INNER FUNCTIONS UNDER UNIFORM TOPOLOGY

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The structure of the space $\mathcal{S}$ of all inner functions in the unit disc $D = \{ z : |z| < 1 \}$ under the metric topology induced by the $H^\infty$-norm is considered. It is proven that if two inner functions $p$ and $q$ belong to the same component of $\mathcal{S}$, then the variation of $p/q$ on each open arc of $\partial D$ (the boundary of $D$ in the complex plane $C$) where they can be continued analytically is bounded by a constant $C = C(p, q)$, independent of the arc. This criterion is used to show that a component of $\mathcal{S}$ can contain nothing but Blaschke products with infinitely many zeroes, exactly one (up to a constant factor) singular inner function or infinitely many pairwise coprime singular inner functions.

The reader is assumed to be familiar with the basic theory of the space $H^\infty$. We recall that the canonical form of an inner function is: $q = \lambda br = \lambda bds$, where $\lambda \in \partial D$ and

\begin{align*}
(1) \quad b(z) &= \prod_k b_k(z) = \prod_k b(z, a_k) = \prod_k \overline{a_k}/(a_k - z)/(1 - \overline{a_k}z), \\
(2) \quad r(z) &= e(z, \mu) = \exp \left\{ \int_0^{2\pi} (z + e^{i\theta})/(z - e^{i\theta})d\mu(\theta) \right\},
\end{align*}

where $\mu$ is a nonnegative singular Borel measure on $\partial D$; $r(z)$ is the singular part of $q$. If $\mu = \nu + \nu_1$, where $\nu$ and $\nu_1$ are the continuous and the purely atomic part, resp., of $\mu$, then

\begin{align*}
(3) \quad s(z) &= e(z, \nu) \text{ is the continuous singular part of } q, \quad \text{and} \\
(4) \quad d(z) &= e(z, \nu_1) = \prod_i d_i(z) = \prod_i d(z; \theta_j, l_j), \quad \text{where } d(z; \theta, l) = \exp \{l(z + e^{i\theta})/(z - e^{i\theta})\}, 0 \leq \theta < 2\pi \text{ and } l > 0; d(z) \text{ is the atomic singular part of } q. \quad \text{Clearly, } \sum_i l_i = \| \nu_1 \| < \infty; \quad \text{it will be assumed that} \\
\theta_j &\neq \theta_k, \text{ whenever } j \neq k.
\end{align*}

The subsets of $\mathcal{S}$ containing all those functions defined by the conditions (1), (2), (3), and (4) (or the constant multiples of such functions) will be denoted by $\mathcal{S}_b$, $\mathcal{S}_s$, $\mathcal{S}_c$, and $\mathcal{S}_a$, resp.

It is well-known that the inner function $q(z)$ can be continued analytically across (a neighborhood of) the point $\lambda \in \partial D$ if and only if $\lambda \in \text{Sp}(q) = \text{supp}(\mu) \cup \text{closure } \{a_k\}$; furthermore, if $\text{Sp}(q) \cap \partial D \neq \text{Sp}(p) \cap \partial D(p, q \in \mathcal{S})$, then $\| q - p \|_\infty = 2$, and $p$ and $q$ belong to different components of $\mathcal{S}$.

Let $\mathcal{S}_\Gamma$ (a closed subset of $\partial D$) be the set of all inner functions such that $\text{Sp}(q) \cap \partial D = \Gamma$. Clearly, $\mathcal{S}_\Gamma$ is the family of all Blaschke
products of finite order, and it is not difficult to see that \( \mathcal{I}_n = \bigcup_{n=0}^{\infty} \mathcal{I}_n \) (disjoint union!), where \( \mathcal{I}_n \) is the family of Blaschke products of order \( n \) and (for each \( n \)) \( \mathcal{I}_n \) is an arcwise connected open and closed component of \( \mathcal{F} \), topologically homeomorphic to \( \partial D \times D^* \) (see [2; 4]).

