STRICTLY LOCAL SOLUTIONS OF DIOPHANTINE EQUATIONS

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For any system $f$ of Diophantine equations, there exist positive integers $C(f)$, $D(f)$ with the following properties:
For any nonnegative integer $n$, for any prime $p$, if $v$ is the $p$-adic valuation, and if a vector $x$ of integers satisfies the inequality

$$v(f(x)) > C(f)n + v(D(f))$$

then there is an algebraic $p$-adic integral solution $y$ to the system $f$ such that

$$v(x - y) > n.$$ 

This theorem is proved by techniques of algebraic geometry in the more general setting of Noetherian domains of characteristic zero. When $f$ is just a single equation, the method of Birch and McCann gives an effective determination of $C(f)$ and $D(f)$.

Let $R$ be a Noetherian integral domain, $K$ its field of fractions. We will consider Henselian discrete valuation rings $R_v$ (see [4]) containing $R$, where $v$ is the valuation normalized so that $v(R_v)$ is the set of nonnegative integers (plus $\infty$). If $f = (f_1, \ldots, f_r)$ is a system of $r$ polynomials in $s$ variables with coefficients in $R$, and $x$ is an $s$-tuple with coordinates in an extension ring of $R$, we set $f(x) = (f_1(x), \ldots, f_r(x))$. We define the valuation of an $r$-tuple (or $s$-tuple) to be the minimum of the valuations of its components.

**THEOREM.** Assume $R$ has characteristic zero. For each system $f$ of polynomials with coefficients in $R$, there exists an integer $C(f) \geq 1$ and an element $D(f) \neq 0$ in $R$ with the following property: For any Henselian discrete valuation ring $R_v$ containing $R$, and any nonnegative integer $n$, if an $s$-tuple $x$ with components in $R$ satisfies the inequality

$$v(f(x)) > C(f)n + v(D(f))$$

then there is a zero $y$ of $f$ in $R_v$ such that

$$v(x - y) > n.$$ 

In particular, if $R$ is the ring of algebraic integers in a number field, and we take $n = 0$, $S = \text{set of primes dividing } D(f)$, then we recover Greenleaf's theorem [3] to the effect that if $p \notin S$, then every
zero of \( f \) mod \( p \) may be refined to an actual zero of \( f \) in the \( p \)-adic integers — in fact, to an actual zero of \( f \) in the algebraic \( p \)-adic integers. The theorem above strengthens Greenleaf's result by giving information about the exceptional primes \( p \in S \) and by providing a precise linear estimate of how close the actual zero \( y \) is to the approximate zero \( x \). The hypothesis that \( R \) have characteristic zero is required by Greenleaf's counterexample ([3], p. 30).

**Proof.** Let \( fR[X] \) be the ideal in the polynomial ring \( R[X, \ldots, X] \) generated by \( f_1(X), \ldots, f_r(X) \), and let \( V \) be the algebraic set in affine \( s \)-space over \( K \) which is the locus of zeroes of \( f \).

**Step 1.** We may assume \( fR[X] \) is equal to its own radical. For let \( g \) be a system of polynomials generating the radical, and suppose the \( m \)th power of the radical is contained in \( fR[X] \). If \( C(g) \) and \( D(g) \) are invariants for \( g \), set
\[
C(f) = mC(g) \quad \text{and} \quad D(f) = D(g)^m.
\]
Then inequality (1) implies that for any polynomial \( h \in fR[X] \), say \( h = h_1f_1 + \cdots + h_rf_r \), we have
\[
v(h(x)) \geq \min_{i} [v(h_i(x)) + v(f_i(x))] \geq \min_{i} v(f_i(x)) = v(f(x)) > C(f)n + v(D(f)).
\]
In particular, for \( h = g_j^n \), with \( g_j \) in \( g \), we get
\[
mv(g_j(x)) > m[C(g)n + v(D(g))]
\]
so that there is a zero \( y \) of \( g \) in \( R \) such that
\[
v(x - y) > n.
\]
Since \( y \) is also a zero of \( f \), we have found the invariants for \( f \).

