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The paper is devoted partly to the study of non-absolute Nörlund summability of Fourier series of $\varphi(t)$ under the condition $\varphi(t)\chi(t) \in AC[0, \pi]$ for suitable $\chi(t)$. The other aspect is to determine the order of variation of the Harmonic mean of the Fourier series whenever $\varphi(t) \log k/t \in BV[0, \pi]$.

1. Let L denote the class of all real functions f with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$ and let the Fourier series of $f \in L$ be given by

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) ,$$

assuming, as we may, the constant term to be zero.

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$

$$g(n, t) = \int_0^t \frac{\cos nu}{\chi(u)} du$$

$$h(n, t) = \int_t^{\pi} \frac{\cos nu}{\chi(u)} du .$$

Let $\{p_n\}$ be a sequence of constants such that $P_n = \sum_{v=0}^{n} p_v \neq 0$ $(n \geqq 0)$ and $P_{-1} = p_{-1} = 0$. For the definition of absolute Nörlund or (N, p) method, see, for example, Pati [9]. When $\sum_{n=0}^{\infty} a_n$ is absolutely (N, p) summable, we shall write, for brevity, $\sum_{n=0}^{\infty} a_n \in |N, p|$.

We define the sequence of constants $\{c_n\}$ formally by $(\sum_{n=0}^{\infty} p_n x^n)^{-1} = \sum_{n=0}^{\infty} c_n x^n$, $c_{-1} = 0$.

2. One of the objects of this paper is to study the non-absolute (N, p) summability factors of Fourier series and generalize the following outstanding result of Pati in Theorems 1-2. Besides, the proof of Theorems 1-2 are short and simple and avoids the direct technique of Pati which is somewhat long and complicated.

If we write

$$G = \left\{f\colon f \in L,\ \varphi(t) \log k/t \in AC[0, \pi] \text{ and } \sum_{n=1}^{\infty} A_n(x) \notin \left|N, \frac{1}{n+1}\right|\right\}$$

then Pati's theorem is in the following form:

THEOREM P [9]. *G is nonempty.*

Mohanty and Ray [8] subsequently constructed an example of $f \in G$.

We now establish

THEOREM 1. *Let χ be a real differentiable function and $\{\varepsilon_n\}$ be a sequence satisfying the following conditions*:

$$\phi(t)\chi(t) \in AC[0, \pi] , \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |g(n, \pi)| < \infty , \tag{2}$$

$$\frac{|\chi^1(t)|}{\chi^2(t)} \nearrow \text{ as } t \searrow 0 , \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n^2 |P_n|} \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} < \infty , \tag{4}$$

$$\sum_{n=1}^{\infty} \left| \Delta\left(\frac{\varepsilon_n}{nP_n}\right)\right| < \infty , \tag{5}$$

$$\varepsilon_n = 0(nP_n) , \tag{6}$$

(7) *$\exists$ a set E: $mE > 0$ and $\exists$ a constant $\eta > 0$ such that $\chi(t)^{-1} > \eta$ $\forall t \in E$.*

Then

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |A_n(t)| = \infty \qquad (\forall t \in E) , \tag{8}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} = \infty . \tag{9}$$

Now, if we denote, $G^* = \{f: f \in L,$ conditions (1) through (7) and (9) hold and $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin |N, p|\}$ then we establish

THEOREM 2. *Let*

$$\sum_{v=0}^{n} |p_v| = O(|P_n|) , \quad \sum_{n=0}^{\infty} |c_n| < \infty . \tag{10}$$

Then G^* is nonempty.

In § 3, we discuss some special cases of interest of Theorem 2.

Since Theorem 2 implies that the total variation of the (N, p) mean of the series $\sum_{n=1}^{\infty} \varepsilon_n A_n(x)$ is unbounded, the natural question now is to determine the order of the variation. And this is achieved in Theorem 3 in § 4.

3. We need the following lemmas for the proof of Theorem 1.

LEMMA 1. (2) *Suppose that $\{f_n(x)\}$ is measurable in (a, b) where $b - a \leqq \infty$, for $n = 1, 2, \cdots$. Then a necessary and sufficient condition that, for every function $\psi(x)$ integrable in the sense of Lebesgue over (a, b), the functions $f_n(x)\psi(x)$ should be integrable L over (a, b) and*

$$\sum_{n=1}^{\infty} \left| \int_a^b \psi(x) f_n(x) dx \right| \leqq K$$

is that

$$\sum_{n=1}^{\infty} |f_n(x)| \leqq K ,$$

where K is an absolute constant for almost every x in (a, b).