During the “Sesquicentennial Seminar in Operator Theory” (Indiana University, Bloomington, Indiana, 1969–1970), R. G. Douglas asked for a characterization of the components of \( \mathcal{I} \setminus \mathcal{I}_n \) containing a singular inner function; in particular, he asked whether or not every component of \( \mathcal{I} \setminus \mathcal{I}_n \) contains a singular inner function. Here we partially answer the first question. A first counterexample to the second question was given by D. Sarason; his unpublished result follows from his paper [5] (some components of \( \mathcal{I} \setminus \mathcal{I}_n \) contain nothing but Blaschke products whose zeroes converge nontangentially to \( z = 1 \)). We shall give another counterexample by using a different approach.

We want to express our gratitude to Professors R. G. Douglas, D. Sarason, and A. E. Tong for several helpful discussions; especially to D. Sarason who made available his unpublished paper [5]. We are also grateful to the referee who made several suggestions.

1. Components of \( \mathcal{F} \) containing exactly one, or \( e(e = \text{the power of the continuum}) \) singular inner functions. We shall need a remarkable classical result due to O. Frostman.

**Theorem 1.1** ([3, § 59, p. 111]). Let \( q \in \mathcal{F} \) and let

\[
q_a(z) = \frac{[q(z) - a]}{[1 - \bar{a}q(z)]}, \quad a \in D.
\]

Then, for all \( a \in D \), except for a subset of logarithmic capacity zero, \( q_a \in \mathcal{F}_n \).

For a precise definition of logarithmic capacity of a subset \( \Lambda \) of the complex plane, see [3; 6]. For our purposes, it is enough to recall that, if \( \log \text{cap} \Lambda = 0 \), then \( \Lambda \) is a “very small” subset of \( C \); in fact, \( \Lambda \) has planar Lebesgue measure zero and, moreover, the projection of \( \Lambda \) on any line of \( C \) has linear measure zero. Thus, Theorem 1.1 implies, in particular, that \( \mathcal{F}_n \) is dense in \( \mathcal{F} \).

We are going to analyze the behavior of an inner function \( q \) on a closed arc \( \Gamma = [e^{i\alpha}, e^{i\beta}] \) \((0 \leq \alpha < 2\pi, \alpha < \beta < \alpha + 2\pi)\), contained in \( \partial D \setminus \text{Sp}(q) \). For an arbitrary continuously differentiable unimodular function \( u(e^{i\theta}) \) defined in a neighborhood (in \( \partial D \)) of \( \Gamma \), we define the variation of \( u \) on \( \Gamma \) as

\[
M(u; \alpha, \beta) = \arg u(e^{i\alpha}) - \arg u(e^{i\beta}) = \int_{\alpha}^{\beta} d \arg u(e^{i\theta}),
\]
where \( \arg u(e^{i\theta}) \) denotes any continuously differentiable determination of the argument of \( u(e^{i\theta}) \).

**Lemma 1.2.** Let \( p \) and \( q \) be two inner functions and let \( \Gamma \) be as above. Then

(i) \( M(q; \alpha, \beta) \geq 0 \);

(ii) If \( \Gamma \subset \partial D \setminus \text{Sp}(p) \), \( M(pq; \alpha, \beta) = M(p; \alpha, \beta) + M(q; \alpha, \beta) \geq M(q; \alpha, \beta) \);

(iii) If \( p \) belongs to the component of \( q \), then \( \text{Sp}(p) \cap \partial D = \text{Sp}(q) \cap \partial D \) and there exists a constant \( C = C(p, q) \) such that \( |M(q; \alpha, \beta) - M(p; \alpha, \beta)| \leq C \), and \( C \) is independent of \( \Gamma \).