**Step 2.** Granted that \( fR[X] \) is its own radical, we may further assume \( fR[X] \) is a prime ideal. Otherwise, it is an intersection of finitely many prime ideals, so by induction on the number of these, we may assume \( fR[X] \) is the intersection of two ideals generated by systems \( g, g' \) for which invariants \( C(g), C(g'), D(g), D(g') \) have already been found. We set
\[
C(f) = \max (2C(g), 2C(g'))
\]
\[
D(f) = D(g)^{2k}D(g')^{2k}.
\]
Then for each \( g_i \in g \) and \( g'_j \in g' \), we have \( g_ig'_j \in fR[X] \), so that as before, inequality (i) implies
Suppose that for one index \( j \), \( v(g_j(x)) < 1/2 \ v(f(x)) \). Fixing that \( j \) and letting \( i \) vary, we get \( v(g_i(x)) > 1/2 \ v(f(x)) \) for all indices \( i \), so that

\[
v(g(x)) > \frac{1}{2} [C(f)n + v(D(f))].
\]

By definition of \( C(f) \) and \( D(f) \), the term on the right is at least as big as \( C(g)n + v(D(g)) \), so that there is a zero \( y \) of \( g \) — a fortiori of \( f \) — in \( R_v \) such that \( v(x - y) > n \). If, on the other hand, \( v(g'(v)) \geq 1/2 \ v(f(x)) \), the same argument gives a zero \( y \) of \( g' \) — a fortiori of \( f \) — in \( R_v \) such that \( v(x - y) > n \).

**Step 3.** Assuming \( fR[X] \) is a prime ideal, we proceed by induction on the dimension \( m \) of the irreducible \( K \)-variety \( V \). If \( V \) is empty, let \( D(f) \) be any nonzero constant in \( fR[X] \), and let \( C(f) = 1 \). Then the inequality (1) is never satisfied for any \( n \), \( v \), and \( x \), so the theorem is vacuously true. Assume now that \( V \) is nonempty and the theorem established in dimensions less than \( m \). Let \( J \) be the Jacobian matrix of \( f \), \( \Delta \) the system of minors \( \Delta_{ij} \) of order \( s - m \) taken from \( J \). Since the characteristic is zero, the locus of common zeros of \( \Delta \) and \( f \) is a proper \( \Gamma \)-closed subset of \( V \) (the singular locus); by inductive hypothesis, there are invariants \( C', D' \) for the system \( \Delta \) plus \( f \).

If \( (i) \) is a collection of \( s - m \) indices \( \leq r \), \( f_{(i)} \) the corresponding system of \( s - m \) polynomials taken out of \( f \), let \( V_{(i)} \) be the algebraic set of zeros of \( f_{(i)} \) and let \( W_{(i)} \) be the union of the \( K \)-irreducible components of \( V_{(i)} \) which have dimension \( m \) and are different from \( V \). Let \( g_{(i)} \) be a system of generators for the ideal of \( W_{(i)} \) in \( R[X] \); by inductive hypothesis, there are invariants \( C_{(i)}, D_{(i)} \) for the system \( g_{(i)} \) plus \( f \) (since \( V \cap W_{(i)} \) is its locus). The results of Zariski (Trans. A.M.S. 62 (1947), pp. 14 and 28–29) tell us that if \( x \) is a point of \( V_{(i)} \) such that for some \( (j) \)

\[
\Delta_{(i)(j)} \neq 0
\]

then \( x \) lies on exactly one component of \( V_{(i)} \), that component having dimension \( m \).

We now set

\[
C(f) = C' + \max \{C', C_{(k)} \ \text{all} \ (k)\}
\]

\[
D(f) = (D')^i \prod_{(k)} D_{(k)}
\]

so that \( v(D(f)) \geq v(D') + \max \{v(D'), v(D_{(k)}) \ \text{all} \ (k)\} \). Assuming inequality (1), we then have three possibilities:
I. \( v(\Delta(x)) > C'n + v(D') \). By inductive hypothesis, there is a singular zero \( y \) of \( f \) in \( R_v \) such that \( v(x - y) > n \).

II. For some \( (i) \), \( v(g_{(i)}(x)) > C_{(i)}n + v(D_{(i)}) \). By inductive hypothesis, there is a zero \( y \) of \( f \) in \( R_v \) (lying on \( V \cap W_{(i)} \)) such that \( v(x - y) > n \).

