

Pacific Journal of Mathematics

**A SIMPLE PROOF OF THE MOY-PEREZ GENERALIZATION
OF THE SHANNON-MCMILLAN THEOREM**

JOHN CRONAN KIEFFER

A SIMPLE PROOF OF THE MOY-PEREZ GENERALIZATION OF THE SHANNON-MCMILLAN THEOREM

J. C. KIEFFER

The Shannon-McMillan Theorem of Information Theory has been generalized by Moy and Perez. The purpose of this paper is to give a simple proof of this generalization.

1. Introduction. Let T be either the semigroup of nonnegative integers N or nonnegative real numbers R^+ . Let $\mathcal{U} = \{U^t: t \in T\}$ be a semigroup of measurable mappings from a given measurable space (Ω, \mathcal{F}) to itself. (We suppose U^0 is the identity map.) If X_0 is a measurable mapping from the space (Ω, \mathcal{F}) to another space, let $(X_t: t \in T)$ be the process generated by \mathcal{U} ; that is, $X_t = X_0 \cdot U^t$, $t \in T$. If $a, b \in T$, $a \leq b$, let \mathcal{F}_{ab} denote the sub-sigmafield of \mathcal{F} generated by the mappings $\{X_t: t \in T, a \leq t \leq b\}$.

Let P, Q be probability measures on \mathcal{F} ; let $P_{ab}(Q_{ab})$ be the restriction of $P(Q)$ to \mathcal{F}_{ab} . We suppose that P_{0t} is absolutely continuous with respect to Q_{0t} , $t \in T$. Then the Radon-Nikodym derivatives $f_{st} = dP_{st}/dQ_{st}$ exist, $s \leq t$. We assume that the entropies $H_{st} = \int_{\Omega} \log f_{st} dP$, $s \leq t$, are all finite. (We use natural logarithms.) It is known that

(1) H_{0t} is a nonnegative, nondecreasing function of t ([6], p. 54); and

(2) If $\|\cdot\|$ denotes the $L^1(P)$ norm, then

$$\|\log(f_{ru}/f_{st})\| \leq H_{ru} - H_{st} + 1, r \leq s \leq t \leq u,$$

([6], inequality (2.4.10), and p. 54).

The Moy-Perez result. The following generalization of the Shannon-McMillan Theorem was proved by Moy [4] for the case $T = N$, and by Perez [5], for the case $T = R^+$.

THEOREM. *Let $(X_t: t \in T)$ be a stationary process with respect to P and a Markov process with stationary transition probabilities with respect to Q . If the sequence $\{n^{-1}H_{0n}: n = 1, 2, \dots\}$ is bounded above, then the functions $\{t^{-1} \log f_{0t}: t > 0, t \in T\}$ converge as $t \rightarrow \infty$ in $L^1(P)$ to a function h which is invariant with respect to \mathcal{U} ; that is, $h \cdot U^t = h$, $t \in T$.*

To prove this theorem, Moy and Perez embedded the process (X_t) in a bilateral process $(X_t: -\infty < t < \infty)$, stationary with respect to P and Markov with respect to Q ; Doob's Martingale Convergence

Theorem was then used. We present a simple proof which requires no such embedding and no martingale theory. The method of proof is a generalization of the method used by Gallager ([2], p. 60) to prove the Shannon-McMillan Theorem, and uses the L^1 version of the Mean Ergodic Theorem.

Proof of the Moy-Perez result. The assumptions made on P and Q imply that:

(3) The sequence $\{H_{0n}: n = 1, 2, \dots\}$ is convex ([4], Theorem 2); (A sequence c_1, c_2, \dots is convex if $c_{n+2} - 2c_{n+1} + c_n \geq 0, n=1, 2, \dots$)

(4) $f_{0t} \cdot U^s = f_{s,s+t}$ a.e. $[P]$, and therefore $H_{0t} = H_{s,s+t}$ ([4], Theorem 1); and

(5) $E_Q(f_{rt} | \mathcal{F}_{0s}) = f_{rs}, r \leq s \leq t$.