**Proof.** (i) This is trivial for finite Blaschke products. If \( q(z) = \prod_{k=1}^{\infty} b(z, a_k) \) is an infinite Blaschke product such that \( \Gamma \subset \partial D \setminus \text{Sp}(q) \), then observe that \( \{ \prod_{k=1}^{n} b(z, a_k) \} \) converges to \( q(z) \) uniformly on \( \Gamma \), from which the result follows. Finally, if \( q \) is any inner function, then we can find (by Frostman's theorem) a sequence \( \{a_n\} \subset D \) converging to zero, such that \( q_n = q_n(a_n) \) (defined by (1)) is a Blaschke product, for all \( n \). Clearly, \( q_n \) belongs to the component of \( q \) and therefore (see the introduction) \( \text{Sp}(q_n) \cap \partial D = \text{Sp}(q) \cap \partial D \). Since the result is true for all \( q_n \), \( n = 1, 2, \cdots \) and \( |q_n(z) - q(z)| \to 0 \), as \( n \to \infty \), uniformly on \( \Gamma \), the result is also true for \( q \).

(ii) This follows immediately from (i).

(iii) Let \( \mathcal{E}(q) = \{q' \in \mathcal{F} : ||q' - q||_\infty < 1\} \) and, by induction, define \( \mathcal{E}_n(q) = \{q'' \in \mathcal{F} : ||q'' - q''||_\infty < 1, \text{ for some } q' \in \mathcal{E}_{n-1}(q)\}, n = 1, 2, \cdots \), and \( \mathcal{E}(q) = \bigcup_{n=0}^{\infty} \mathcal{E}_n(q) \). Clearly, \( \mathcal{E}(q) \) is open and closed in \( \mathcal{F} \) and contains the component of \( q \). Moreover, \( \text{Sp}(q') \cap \partial D = \text{Sp}(q) \cap \partial D \), for all \( q' \in \mathcal{E}(q) \).

Let \( q' \in \mathcal{E}(q) \); then for every \( e^{i\theta} \in \Gamma \), \( |q(e^{i\theta}) - q'(e^{i\theta})| \leq ||q - q'||_\infty < 1 \), and therefore, for a suitably continuously differentiable definition of the argument, we have \( |\arg q(e^{i\theta})/q'(e^{i\theta})| < \pi/3 \). Hence, \( |M(q/q'; \alpha, \beta)| = |\arg q(e^{i\theta})/q'(e^{i\theta}) - \arg q(e^{i\theta})/q'(e^{i\theta})| \leq |\arg q(e^{i\theta})/q'(e^{i\theta})| + |\arg q(e^{i\theta})/q'(e^{i\theta})| < 2\pi/3 \).

By an elementary inductive argument, it follows that \( |M(q/q''; \alpha, \beta)| < 2\pi(n + 1)/3 \), for all \( q'' \in \mathcal{E}_n(q) \), \( n = 0, 1, 2, \cdots \), whence the result follows.

**Theorem 1.3.** Let \( d(z) = d(z; 0, 1) \). Then, for each positive \( r \), the component of \( d \) is isometric to the component of \( d' \); however, if \( r \neq 1 \), \( d \) and \( d' \) belong to different components of \( \mathcal{F} \). Thus, if \( 0 < r < 1 \), then the subset \( \{qd' \in \text{component of } d\} \) is isometric to the whole component of \( d \cdot d' \) is the only (up to a constant factor) singular inner function in its own component, for each \( r > 0 \).
Proof. If $b(z)$ is a Blaschke factor, then the mapping $q(z) \to q(b(z))$ is an isometry from $\mathbb{F}$ onto itself. If the zero of $b(z)$ is the point $a = (1 - r)/(1 + r) \in D(r > 0)$, then $d(b(z)) = d'(z)$, which proves the first part.

Observe that $\text{Sp}(d) = \{1\}$ and $M(d^t; 1/n, \pi) \to +\infty$, as $n \to \infty$, for each positive $t$; hence, $d^r$ and $d^{r+t}$ cannot belong to the same component, as follows from Lemma 1.2. Since $\mathcal{F}_s \cap \mathcal{F}_l = \{\lambda d^t: \lambda \in \partial D, t > 0\}$, we conclude that every singular inner function in the component of $d^r$ is a constant multiple of $d^r$.

The remaining statements are clear now.

Corollary 1.4. Let $q \in \mathcal{F}_s$ and assume that $\{\theta: e^{i\theta} \in \text{Sp}(q)\}$ is well-ordered (with the usual order of the interval $[0, 2\pi]$). Then the only singular inner functions in the component of $q$ are the constant multiples of $q$.

