now, if we denote,  $G^* = \{f: f \in L, \text{ conditions (1) through (7) and (9) hold and } \sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin |N, p|\}$  then we establish

THEOREM 2. Let

$$(10) \quad \sum_{v=0}^n |p_v| = O(|P_n|), \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

Then  $G^*$  is nonempty.

In §3, we discuss some special cases of interest of Theorem 2.

Since Theorem 2 implies that the total variation of the  $(N, p)$  mean of the series  $\sum_{n=1}^{\infty} \epsilon_n A_n(x)$  is unbounded, the natural question now is to determine the order of the variation. And this is achieved in Theorem 3 in § 4.

3. We need the following lemmas for the proof of Theorem 1.

LEMMA 1. (2) Suppose that  $\{f_n(x)\}$  is measurable in  $(a, b)$  where  $b - a \leq \infty$ , for  $n = 1, 2, \dots$ . Then a necessary and sufficient condition that, for every function  $\psi(x)$  integrable in the sense of Lebesgue over  $(a, b)$ , the functions  $f_n(x)\psi(x)$  should be integrable  $L$  over  $(a, b)$  and

$$\sum_{n=1}^{\infty} \left| \int_a^b \psi(x) f_n(x) dx \right| \leq K$$

is that

$$\sum_{n=1}^{\infty} |f_n(x)| \leq K,$$

where  $K$  is an absolute constant for almost every  $x$  in  $(a, b)$ .

LEMMA 2. Let condition (3) hold. Then

$$h(n, t) = \frac{\sin nt}{n\lambda(t)} + O\left(\frac{1}{n^2}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)}.$$

*Proof.* We have, by integration by parts, and second mean value theorem,

$$\begin{aligned} h(n, t) &= \left( \int_{\pi/n}^{\pi} - \int_{\pi/n}^t \right) \frac{\cos nu}{\chi(u)} du \\ &= \frac{\sin nt}{n\lambda(t)} + \frac{1}{n} \left( \int_{\pi/n}^{\pi} - \int_{\pi/n}^t \right) \frac{\chi^1(u)}{\chi^2(u)} \sin nudu \\ &= \frac{\sin nt}{n\lambda(t)} + O\left(\frac{1}{n}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} \left( \int_{\pi/n}^{\zeta^1} - \int_{\pi/n}^{\zeta} \right) \sin nudu \\ &= \frac{\sin nt}{n\lambda(t)} + O\left(\frac{1}{n^2}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)}, \end{aligned}$$

where  $\pi/n \leq \zeta \leq \pi$ ,  $\pi/n \leq \zeta^1 \leq t$ .

This completes the proof.

*Proof of Theorem 1.* We have, by integration by parts,

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos ntdt = F(0)g(n, \pi) + \int_0^{\pi} F'(t)h(n, t)dt,$$

where  $F(t) \equiv \phi(t)\lambda(t)$ . Hence by condition (2) the statement (8) is

equivalent to proving that:

$$(11) \quad \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} \left| \int_0^{\pi} F'(t) h(n, t) dt \right| = \infty \quad (\forall t \in E).$$

Since, by hypothesis (1)

$$\int_0^{\pi} |F'(t)| dt < \infty,$$

by Lemma 1, the statement (11) is equivalent to proving that  $\exists$  a set  $E: mE > 0$  and

$$(12) \quad \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |h(n, t)| = \infty \quad (\forall t \in E).$$

Whenever conditions (3) and (4) hold, by virtue of Lemma 2, the statement (12) is easily seen to be equivalent to proving that

$$(13) \quad M(t) = \frac{1}{|\chi(t)|} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} |\sin nt| = \infty \quad (\forall t \in E).$$

Now, since

$$|\sin nt| \geq \sin^2 nt = \frac{1}{2}(1 - \cos 2nt),$$

we have

$$M(t) \geq \frac{1}{2\chi(t)} \left( \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} - \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} \cos 2nt \right).$$

Using conditions (5) and (6) and using Dedekind's theorem we observe that the series

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} \cos 2nt$$

is convergent for  $0 < t < \pi$ . Hence (13) is equivalent to showing that

$$(14) \quad \frac{1}{\chi(t)} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} = \infty \quad (\forall t \in E).$$

Now the result follows from (14) by using the conditions (7) and (8).