III. For some \( (i) \) and \( (j) \),
\[
v(\Delta_{(i)(j)}(x)) \leq C'n + v(D')
\]
and for every \( (k) \), there is a polynomial \( \gamma_{(k)} \) in the system \( g_{(k)} \) for which
\[
v(\gamma_{(k)}(x)) \leq C_{(k)}n + v(D_{(k)}) .
\]
By Hensel’s Lemma, there is a zero \( y \) of the system \( f_{(i)} \) in \( R_v \) such that
\[
v(y - x) > \max \{C'n + v(D'), C_{(i)}n + v(D_{(i)}) \text{ all } k\} .
\]
In that case \( g_{(k)}(y) \neq 0 \), for all \( (k) \), since
\[
v(\gamma_{(k)}(y)) = v(\gamma_{(k)}(x)) .
\]
Thus \( y \in W_{(k)} \) for any \( (k) \). As we also have
\[
\Delta_{(i)(j)}(y) \neq 0
\]
y must lie on \( V \), so \( y \) is a zero of \( f \).

Note 1. In the last part of the above argument we used a version of Hensel’s Lemma which is a strengthening of Lemma 2, p. 63 of [2]. It says that if \( R_v \) is a Henselian discrete valuation ring with maximal ideal \( m \), \( F \) a system of \( r \) polynomials in \( s \) variables with coefficients in \( R_v \), \( r \leq s \), \( J \) its Jacobian matrix, \( x \in R_v^r \), \( a \in R_v \) so that
\[
F(x) \equiv 0 \pmod{ae^m}
\]
where \( e = D(x) \), \( D \) being a minor of order \( r \) taken from \( J \), then there exists \( y \in R_v^r \) such that \( F(y) = 0 \) and
\[
y \equiv x \pmod{ae^m} .
\]
(Since \( h = v(a) \) is an arbitrary integer, we have applied this lemma by taking \( F = f_{(i)} \) and
\[
h = \max \{C'n + D', C_{(k)}n + D_{(k)} \text{ all } k\} - v(\Delta_{(i)(j)}(x))
\]
in part III above.) The idea for proving this stronger Hensel’s Lemma is the same as in [2], pp. 63–64, reducing to the case \( r = s \), applying Taylor’s formula to \( F(aeX) \), obtaining \( F(aeX) = aeJ(0)H(X) \), and if \( y' \in m' \) is zero of \( H \) as in Lemma 1 of [2], then \( y = ae y' \) is the zero we seek.
Note 2. Birch and McCann [1] proved the special case of the theorem where $R$ is a unique factorization domain, and $f$ is a single polynomial (in several variables). Their method has the advantage of providing an effective (but impractical) method of calculating $D(f)$ when $f$ is a single polynomial. If $f$ involves $s$ variables, they use the notation $D_s(f)$ because their invariant is constructed by induction on $s$. They omit the definition of $C_s(f) = C(f)$, which can be given inductively on $s$ as follows: If $s = 1$, $C_s(f) = d(f)$, where $d(f)$ is the degree of $f$. If $s > 1$, denote by $f_i$ the polynomial $f$ regarded as having coefficients in $R[X_i]$ and involving the other $s - 1$ variables. Then

$$C_s(f) = \max_{1 \leq i \leq s} \{C_{s-i}(f_i) + d(D_{s-i} f_i)\}$$

with $d(D_{s-i} f_i)$ being the degree in $X_i$ of $D_{s-i} f_i \in R[X_i]$.

The proof by Birch and McCann then goes by induction on $s$. However, there is an error in the inductive step (their equation $D_{s-1}(\phi) = g(a_i)$ does not always hold, as is shown by the polynomial $f(X_1, X_2) = X_1^3 - X_2^3$, with $a_i = 0$, where $g_i(a_i) = 0$ while $D_i(\phi) = 1$). This error can be rectified by proving the following result and its corollary, since the inequality in the corollary is all they really need for their argument.

**Specialization Theorem.** Let $R$ be a unique factorization domain of characteristic zero. Given $f \in R[X_0, X_1, \ldots, X_s]$ and $a_o \in R$. Denote by a bar the specialization obtained by substituting $a_o$ for $X_0$. Let $f_o$ be $f$ regarded as a polynomial in the variables $X_1, \ldots, X_s$ with coefficients in $R[X_o]$. Let $D_s f_o \in R[X_o] and D_s f_0 \in R$ be the invariants defined by Birch-McCann. If

$$D_s f_o \neq 0$$

then $D_s f_o$ is divisible by $D_s f_0$ and they have the same irreducible factors.