Proof. Clearly, $\text{Sp}(q)$ is countable and therefore $q$ has the form $q(z) = \lambda \prod_j d(z; \theta_j, l_j) = \lambda \prod_j d_j(z)$ (i.e., $q \in \mathcal{F}_d$). Moreover, if $p \in \mathcal{F}_s$ belongs to the component of $q$, then $p(z) = \lambda' \prod_j d(z; \theta'_j, l'_j)$, where $\lambda' \in \partial D$ and $\text{clos}\{e^{i\theta}\} = \text{clos}\{e^{i\theta'}\} = \text{Sp}(q)$.

Let $q(z) = \lambda q_j(z)q_j'(z)$, where $q_j(z)$ is the product of all $d_j$'s corresponding to the isolated points of $\text{Sp}(q)$. It is not difficult to see, by using Lemma 1.2 and the arguments of the proof of Theorem 1.3, that $p(z) = \lambda' q_j(z)p_j'(z)$, where $\text{Sp}(p_j') = \text{Sp}(q_j') \subset \text{Sp}(q_j)$ (here $I''$ denotes the derived set of the set $I'$).

Now, let $q(z) = \lambda q_j(z)q_j(z)q_j(z)$, where $q_j(z)$ is the product of the $d_j$'s corresponding to the isolated points of $\text{Sp}(q_j)$. Using the above arguments and the fact that, if $e^{i\theta} \in \text{Sp}(q_j)$, then the arc $(e^{i\theta}, e^{i\theta+\varepsilon}]$ (for some $\varepsilon = \varepsilon(\theta) > 0$) does not intersect $\text{Sp}(q)$ (here we are using the "well-order property"!), we see that $p(z) = \lambda' q_j(z)q_j(z)p_j'(z)$.

The result follows by a transfinite inductive argument.

It is clear that, if $q \in \mathcal{F}$ is nonconstant, then $q(D)$ is an open subset of $D$; in general, $q(D) \neq D$ (namely, if $q$ is singular, then $0 \in q(D)$). In [6, Theorem 10], W. Seidel proved that if $q$ is a nonconstant inner function and $D \backslash q(D)$ contains more than one point, then $\text{Sp}(q) \cap \partial D$ is a nonempty perfect subset of $\partial D$. The set $D \backslash q(D)$ has been completely determined by O. Frostman ([3, § 61, p. 113]; see also [6, Theorem 13]): If $q \in \mathcal{F}$ is nonconstant, then $D \backslash q(D)$ is a closed subset of $D$ of logarithmic capacity zero; conversely, if $A$ is a closed subset of $D$ and log cap $A = 0$, then the uniformizer of the Riemann surface $D \backslash A$ is an inner function $q$ such that $D \backslash q(D) = A$. Moreover, if $A$ is compact, then $\text{Sp}(q) \cap \partial D$ has linear measure zero (see [6, p. 218]). The uniformizer of $D \backslash (a)(a \in D)$ can be taken equal
to $d(z; \theta, l) = q(z)$ (defined by (1); $\theta$ and $l > 0$ can be arbitrarily chosen); in this case $Sp(q) = \{e^{i\theta}\}$ consist of a single point and we know (by Theorem 1.3) that the component of $q$ contains exactly one (up to a constant factor) singular inner function. On the other hand, if $A$ is a compact subset of logarithmic capacity zero of $D$, and $A$ contains more than one point, the uniformizer of $D \setminus A$, $p(z)$ is an inner function such that $Sp(p)$ is a nonempty perfect subset of $\partial D$ of linear measure zero; since, for every $a \in A$, $p_a \in \mathcal{F}$, we conclude that the component of $p$ contains (at least!) two coprime singular inner functions. In fact, if $q = rs$ and $q_a = rt$, where $q, r, s, t \in \mathcal{F}$ and $a \neq 0$, then $q - a = rs - a = (1 - \alpha q)rt$; hence $t(1 - \alpha q) = s - a/r \in H^\infty$. It follows that $r$ and $a/r$ belong to $H^\infty$, but this is impossible unless $r$ is a constant. In other words, $q$ and $q_a$ are coprime (this example and Theorem 1.5 below are due to D. Sarason).