*Proof of Theorem 2.* Das [4], in particular, proved that whenever condition (10) holds, then

$$\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in |N, p| \implies \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |A_n(x)| < \infty.$$

Now the result follows from Theorem 1.

4. In this section we apply Theorem 2 to some special cases. If we take  $\chi(t) = \log k/t$ ,  $E = \{t: k/e \leq t < \pi\}$  we get

COROLLARY 1. Let  $\{\varepsilon_n\}$  satisfy the conditions:

- (i)  $\varepsilon_n = O(\log n)$ ,
- (ii)  $\sum_{n=1}^{\infty} |\varepsilon_n|/n \log^3(n+1) < \infty$ ,
- (iii)  $\sum_{n=1}^{\infty} |\Delta\varepsilon_n|/n \log(n+1) < \infty$ ,
- (iv)  $\sum_{n=1}^{\infty} |\varepsilon_n|/n \log(n+1) = \infty$ .

Then

$$\varphi(t) \log k/t \in AC[0, \pi] \implies \sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin \left| N, \frac{1}{n+1} \right|.$$

*Proof.* Since the Fourier series of the even periodic function  $(\log k/|t|)^{-1}$  is absolutely convergent (see Mohanty [7]) we get that

$$(15) \quad \sum_{n=1}^{\infty} \left| \int_0^{\pi} \frac{\cos nu}{\log k/u} du \right| < \infty.$$

It may be observed that if we take  $\varepsilon_n = 1$ ,  $p_n = 1/(n+1)$  in Corollary 1, then we get Theorem P.

COROLLARY 2. Let  $\varphi(t) \in BV[0, \pi]$  and let conditions (5), (6), and (9) hold. Then  $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin |N, p|$ .

Take  $\chi(t) \equiv 1$ ,  $E = [0, \pi]$  in Theorem 2. In this case  $g(n, \pi) = 0$ .

REMARK. Corollary 2 in the case  $\varepsilon_n = 1$  gives that

$$\varphi(t) \in BV[0, \pi] \implies \sum_{n=1}^{\infty} A_n(x) \notin \left| N, \frac{1}{n+1} \right|.$$

This interalia establishes the result that  $\varphi(t) \in BV[0, \pi]$  is not sufficient to guarantee the absolute convergence of the series  $\sum_{n=1}^{\infty} A_n(x)$ . See Bosanquet (1) who showed this by taking an example.

5. Throughout this section we consider the case  $p_n = 1/(n+1)$  only. We write  $t_n$  and  $\tau_n$  respectively for the  $(N, 1/(n+1))$  means of the sequences  $\{\sum_{v=1}^n \varepsilon_v A_v(x)\}$  and  $\{n\varepsilon_n A_n(x)\}$ . It follows from a result of Das [4] Theorem 6 on general infinite series that

$$(16) \quad \sum_{n=1}^m \frac{|\tau_n|}{n} = O(1) \text{ if and only if } \sum_{n=1}^m |t_n - t_{n-1}| = O(1).$$

Proceeding as in the proof of above result we in fact get that for any positive nondecreasing sequence  $\{\lambda_n\}$

$$(17) \quad \sum_{n=1}^m \frac{|\tau_n|}{n} = O(\lambda_m) \text{ if and only if } \sum_{n=1}^m |t_n - t_{n-1}| = O(\lambda_m).$$

Since Theorem P implies that the variation of  $\{t_n\}$  is of unbounded order, we are immediately confronted with the problem of determining the order of variation of  $\{t_n\}$ . Because of relation (17) this problem simplifies to determining the order of  $\sum_{n=1}^m |\tau_n|/n$  and this is achieved in

