A SIMPLE PROOF OF THE MOY-PEREZ GENERALIZATION OF THE SHANNON-MCMILLAN THEOREM

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The Shannon-McMillan Theorem of Information Theory has
been generalized by Moy and Perez. The purpose of this
paper is to give a simple proof of this generalization.

1. Introduction. Let $T$ be either the semigroup of nonnegative
integers $\mathbb{N}$ or nonnegative real numbers $\mathbb{R}^+$. Let $\mathcal{U} = \{U^t : t \in T\}$ be
a semigroup of measurable mappings from a given measurable space
$(\Omega, \mathcal{F})$ to itself. (We suppose $U^0$ is the identity map.) If $X_0$ is a
measurable mapping from the space $(\Omega, \mathcal{F})$ to another space, let
$(X_t : t \in T)$ be the process generated by $\mathcal{U}$; that is, $X_t = X_0 \cdot U^t, t \in T$.
If $a, b \in T, a \leq b$, let $\mathcal{F}_{ab}$ denote the sub-sigmafield of $\mathcal{F}$ generated
by the mappings $\{X_t : t \in T, a \leq t \leq b\}$.

Let $P, Q$ be probability measures on $\mathcal{F}$; let $P_{ab}(Q_{ab})$ be the re-
striction of $P(Q)$ to $\mathcal{F}_{ab}$. We suppose that $P_{st}$ is absolutely continuous
with respect to $Q_{st}, t \in T$. Then the Radon-Nikodym derivatives $f_{st} =
\frac{dP_{st}}{dQ_{st}}$ exist, $s \leq t$. We assume that the entropies $H_{st} = \int_{\Omega} \log f_{st}dP,$
$s \leq t$, are all finite. (We use natural logarithms.) It is known that
(1) $H_{st}$ is a nonnegative, nondecreasing function of $t$ ([6], p. 54); and

(2) If $\| \cdot \|$ denotes the $L'(P)$ norm, then
$$\| \log f_{ru}/f_{st} \| \leq H_{ru} - H_{st} + 1, \quad r \leq s \leq t \leq u,$$
([6], inequality (2.4.10), and p. 54).

The Moy-Perez result. The following generalization of the Shannon-
McMillan Theorem was proved by Moy [4] for the case $T = \mathbb{N}$, and by
Perez [5], for the case $T = \mathbb{R}^+$.

THEOREM. Let $(X_t : t \in T)$ be a stationary process with respect to
$P$ and a Markov process with stationary transition probabilities with
respect to $Q$. If the sequence $\{n^{-1}H_{nt} : n = 1, 2, \cdots\}$ is bounded above,
then the functions $\{t^{-1} \log f_{ut} : t > 0, t \in T\}$ converge as $t \to \infty$ in $L'(P)$
to a function $h$ which is invariant with respect to $\mathcal{U}$; that is, $h \cdot U^t = h, t \in T$.

To prove this theorem, Moy and Perez embedded the process $(X_t)$
in a bilateral process $(X_t : -\infty < t < \infty)$, stationary with respect to
$P$ and Markov with respect to $Q$; Doob’s Martingale Convergence
Theorem was then used. We present a simple proof which requires no such embedding and no martingale theory. The method of proof is a generalization of the method used by Gallager ([2], p. 60) to prove the Shannon-McMillan Theorem, and uses the $L^1$ version of the Mean Ergodic Theorem.

Proof of the Moy-Perez result. The assumptions made on $P$ and $Q$ imply that:

1. The sequence $\{H_{n}: n = 1, 2, \cdots\}$ is convex ([4], Theorem 2); (A sequence $c_1, c_2, \cdots$ is convex if $c_{n+2} - 2c_{n+1} + c_n \geq 0$, $n = 1, 2, \cdots$)

2. $f_{ot} \cdot U^s = f_{s,t+s}$ a.e. $[P]$, and therefore $H_{ot} = H_{s,t+s}$ ([4], Theorem 1); and

3. $E_Q(f_{ot} | \mathcal{F}_s) = f_{rs} \quad r \leq s \leq t$.

Because of (3), $H_{on} - H_{o,n-1}$ is an increasing sequence and so has a limit $H$, possibly infinite. Since

$$n^{-1}H_{on} = n^{-1} \sum_{t=1}^{n-1} (H_{ot} - H_{o,t-1}) + n^{-1}H_{oo}, \quad \text{and} \quad \left\{ \frac{1}{n} H_{on} \right\}$$

is bounded,

$$\lim_{n \to \infty} n^{-1}H_{on} = \lim_{n \to \infty} (H_{on} - H_{o,n-1}) = H < \infty .$$

From (1), we have

$$[t]^{-1}H_{o(t)}[t]t^{-1} \leq t^{-1}H_{ot} \leq ([t] + 1)^{-1}H_{0,[t]+1}([t] + 1)t^{-1} ,$$

which implies that $\lim_{t \to \infty} t^{-1}H_{ot} = H$.