Corollary 1.4 may be considered an improvement of the above mentioned result of W. Seidel for a very particular class of singular inner functions (in fact, Seidel’s result can be reformulated as: If $Sp(q) \cap \partial D$ is not perfect, then $D \setminus q(D)$ contains, at most, one point) and we guess that the “well-order” condition could be replaced by the weaker condition “$Sp(q) \cap \partial D$ is countable”; however, Lemma 1.2 is not sufficient to prove this stronger conjecture. On the other hand, an analysis of the function $p_a$ of the above example shows that no “reasonable” condition weaker than “countable” can work to get the same result.

Let $p(z)$ be the uniformizer of $D \setminus A$, where $A$ is any nonempty perfect subset of $D$ of logarithmic capacity zero (e.g., take as $A$ a suitable “Cantor type” subset of the real interval $[0, 1/2]$; see [3; 6]). Then, for each $a \in A$, $p_a$ is a singular inner function. Since $c(A) = c$, the power of the continuum, it follows from the previous observations that

**Theorem 1.5 (D. Sarason).** There exists a compact of $\mathcal{F}$ containing $c$ pairwise coprime singular inner functions.

The perfect set $A$ can be replaced by a finite subset or by a sequence of points in $D$ converging to $\partial D$. This suggests that, for each $n = 0, 1, 2, \ldots$, there exists a component of $\mathcal{F}$ containing exactly $n$ coprime singular inner functions or, at least, exactly $n$ “essentially different” (i.e., $p/q$ is not a constant) singular inner functions, but we have been unable to prove it.

From Theorem 1.1, we obtain
COROLLARY 1.6.

(i) \( \mathcal{F}_S \) is a closed nowhere dense subset of \( \mathcal{F} \).

(ii) \( \mathcal{F}_B \) is a dense, but not open subset of \( \mathcal{F} \).

Proof. By the observations following Theorem 1.1, we only have to prove that \( \mathcal{F}_S \) is closed and \( \mathcal{F}_B \) is not open. The first fact is immediate, because \( \mathcal{F}_S \) is clearly closed with respect to the compact-open topology restricted to \( \mathcal{F} \) and the norm-topology in \( H^\infty \) is stronger than the compact-open topology.

To see that \( \mathcal{F}_B \) is not open, write \( d(z; 0, 1) = \prod_{j=1}^\infty d_j(z) \), where \( d_j(z) \) is the \( 2^j \)-root of \( d(z; 0, 1) \), and set \( b(z) = \prod_{j=1}^\infty p_j(z) \), \( p_j(z) = [d_j(z) - 1/(2^j)]/[1 - (1/2^j)d_j(z)] \). It follows from Theorem 1.3 that \( p_j \in \mathcal{F}_B \), for all \( j \), and \( b = \prod_j p_j \in \mathcal{F}_B \). Given any \( \varepsilon > 0 \), choose \( n \) so that \( 2/n < \varepsilon \). Then \( \prod_{j=1}^n p_j \in \mathcal{F}_B \) and \( \| b - \prod_{j=n+1}^\infty p_j \|_\infty = \| (d_n)_{1/2n} - d_n \|_\infty < 2/n < \varepsilon \).

Therefore, \( b(z) \) does not belong to the interior of \( \mathcal{F}_B \).

We close this section with two conjectures:

1) The component, in \( \mathcal{F}_S \), of a singular inner function \( p \) is the set of the constant multiples of \( p \) (i.e., \( \mathcal{F}_S \) is “essentially” a totally disconnected space).

2) \( \mathcal{F}_C \) and \( \mathcal{F}_A \) are closed in \( \mathcal{F} \).