**Corollary.** For any valuation $v$ nonnegative on $R$,

$$v(D_s f_o) \leq v(D_s f_0).$$

2. Proof of the specialization theorem and the main theorem for the invariant of Birch-McCann. Recall how $D_s(f)$ is defined: For any polynomial $g$ in one variable, $A(g)$ is the leading coefficient of $g$, $d(g)$ is its degree, and

$$rg = g/(g, g')$$
where \((g, g')\) is the greatest common divisor of \(g\) and its derivative \(g'\). Thus \(rg\) is the primitive polynomial having the same roots as \(g\) but all taken with multiplicity one. \(\Delta(g)\) is the discriminant of \(g\); if \(g\) has the linear factorization

\[ g(X) = A(g) \prod_{i=1}^{d} (X - \alpha_i) \]

then

\[ \Delta(g) = A^{2(d-1)} \prod_{i < j} (\alpha_i - \alpha_j)^2. \]

Suppose \(f\) is a polynomial in \(s\) variables \(X_1, \ldots, X_s\) and \(g_i\) is a polynomial in \(X_i\) only. Let \(d(g_i) = d_i\), and let \(\alpha_{i,j}\) with \(1 \leq j \leq d_i\), be the roots of \(g_i\), counted with their multiplicities. Then the eliminant

\[ E(Z) = E(f; g_1, \ldots, g_s)(Z) \]

is the polynomial in \(Z\) of degree \(d(E) = \prod d\), given by

\[ E(Z) = \prod A(g_i)^{d(E)\delta_i} \prod_{(j)} (Z - f(\alpha_{i,j}, \ldots, \alpha_{i,j})). \]

Inductively, \(D_s(f)\) is then defined as follows: If \(s = 1\), \(D_1(f) = A(f)^{d-1}d^2\Delta(rf)^d\). If \(s > 1\), set \(g_s = D_{s-1}(f_s)\), where \(f_s\) has been defined before as \(f\) regarded as a polynomial in the \(s - 1\) variables other than \(X_s\); let \(E\) be \(E(f; g_1, \ldots, g_s)\). Then

\[ D_s(f) = \begin{cases} \prod_i D_i(g_i)(A(E)^{d(E)}E(0))^{d(s_i)} & \text{if } E(0) \neq 0 \\ \prod_i D_i(g_i)D_i(E)^{d(s_i)} & \text{if } E(0) = 0. \end{cases} \]

We will prove the Specialization Theorem by induction on \(s\).

**Case \(s = 1\).** Let \(f_s(X_s) = A(f_0)X_s^d + \cdots\), and let \((rf_0)(X_i) = A(rf_0)X_s^d + \cdots\), so that \(\delta \leq d\) and \(A(rf_0)\) divides \(A(f_0)\). Since by hypothesis \(D_1f_0 \neq 0\), we have \(\overline{A(f_0)} \neq 0\), so \(\overline{A(f_0)} = \overline{A(f_0)}\) and \(f_0\) has the same degree \(\delta\) in \(X_s\). Also \(\overline{A(rf_0)} = \overline{A(rf_0)} \neq 0\), so \(\overline{rf_0}\) has the same degree \(\delta\) and only simple roots, but may not be primitive. Let \(c\) be the greatest common divisor of the coefficients of \(\overline{rf_0}\); then \(\overline{rf_0} = c(\overline{rf_0})\). Now \(\overline{A(rf_0)}\) is homogeneous of degree \(2(\delta - 1)\) in the coefficients of \(\overline{rf_0}\). Thus

\[ \overline{D_1f_0} = \overline{A(f_0)}^{(d-1)d^2\Delta(c(\overline{rf_0}))^d} = c^{2d(d-1)}\overline{D_1f_0}. \]

The theorem then follows from the fact that \(c\) divides \(A(\overline{rf_0})\) which divides \(A(\overline{f_0})\) which divides \(D_1\overline{f_0}\).

To carry out the induction, we will need to strengthen our result for \(s = 1\) with the following lemma.

**Lemma 1.** Let \(g, h\) be polynomials in one variable \(Y\) which satisfy
\[ g = c_1^{k_1} \cdots c_i^{k_i} h \]

with each \( c_i \) dividing \( h \), and \( k_i \geq 1 \). Then \( D_t g \) and \( D_t h \) satisfy the same type of relationship:

\[ D_t g = C_1^{n_1} \cdots C_i^{n_i} D_t h \]

with each \( C_i \) dividing \( D_t h \).