LEMMA 2. *Let condition* (3) *hold. Then*

$$h(n, t) = \frac{\sin nt}{n\chi(t)} + O\left(\frac{1}{n^2}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} .$$

Proof. We have, by integration by parts, and second mean value theorem,

$$\begin{aligned} h(n, t) &= \left(\int_{\pi/n}^{\pi} - \int_{\pi/n}^{t} \right) \frac{\cos nu}{\chi(u)} du \\ &= \frac{\sin nt}{n\chi(t)} + \frac{1}{n} \left(\int_{\pi/n}^{\pi} - \int_{\pi/n}^{t} \right) \frac{\chi^1(u)}{\chi^2(u)} \sin nudu \\ &= \frac{\sin nt}{n\chi(t)} + O\left(\frac{1}{n}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} \left(\int_{\pi/n}^{\zeta^1} - \int_{\pi/n}^{\zeta} \right) \sin nudu \\ &= \frac{\sin nt}{n\chi(t)} + O\left(\frac{1}{n^2}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} , \end{aligned}$$

where $\pi/n \leqq \zeta \leqq \pi$, $\pi/n \leqq \zeta^1 \leqq t$.

This completes the proof.

Proof of Theorem 1. We have, by integration by parts,

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos ntdt = F(0) g(n, \pi) + \int_0^{\pi} F'(t) h(n, t) dt ,$$

where $F(t) \equiv \phi(t)\chi(t)$. Hence by condition (2) the statement (8) is equivalent to proving that:

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} \left| \int_0^{\pi} F'(t) h(n, t)\, dt \right| = \infty \qquad (\forall t \in E) . \tag{11}$$

Since, by hypothesis (1)

$$\int_0^{\pi} | F'(t) | dt < \infty ,$$

by Lemma 1, the statement (11) is equivalent to proving that $\exists$ a set E: $mE > 0$ and

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} | h(n, t) | = \infty \qquad (\forall t \in E) . \tag{12}$$

Whenever conditions (3) and (4) hold, by virtue of Lemma 2, the statement (12) is easily seen to be equivalent to proving that

$$M(t) = \frac{1}{|\chi(t)|} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n |P_n|} | \sin nt | = \infty \qquad (\forall t \in E) . \tag{13}$$

Now, since

$$| \sin nt | \geqq \sin^2 nt = \frac{1}{2}(1 - \cos 2nt) ,$$

we have

$$M(t) \geqq \frac{1}{2\chi(t)} \left(\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n |P_n|} - \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n |P_n|} \cos 2nt \right) .$$

Using conditions (5) and (6) and using Dedekind's theorem we observe that the series

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n |P_n|} \cos 2nt$$

is convergent for $0 < t < \pi$. Hence (13) is equivalent to showing that

$$\frac{1}{\chi(t)} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n |P_n|} = \infty \qquad (\forall t \in E) . \tag{14}$$

Now the result follows from (14) by using the conditions (7) and (8).

Proof of Theorem 2. Das [4], in particular, proved that whenever condition (10) holds, then

$$\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in | N, p | \Longrightarrow \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} | A_n(x) | < \infty .$$

Now the result follows from Theorem 1.

4. In this section we apply Theorem 2 to some special cases. If we take $\chi(t) = \log k/t$, $E = \{t: k/e \leqq t < \pi\}$ we get

COROLLARY 1. *Let* $\{\varepsilon_n\}$ *satisfy the conditions*:
(i) $\varepsilon_n = O(\log n)$,
(ii) $\sum_{n=1}^{\infty} |\varepsilon_n|/n \log^3(n+1) < \infty$,
(iii) $\sum_{n=1}^{\infty} |\Delta\varepsilon_n|/n \log(n+1) < \infty$,
(iv) $\sum_{n=1}^{\infty} |\varepsilon_n|/n \log(n+1) = \infty$.
Then

$$\varphi(t) \log k/t \in AC[0, \pi] \Longrightarrow \sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin \left| N, \frac{1}{n+1} \right| .$$

Proof. Since the Fourier series of the even periodic function $(\log k/|t|)^{-1}$ is absolutely convergent (see Mohanty [7]) we get that

$$\sum_{n=1}^{\infty} \left| \int_0^{\pi} \frac{\cos nu}{\log k/u} du \right| < \infty . \tag{15}$$

It may be observed that if we take $\varepsilon_n = 1$, $p_n = 1/(n+1)$ in Corollary 1, then we get Theorem P.

COROLLARY 2. *Let* $\varphi(t) \in BV[0, \pi]$ *and let conditions* (5), (6), *and* (9) *hold. Then* $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin |N, p|$.

Take $\chi(t) \equiv 1$, $E = [0, \pi]$ in Theorem 2. In this case $g(n, \pi) = 0$.

REMARK. Corollary 2 in the case $\varepsilon_n = 1$ gives that

$$\varphi(t) \in BV[0, \pi] \Longrightarrow \sum_{n=1}^{\infty} A_n(x) \notin \left| N, \frac{1}{n+1} \right| .$$

This interalia establishes the result that $\varphi(t) \in BV[0, \pi]$ is *not sufficient* to guarantee the absolute convergence of the series $\sum_{n=1}^{\infty} A_n(x)$. See Bosanquet (1) who showed this by taking an example.