2. Components contained in \( \mathcal{F}_B \).

THEOREM 2.1. Given any sequence \( 0 \leq r_1 \leq r_2 \leq \cdots \leq r_k \leq \cdots < 1 \) of radii such that \( \sum_{k=1}^\infty (1 - r_k) < \infty \), it is possible to choose \( c \) sequences \( \{\theta_k(t); 0 \leq t < \omega\} \) of arguments such that, for each \( t \in [0, \omega) \), the component of

\[
\begin{align*}
\text{b}_t(z) = \prod_{k=1}^\infty b(z, r_k \exp \{i\theta_k(t)\})
\end{align*}
\]

in \( \mathcal{F} \) is contained in \( \mathcal{F}_B \). Moreover, if \( t \neq t' \), then \( \text{b}_t \) and \( \text{b}_{t'} \) belong to different components.

Proof. First of all observe that \( \prod_{k=1}^\infty b(z, r_k) \) converges uniformly on each of the subsets \( A_n = \{z: |z| \leq 1, |1 - z| \geq 2^{-n}\} \). Therefore, we can choose \( k_0 = 0 < k_1 < k_2 < \cdots \) in such a way that

\[
|M\left(\prod_{k=k_1}^{m_2} b(\epsilon^{\theta}, r_k); 2^{-n}, 2\pi - 2^{-n}\right)| < 2^{-n}, \quad n = 1, 2, \cdots, \quad \text{and}
\]

(2)

\[
2\pi(m_2 - m_1 + 1) - 2^{-n} < \left|M\left(\prod_{k=m_1}^{m_2} b(\epsilon^{\theta}, r_k); -2^{-n}, 2^{-n}\right)\right|
\]

\[
< 2\pi(m_2 - m_1 + 1), \quad n = 1, 2, \cdots,
\]

whenever \( k_n \leq m_1 \leq m_2 < \infty \).
Define \( b(z) = b_0(z) = \prod_{i=1}^{n} K_i(z) \), where

\[
K_i(z) = \exp\left[i2^{-h}\right]: k_i < k_i \leq k_i, h = 1, 2, \ldots
\]

Then the first inequality of (2) implies that \( M(b_0(\varepsilon^i); \pi, 2\pi - \varepsilon) \) is uniformly bounded with respect to \( \varepsilon, 0 < \varepsilon < \pi \). Since \( \text{Sp}(b_0) \cap \partial D = \{1\} \), it is easy to see from Lemma 1.2 and the properties of the functions \( d(z; 0, t), t > 0 \), that the component of \( b_0 \) in \( \mathcal{F} \) only contains Blaschke products. It is also clear that this property of \( b_0 \) only depends on the fact that the sequence \( \{k_i\}_{i=0}^\infty \) tends to infinity rapidly enough.

Thus, if \( k_i(t) \) denotes the integer part of \( k_i h_i, 0 \leq t < \infty \), and \( b_t \) is defined by \( b_t(z) = \prod_{i=1}^{n} b_i'(z), \) where

\[
b_i'(z) = \prod_{k=k_i}^{k_{i+1}} b_i(z, r_k \exp[i2^{-h}]): k_{i+1} < k \leq k_i, h = 1, 2 \cdots.
\]

Then the component \( b_t \) is also contained in \( \mathcal{F}_b \) and \( \text{Sp}(b_t) \cap \partial D = \{1\} \).

Finally observe that, if \( 0 \leq t < t' < \infty \), then the second inequality of (2) implies that \( M(b_t/b_{t'}; 2^{-n}, \pi) \) is uniformly bounded with respect to \( \varepsilon, 0 < \varepsilon < \pi \), and \( a_k \neq 0 \) for all \( k \) and \( \pi > \arg a_1 > \arg a_2 > \cdots > \arg a_k > \arg a_{k+1} > \cdots > 0 \), and \( \lim \arg a_k = 0 \), and

(3) \( M(b; \pi, 2\pi - \varepsilon) \) is uniformly bounded with respect to \( \varepsilon, 0 < \varepsilon < \pi \); then there exists \( c \) subproducts \( b_\omega(0 \leq \omega \leq \pi/4) \) of \( b \) enjoying the properties (1), (2), and (3); moreover, if \( \omega \neq \omega' \), then \( b_\omega \) and \( b_{\omega'} \) belong to different component of \( \mathcal{F} \).