**Proof.** Let \( e = \text{degree } h \), \( \gamma_i = \text{degree } c_i \), so that degree \( g = \varepsilon = e + \sum_i k_i \gamma_i \), and

\[ A(g) = A(c_i)^{k_1} \cdots A(c_i)^{k_i} A(h) \, . \]

Since each \( c_i \) divides \( h \), \( g \), and \( h \) have the same irreducible factors, so that \( rg = rh \). Hence

\[ D_t g = A(g)^{(e-1)\varepsilon} A(rg)^{\varepsilon} = (\prod_i A(c_i)^{k_i})^{(e-1)\varepsilon} A(h)^{\varepsilon} \Delta(rg)^{\varepsilon} \, . \]

Now \( D_t h = A(h)^{(e-1)\varepsilon} \Delta(rg)^{\varepsilon} \), and if we write \((e - 1)e^2 = (e - 1)e^2 + m\) we get

\[ D_t g = (\prod_i A(c_i)^{k_i})^{(e-1)\varepsilon} A(h)^{\varepsilon} \Delta(rg)^{\varepsilon} D_t h \, . \]

Since \( A(c_i) \), \( A(h) \), \( \Delta(rg) \) each divide \( D_t h \), the lemma is proved.

The inductive step: By definition,

\[ D_i f_o = \sum_{i=1}^i D_i(g_i) M^{d(g_i)} \]

\[ D_i f_o = \sum_{i=1}^i D_i(g_i) M^{*d(g_i)} \]

where \( g_i = D_{i-1} f_o \), \( f_o \) being \( f_o \) regarded as a polynomial in the variables \( X_j \) with \( j \neq i \), \( j \geq 1 \) (so that the coefficients of \( f_o \) are polynomials in \( X_0 \) and \( X_i \)); \( g_i = D_{i-1}(f_0) \) is defined similarly. Also,

\[ M = \begin{cases} A(E)^d E(0) & \text{if } E(0) \neq 0 \\ D_i(E) & \text{if } E(0) = 0 \end{cases} \]

where \( E = E(f_o; g_j, \cdots, g_i) \); and

\[ M^* = \begin{cases} A(E^*)^d E^*(0) & \text{if } E^*(0) \neq 0 \\ D_i(E^*) & \text{if } E^*(0) = 0 \end{cases} \]

where \( E^* = E(f_o; g_i^*, \cdots, g_i^*) \). Our hypothesis is \( D_i f_o \neq 0 \), so that \( D_i(g_i) \neq 0 \) for all \( i \) and \( M \neq 0 \).

Since \( g_i \neq 0 \) (because \( A(g_i) \), which is a factor of \( D_i g_i \), is not zero), and \( f_o = (f_0)_i \), the inductive hypothesis provides us with \( c_i \in R[X_i] \) such that
\[ \overline{g}_i = c_i g_i^* \]

with each irreducible factor of \( c_i \) being a factor of \( g_i^* \). By Lemma 1,
\[ D_i \overline{g}_i = C_i D_i g_i^* \]

with each irreducible factor of \( C_i \) dividing \( D_i g_i^* \). The step \( n = 1 \)
already proved yields
\[ \overline{D}_i g_i = B_i D_i \overline{g}_i \]

with each irreducible factor of \( B_i \) dividing \( D_i \overline{g}_i \). Combining gives
\[ \overline{D}_i g_i = B_i C_i D_i g_i^* \]

so that \( \overline{D}_i g_i \) and \( D_i g_i^* \) have the same irreducible factors.

The condition \( A(\overline{g}_i) \neq 0 \) implies \( d(g_i) = d(\overline{g}_i) \), and since \( g_i^* \) divides \( \overline{g}_i \), \( d(\overline{g}_i) \geq d(g_i^*) \). As
\[ \overline{D}_i f_0 = \prod_{i=1}^{s} \overline{D}_i g_i \overline{M} d(\tau_i) \]

the theorem will be proved if we can show \( M^* \) divides \( \overline{M} \) and they have the same irreducible factors.