**THEOREM 3.** *If  $g(t) \equiv \varphi(t) \log k/t \in BV[0, \pi]$ . Then*

$$\sum_{n=1}^m \frac{|\tau_n|}{n} = O(\log \log m).$$

*Proof.* We have

$$\tau_n = \frac{2}{\pi P_n} \sum_{\nu=1}^n p_{n-\nu} \nu \int_0^\pi \varphi(t) \cos \nu t dt.$$

Since

$$\int_0^\pi \varphi(t) \cos \nu t dt = g(0) \int_0^\pi \frac{\cos \nu t}{\log k/t} dt + \int_0^\pi dg(t) \int_t^\pi \frac{\cos \nu u}{\log k/u} du,$$

we get

$$\begin{aligned} \sum_{n=1}^m \frac{|\tau_n|}{n} &\leq \frac{2}{\pi} |g(0)| \sum_{n=1}^m \frac{1}{nP_n} \left| \int_0^\pi \frac{dt}{\log k/t} \left( \sum_{\nu=1}^n p_{n-\nu} \nu \cos \nu t \right) \right| \\ &\quad + \frac{2}{\pi} \int_0^\pi |dg(t)| \sum_{n=1}^m \frac{1}{nP_n} \left| \int_t^\pi \frac{dt}{\log k/t} \left( \sum_{\nu=1}^n p_{n-\nu} \nu \cos \nu t \right) \right| \\ &= \frac{2}{\pi} \{ |g(0)| G_m + H_m \}. \end{aligned}$$

Since the series  $\sum_{n=1}^\infty \int_0^\pi \cos nu / \log k/u du$  is absolutely convergent (see (15)) and therefore it is absolutely  $(N, 1/(n+1))$  summable, we get that  $G_m = O(1)$  by using relation (16).

Since  $\int_0^\pi |dg(t)| < \infty$ , using Lemma 2 with  $\log k/t$  in place of  $\chi(t)$  we get that

$$\begin{aligned} H_m &= O(1) \sum_{n=1}^m \frac{1}{n \log(n+1)} \left| \sum_{\nu=1}^n \frac{\sin \nu t}{n - \nu + 1} \right| \\ &\quad + O(1) \sum_{n=1}^m \frac{1}{n \log(n+1)} \sum_{\nu=1}^n \frac{1}{(n - \nu + 1) \log^2(\nu + 1)} = H_m^{(1)} + H_m^{(2)}. \end{aligned}$$

By a result of McFadden ([6], Lemma 5.10) we get

$$\sum_{\nu=1}^n \frac{\sin \nu t}{n - \nu + 1} = O(\log \tau), \quad (\tau = [k/t])$$

and consequently

$$H_m^{(1)} = O(1) \frac{\log \tau}{\log k/t} \sum_{n=1}^m \frac{1}{n \log(n+1)} = O(\log \log m).$$

On change of order of summation in  $H_m^{(2)}$  and by use of the fact that

$$\sum_{n=\nu}^m \frac{1}{(n-\nu+1)n \log(n+1)} = O\left(\frac{1}{\nu+1}\right),$$

we get

$$H_m^{(2)} = O(1) \sum_{\nu=1}^m \frac{1}{\nu \log^2(\nu+1)} = O(1) \quad (m \rightarrow \infty);$$

and this completes the proof.

REMARKS. In view of Corollary 1, one is naturally led to determine suitable sequences  $\{\varepsilon_n\}$  such that  $g(t) \in BV[0, \pi] \Rightarrow \sum \varepsilon_n A_n(x) \in |N, 1/(n+1)|$ . But in view of Theorem 3 it is enough to determine the sequence of factors  $\{\varepsilon_n\}$  such that  $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in |N, 1/(n+1)|$  whenever  $\sum_{n=1}^m |\tau_n|/n = O(\log \log m)$ . Such a result is contained in the more general result of Das [5].

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