5. Throughout this section we consider the case $p_n = 1/(n+1)$ only. We write t_n and τ_n respectively for the $(N, 1/(n+1))$ means of the sequences $\{\sum_{v=1}^{n} \varepsilon_v A_v(x)\}$ and $\{n\varepsilon_n A_n(x)\}$. It follows from a result of Das [4] Theorem 6 on general infinite series that

$$\sum_{n=1}^{m} \frac{|\tau_n|}{n} = O(1) \text{ if and only if } \sum_{n=1}^{m} |t_n - t_{n-1}| = O(1) . \tag{16}$$

Proceeding as in the proof of above result we in fact get that for any positive nondecreasing sequence $\{\lambda_n\}$

(17) $$\sum_{n=1}^{m} \frac{|\tau_n|}{n} = O(\lambda_m) \text{ if and only if } \sum_{n=1}^{m} |t_n - t_{n-1}| = O(\lambda_m) .$$

Since Theorem P implies that the variation of $\{t_n\}$ is of unbounded order, we are immediately confronted with the problem of determining the order of variation of $\{t_n\}$. Because of relation (17) this problem simplifies to determining the order of $\sum_{n=1}^{m} |\tau_n|/n$ and this is achieved in

THEOREM 3. *If* $g(t) \equiv \varphi(t) \log k/t \in BV[0, \pi]$. *Then*

$$\sum_{n=1}^{m} \frac{|\tau_n|}{n} = O(\log \log m) .$$

Proof. We have

$$\tau_n = \frac{2}{\pi P_n} \sum_{\nu=1}^{n} p_{n-\nu}\nu \int_0^\pi \varphi(t) \cos \nu t dt .$$

Since

$$\int_0^\pi \varphi(t) \cos \nu t dt = g(0) \int_0^\pi \frac{\cos \nu t}{\log k/t} dt + \int_0^\pi dg(t) \int_t^\pi \frac{\cos \nu u}{\log k/u} du ,$$

we get

$$\begin{aligned}\sum_{n=1}^{m} \frac{|\tau_n|}{n} &\leqq \frac{2}{\pi} |g(0)| \sum_{n-1}^{m} \frac{1}{nP_n} \left| \int_0^\pi \frac{dt}{\log k/t} \left(\sum_{\nu=1}^{n} p_{n-\nu}\nu \cos \nu t \right) \right| \\ &\quad + \frac{2}{\pi} \int_0^\pi |dg(t)| \sum_{n=1}^{m} \frac{1}{nP_n} \left| \int_t^\pi \frac{dt}{\log k/t} \left(\sum_{\nu=1}^{n} p_{n-\nu}\nu \cos \nu t \right) \right| \\ &= \frac{2}{\pi} \{ |g(0)| G_m + H_m \} .\end{aligned}$$

Since the series $\sum_{n=1}^{\infty} \int_0^\pi \cos nu/\log k/u \, du$ is absolutely convergent (see (15)) and therefore it is absolutely $(N, 1/(n+1))$ summable, we get that $G_m = O(1)$ by using relation (16).

Since $\int_0^\pi |dg(t)| < \infty$, using Lemma 2 with $\log k/t$ in place of $\chi(t)$ we get that

$$\begin{aligned}H_m &= O(1) \sum_{n=1}^{m} \frac{1}{n \log (n+1)} \left| \sum_{\nu=1}^{n} \frac{\sin \nu t}{n - \nu + 1} \right| \\ &\quad + O(1) \sum_{n=1}^{m} \frac{1}{n \log (n+1)} \sum_{\nu=1}^{n} \frac{1}{(n - \nu + 1) \log^2 (\nu + 1)} = H_m^{(1)} + H_m^{(2)} .\end{aligned}$$

By a result of McFadden ([6], Lemma 5.10) we get

$$\sum_{\nu=1}^{n} \frac{\sin \nu t}{n - \nu + 1} = O(\log \tau), \ (\tau = [k/t])$$

and consequently

$$H_m^{(1)} = O(1)\frac{\log \tau}{\log k/t} \sum_{n=1}^{m} \frac{1}{n \log (n+1)} = O(\log \log m) .$$

On change of order of summation in $H_m^{(2)}$ and by use of the fact that

$$\sum_{n=\nu}^{m} \frac{1}{(n-\nu+1) n \log (n+1)} = O\left(\frac{1}{\nu+1}\right),$$

we get

$$H_m^{(2)} = O(1) \sum_{\nu=1}^{m} \frac{1}{\nu \log^2 (\nu+1)} = O(1) \qquad (m \longrightarrow \infty) ;$$

and this completes the proof.

REMARKS. In view of Corollary 1, one is naturally led to determine suitable sequences $\{\varepsilon_n\}$ such that $g(t) \in BV[0, \pi] \Rightarrow \sum \varepsilon_n A_n(x) \in |N, 1/(n+1)|$. But in view of Theorem 3 it is enough to determine the sequence of factors $\{\varepsilon_n\}$ such that $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in |N, 1/(n+1)|$ whenever $\sum_{n=1}^{m} |\tau_n|/n = O(\log \log m)$. Such a result is contained in the more general result of Das [5].

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