\( \overline{M} \) is the specialization of \( M \) and is given by the same formula
as \( M \) with the specialization \( \overline{E} \) of \( E \) taking the place of \( E \). Now
the function \( E \), like \( \Delta \), commutes with specialization, so we have
\[ \overline{E} = E(\overline{f}_0; \overline{g}_0, \cdots, \overline{g}_s) = E(f_0; c_i g_i^*, \cdots, c_s g_s^*) \]

Notice also that if \( E(0) \neq 0 \) so \( M = A(E)^{d(E)} E(0) \), \( \overline{M} \neq 0 \) implies \( \overline{A(E)} \neq 0 \), so \( \overline{A(E)} = A(\overline{E}) \), and \( \overline{E}(0) \neq 0 \), so \( \overline{E}(0) = 0 \). On the other hand, if
\( E(0) = 0 \), then \( M = D_i(\overline{E}) \), and \( \overline{M} \neq 0 \) implies again \( \overline{A(E)} \neq 0 \), so
again \( \overline{A(E)} = A(\overline{E}) \) and \( d(E) = d(\overline{E}) \).

The problem reduces to examining the relation between \( \overline{E} = E(\overline{f}_0; c_i g_i^*, \cdots, c_s g_s^*) \) and \( E^* = E(\overline{f}_0; g_i^*, \cdots, g_s^*) \) given that every root
of \( c_i \) is a root of \( g_i^* \).

Note first that \( A(E^*) = \prod A(g_i^*)^{\delta_i(\overline{f}_0)} \), where \( \delta_i = \prod_{j \neq i} d(g_j^*) \).
If \( \varepsilon_i = \prod_{j \neq i} (d(g_j^*) + d(c_j)) \), then write \( \varepsilon_i = \delta_i + \gamma_i \), so that
\[ A(\overline{E}) = A(E^*) \prod A(c_i)^{\delta_i(\overline{f}_0)} A(g_i^*)^{\varepsilon_i(\overline{f}_0)} \]

Since every irreducible factor of \( c_i \) is an irreducible factor of \( g_i^* \),
every irreducible factor of \( A(c_i) \) is an irreducible factor of \( A(g_i^*) \), so
the above expression shows that \( A(\overline{E}) \) and \( A(E^*) \) have the same irreducible factors.

Thus in the case where \( M = A(E)^{d(E)} E(0) \), we are reduced to proving that \( \overline{E}(0) \) is divisible by \( E^*(0) \) and they have the same irreducible factors. This will follow from the formula.
whose proof is an easy exercise. From this formula we see that the constant term of \( E(f; g_1, g_2, \ldots, g_s) \) is just a product of the constant terms of the various \( E(f; p_i, p_j, \ldots, p_s) \), where \( p_i \) runs through the irreducible factors of \( g_i \) for each \( i = 1, \ldots, s \). Hence \( E(0) \) is divisible by \( E^*(0) \) with the same irreducible factors.

Consider finally the case where \( M = D_1(E) \). Since \( E \) is divisible by \( E^* \) with the same irreducible factors, it follows from Lemma 1 that \( D_1(E) \) is divisible by \( D_1(E^*) \) with the same irreducible factors. The proof for the case \( s = 1 \) showed that \( D_1(E) \) is divisible by \( D_1(E) \) with the same irreducible factors.

Thus in both cases \( M \) is divisible by \( M^* \) with the same irreducible factors.

Having demonstrated the Specialization Theorem, we can now prove that the Birch-McCann invariant \( D_s(f) \) and the other invariant \( C_s(f) \) defined inductively by

\[
C_1(f) = d(f) \quad \text{if} \quad s = 1
\]

\[
C_s(f) = \max_{i \leq t \leq s} \{ C_{s-t}(f_i) + d(D_{s-t}f_i) \}
\]

satisfy our main theorem, if \( R \) is a unique factorization domain.

Proof. For \( s = 1 \) this is Birch-McCann’s Theorem with \( \mathbb{Z} \) and \( v_p \) replaced by \( R \) and \( R_\nu \). The proof goes over word-for-word because \( v \) has a unique extension to the algebraic closure of the field of fractions of \( R_\nu \) (as follows from Nagata, Local Rings, statement (30.5), p. 105). Notice also that in this case (\( s = 1 \)), the zero \( y = b \) is unique.