It is completely apparent that, if \( b(z) \) is a Blaschke product satisfying any of the properties (1), (2), or (3), then the same result is true for every subproduct of \( b \) with infinitely many zeroes. Therefore, we only have to show that the subproducts of \( b \) can be chosen in such a way that they belong to different components. The proof of the next lemma is a minor modification of the proof given in [1].

**Lemma 2.3.** Let \( N \) be the set of natural numbers. There exist \( c \) subsets \( \{A_\omega\}: 0 \leq \omega \leq \pi/4 \) of \( N \) such that, if \( \omega \neq \omega' \), then

(i) \( A_\omega \cap A_{\omega'} \) is finite and

(ii) given any \( N \in N, A_\omega \) contains a finite sequence of \( N \) consecutive numbers \( n_N, n_N + 1, \ldots, n_N + N - 1 \) which are not in \( A_{\omega'} \).
Proof. Enumerate the points with integral coordinates in \{(x, y): 0 \leq x \leq y\} as follows:

1 \rightarrow (0, 0),
2 \rightarrow (0, 1), 3 \rightarrow (1, 1),
4 \rightarrow (0, 2), 5 \rightarrow (1, 2), 6 \rightarrow (2, 2),
7 \rightarrow (0, 3), 8 \rightarrow (1, 3), 9 \rightarrow (2, 3), 10 \rightarrow (3, 3), \text{etc.}

Let \( P_0 = \{(x, y): y > x^2\} \) and let \( P_\omega, 0 \leq \omega \leq \pi/4, \) be the result of rotating \( P_0 \) in \(-\omega\) about the origin. Now it is enough to take \( A_\omega \) equal to the subset of numbers in the above lattice lying inside \( P_\omega. \)

Proof of Theorem 2.2. Properties (2) and (3) show that \( b(z) \) can be factored as \( b(z) = b'(z) \cdot b''(z), \) where \( b'(z) \) is a finite or infinite subproduct and \( b''(z) = \prod_{k=1}^{\infty} b(z, a'_k) \) satisfies (1), (2) (with \( a_k \) replaced by \( a''_k \)), and (3), and moreover,

\[
| M(b''; \arg a''_k, \pi) - 2(k - 1/2)\pi | < 2\pi
\]

Set \( b_\omega(z) = b'(z) \cdot \prod \{b(z, a'_k): k \in A_\omega\}, 0 \leq \omega \leq \pi/4, \) where \( A_\omega \) is the subset of \( N \) defined in Lemma 2.3. Now observe that the definition of \( A_\omega \) and (3) imply that, if \( 0 \leq \omega < \omega' \leq \pi/4, \) then

\[
\limsup \{M(b_\omega/b_\omega'; \arg a''_k, \pi); k \rightarrow \infty\} = +\infty
\]
or \( \liminf \{M(b_\omega/b_\omega'; \arg a''_k, \pi); k \rightarrow \infty\} = -\infty. \) Thus, the result follows from Lemma 1.2.

Remark. It is completely apparent that the same argument can be used to prove, e.g., that \( \mathcal{F} \) has \( c \) components containing nothing but Blaschke products whose sequences of zeroes converges non-tangentially to \( z = 1. \) To see this, set \( b(z) = \prod_{k=1}^{\infty} b(z, r_k), \) where \( \{r_k\}_{k=1}^{\infty} \) is a sequence of radii converging “very rapidly” to 1 so that, for a suitable sequence of arguments \( \{\theta_n\}_{n=1}^{\infty}, \) decreasing to zero, \( M(d(e^{i\theta}; 0, 1/n)/b(e^{i\theta}); \theta_n, \pi) > n, n = 1, 2, \cdots. \) Clearly, the same result is true for any of the subproducts \( b_\omega \) of \( b \) (defined as in the proof of Theorem 2.2), and the result follows from Lemma 1.2.

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Received October 9, 1972, and in revised form April 23, 1973. This research was supported by National Science Foundation Grant GP–14255 and GU–3173.

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Pacific Journal of Mathematics
Vol. 51, No. 1 November, 1974

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