Also, since

$$\| t^{-1} \log f_{ot} - [t]^{-1}f_{o,t} \| \leq \| t^{-1} \log f_{o(t)} - [t]^{-1} \log f_{o(t)} \|$$

$$\quad + \| t^{-1} \log \left(\frac{f_{ot}}{f_{o(t)}}\right) \| ,$$

and by (2)

$$\| t^{-1} \log \left(\frac{f_{ot}}{f_{o(t)}}\right) \| \leq t^{-1}(H_{ot} - H_{o(t)} + 1) ,$$

we see that the convergence of $n^{-1}\log f_{on}$ in $L'(P)$ as $n \to \infty$ would imply the convergence of $t^{-1}\log f_{ot}$ in $L'(P)$ as $t \to \infty$ to the same limit.

Now, for fixed $s \in T$, we have for $t \geq s$,

$$\| t^{-1} \log f_{ot} - t^{-1} \log f_{s,s+t} \| \leq \| t^{-1} \log \left(\frac{f_{ot}}{f_{st}}\right) \|$$

$$\quad + \| t^{-1} \log \left(\frac{f_{s,s+t}}{f_{st}}\right) \| \leq \frac{2}{t} (H_{ot} - H_{o,s-t} + 1) ,$$

using (2) and (4). Consequently if $\lim_{t \to \infty} t^{-1} \log f_{ot} = h$, then

$$\lim_{t \to \infty} t^{-1} \log f_{s,t+s} = h .$$
It follows then that \( h = h \cdot U^s \) because
\[
\lim_{t \to \infty} t^{-1} \log f_{s,t} = \lim_{t \to \infty} (t^{-1} \log f_{st}) \cdot U^s = h \cdot U^s,
\]
where we used (4).

These considerations show that it suffices to prove the \( L(P) \) convergence as \( n \to \infty \) of \( \{ n^{-1} \log f_{0n} : n = 1, 2, \cdots \} \). This we now do.

Given \( \varepsilon > 0 \), pick \( N \) to be a positive integer so large that \( |N^{-1}H_{0N} - H| < \varepsilon \), and \( |H_{0,N+1} - H_{0N} - H| < \varepsilon \). Define the sequence of functions \( h_n, n = N + 1, N + 2, \cdots \), as follows:
\[
h_n = f_{0N} \prod_{i=0}^{n-N-1} (f_{i,N+i+1}/f_{i,N+i}) I(f_{i,N+i+1}),
\]
where for a given function \( f \), \( I(f) \) we define to be the function such that \( I(f) = 1 \) if \( f > 0 \), and \( I(f) = 0 \), otherwise.

Now, using (5), we have
\[
E_Q(h_n | \mathcal{F}_{0,n-1}) = h_{n-1}[I(f_{n-N-1,n-1})/f_{n-N-1,n-1}] E_Q(f_{n-N-1,n} | \mathcal{F}_{0,n-1}) \leq h_{n-1}.
\]
Since \( h_n \) is \( \mathcal{F}_{0n} \)-measurable, it follows that
\[
E_P(h_n/f_{0n}) \leq E_Q(h_n) \leq E_Q(h_{N+1}) \leq E_Q(f_{0,N+1}) = 1.
\]

Now
\[
| \log x | = 2 \log^+ x - \log x \leq 2x - \log x;
\]
therefore,
\[
| n^{-1} \log (h_n/f_{0n}) | \leq 2 n^{-1}(h_n/f_{0n}) - n^{-1} \log (h_n/f_{0n}),
\]
a.e. \([P]\). Integrating with respect to \( P \), we obtain
\[
\| n^{-1} \log f_{0n} - n^{-1} \log h_n \| \leq 2 n^{-1} - n^{-1}E_P[\log (h_n/f_{0n})].
\]
However,
\[
- E_P[\log (h_n/f_{0n})] = - H_{0n} - (n - N)(H_{0,N+1} - H_{0N}) + H_{0n} \\
\leq - N(H - \varepsilon) - (n - N)(H - \varepsilon) \\
+ H_{0n} = - n(H - \varepsilon) + H_{0n},
\]
and so \( \lim_{n \to \infty} \| n^{-1} \log f_{0n} - n^{-1} \log h_n \| \leq \varepsilon \).

Using (4), we have, a.e. \([P]\),
\[
n^{-1} \log h_n = n^{-1} \log f_{0N} + n^{-1} \sum_{i=0}^{N-n} \log (f_{0,N+i}/f_{0N}) \cdot U^i,
\]
which converges as \( n \to \infty \) in \( L(P) \) to a function \( h_e \) by the Mean Ergodic Theorem (\([1]\), p. 667). This gives
\[
\lim_{n \to \infty} \| n^{-1} \log f_{0n} - h_e \| \leq \varepsilon, \text{ for every } \varepsilon > 0,
\]
which makes $n^{-1} \log f_n$ a Cauchy sequence in $L'(P)$, and therefore a convergent sequence.

**Final Remark.** For the reader who may wish to consult [5], we point out that the proof of the Moy-Perez Theorem given in [5] is erroneous. The Theorem 2.3 of [5] states that the Moy-Perez result holds as well for the case when $(X_t: t \in T)$ is stationary with respect to $P$ and $Q$, with no Markov assumption made. This is false; a counterexample is given in [3].

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