For \( s > 1 \), we proceed by induction on \( s \). Take \( f \in R[\{X_i \} \ldots, a \in R^{r+1} \}, \) and let \( \overline{f}_0(X_i, \ldots, X_s) = f(a_0, X_1, \ldots, X_s) \), and similarly denote throughout by a bar the result of substituting \( a_0 \) for \( X_0 \). Now \( D_s\overline{f}_0 \in R[\{X_i \} \ldots, \) so can be written \( g_s(X_0) \). Suppose

\[
v(f(a)) > C_s(\overline{f}_0)n + v(D_s\overline{f}_0).
\]

Then the inductive hypothesis gives us a zero \( b \in R_\nu \) of \( \overline{f}_0 \) such that \( v(a_i - b_i) > n \) for \( i = 1, \ldots, s \); hence \( (a_0, b_1, \ldots, b_s) \) is the required zero for \( f \). Otherwise

\[
v(f(a)) \leq C_s(\overline{f}_0)n + v(D_s\overline{f}_0).
\]

In this inequality we propose to replace \( C_s(\overline{f}_0) \) by \( C_s(f_0) \) and \( D_s\overline{f}_0 \) by \( D_s\overline{f}_0 = g_s(a_0) \). If \( g_s(a_0) = 0 \), we get infinity on the right side. So
suppose \( g_0(a_0) \neq 0 \). Then by the corollary to the Specialization Theorem, \( v(D_s f_0) \leq v(D_s f_0) = v(g_0(a_0)) \). We need

**Addendum to Specialization Theorem.** Under the same hypotheses, \( C_s(f_0) \leq C_s(f_0) \).

Proof by induction on \( s \): For \( s = 1 \), \( C_1 \) is just the degree in the variable \( X_1 \), which stays the same or decreases under specialization. Assume the result for \( s - 1 \). Then \( C_{s-1}(f_0) \leq C_{s-1}(f_0) \) for all \( i = 1, \ldots, s \). In the notation of the proof of the Specialization Theorem,

\[
\bar{g}_i = c_i g_i^2 = c_i D_{s-1} f_0, \quad \text{so that}
\]

\[
d(D_{s-1} f_0) = d(g_i) = d(g_i) = d(D_{s-1} f_0).
\]

So by definition of \( C_s \), \( C_s(f_0) \leq C_s(f_0) \), proving the addendum.

We have thus obtained, arguing with respect to any other variable \( X_i \) as we have for \( X_0 \), the inequality

\[
(2) \quad v(f(a)) \leq C_s(f_0) n + v(g_0(a_0))
\]

for all \( i = 0, 1, \ldots, s \). Combining with our hypothesis (1) on \( v(f(a)) \), with \( a = x \), we obtain

\[
(3) \quad [C_{s+1}(f) - C_s(f)] n + v(D_{s+1} f) < v(g_0(a_0))
\]

for all \( i = 0, 1, \ldots, s \), where by definition of \( C_{s+1}(f) \), the coefficient of \( n \) in the left side is nonnegative, hence

\[
(4) \quad v(D_{s+1} f) < v(g_0(a_0))
\]

for all \( i = 0, 1, \ldots, s \).

Arguing exactly as in Birch-McCann, we next show that inequality (4) implies that for every root \( \alpha = (\alpha_0, \ldots, \alpha_s) \) of \( (g_0, \ldots, g_s) \) such that \( v(\alpha - \alpha) > v(M) \) — and there exist such roots by (4) and the theorem for 1 variable applied \( s + 1 \) times — we must have \( f(\alpha) = 0 \). Thus \( E(0) = 0 \), and hence \( M = D_1(E) \).

By definition of \( C_{s+1} \), the coefficient of \( n \) in inequality (3) is at least equal to \( d(g_i) \), and by definition of \( D_{s+1} \), we have \( v(D_{s+1} f) \geq v(D_i g_i) \) for all \( i \). So we can apply the theorem for one variable to obtain a unique zero \( \alpha_i \) of \( g_i \) such that \( v(\alpha_i - \alpha) > n \), for each \( i = 0, 1, \ldots, s \).

Applying the definition of \( D_{s+1} \) again and using inequality (4), we obtain

\[
d(g_i) v(M) + v(D_i g_i) < v(g_i(a_i))
\]

for all \( i \), hence by the theorem for one variable again there is a
unique zero $\beta_i$ of $g_i$ in $R_v$ such that $v(a_i - \beta_i) > v(M)$ for each $i$. Define

$$\gamma_i = \begin{cases} 
\alpha_i & \text{if } n \geq v(M) \\
\beta_i & \text{if } n \leq v(M) 
\end{cases}.$$ 

Then, as remarked before, we must have $f(\gamma) = 0$, which proves the theorem.

**References**


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