Because of (3), $H_{0n} - H_{0,n-1}$ is an increasing sequence and so has a limit H , possibly infinite. Since

$$n^{-1}H_{0n} = n^{-1} \sum_{i=1}^n (H_{0i} - H_{0,i-1}) + n^{-1}H_{00}, \text{ and } \left\{ \frac{1}{n} H_{0n} \right\}$$

is bounded,

$$\lim_{n \rightarrow \infty} n^{-1}H_{0n} = \lim_{n \rightarrow \infty} (H_{0n} - H_{0,n-1}) = H < \infty .$$

From (1), we have

$$[t]^{-1}H_{0[t]}[t]t^{-1} \leq t^{-1}H_{0t} \leq ([t] + 1)^{-1}H_{0,[t]+1}([t] + 1)t^{-1} ,$$

which implies that $\lim_{t \rightarrow \infty} t^{-1}H_{0t} = H$.

Also, since

$$\begin{aligned} \| t^{-1} \log f_{0t} - [t]^{-1} f_{0[t]} \| &\leq \| t^{-1} \log f_{0[t]} - [t]^{-1} \log f_{0[t]} \| \\ &+ \| t^{-1} \log (f_{0t}/f_{0[t]}) \| , \end{aligned}$$

and by (2)

$$\| t^{-1} \log (f_{0t}/f_{0[t]}) \| \leq t^{-1}(H_{0t} - H_{0[t]} + 1) ,$$

we see that the convergence of $n^{-1} \log f_{0n}$ in $L^1(P)$ as $n \rightarrow \infty$ would imply the convergence of $t^{-1} \log f_{0t}$ in $L^1(P)$ as $t \rightarrow \infty$ to the same limit.

Now, for fixed $s \in T$, we have for $t \geq s$,

$$\begin{aligned} \| t^{-1} \log f_{0t} - t^{-1} \log f_{s,s+t} \| &\leq \| t^{-1} \log (f_{0t}/f_{st}) \| \\ &+ \| t^{-1} \log (f_{s,s+t}/f_{st}) \| \leq \frac{2}{t} (H_{0t} - H_{0,t-s} + 1) , \end{aligned}$$

using (2) and (4). Consequently if $\lim_{t \rightarrow \infty} t^{-1} \log f_{0t} = h$, then

$$\lim_{t \rightarrow \infty} t^{-1} \log f_{s,t+s} = h .$$

It follows then that $h = h \cdot U^s$ because

$$\lim_{t \rightarrow \infty} t^{-1} \log f_{s, s+t} = \lim_{t \rightarrow \infty} (t^{-1} \log f_{0t}) \cdot U^s = h \cdot U^s ,$$

where we used (4).

These considerations show that it suffices to prove the $L^1(P)$ convergence as $n \rightarrow \infty$ of $\{n^{-1} \log f_{0n} : n = 1, 2, \dots\}$. This we now do.

Given $\varepsilon > 0$, pick N to be a positive integer so large that $|N^{-1}H_{0N} - H| < \varepsilon$, and $|H_{0, N+1} - H_{0N} - H| < \varepsilon$. Define the sequence of functions h_n , $n = N + 1, N + 2, \dots$, as follows:

$$h_n = f_{0N} \prod_{i=0}^{n-N-1} (f_{i, N+i+1}/f_{i, N+i}) I(f_{i, N+i}) ,$$

where for a given function f , $I(f)$ we define to be the function such that $I(f) = 1$ if $f > 0$, and $I(f) = 0$, otherwise.

Now, using (5), we have

$$E_Q(h_n | \mathcal{F}_{0, n-1}) = h_{n-1} [I(f_{n-N-1, n-1})/f_{n-N-1, n-1}] E_Q(f_{n-N-1, n} | \mathcal{F}_{0, n-1}) \leq h_{n-1} .$$

Since h_n is \mathcal{F}_{0n} -measurable, it follows that

$$E_P(h_n/f_{0n}) \leq E_Q(h_n) \leq E_Q(h_{N+1}) \leq E_Q(f_{0, N+1}) = 1 .$$

Now

$$|\log x| = 2 \log^+ x - \log x \leq 2x - \log x ;$$

therefore,

$$|n^{-1} \log (h_n/f_{0n})| \leq 2n^{-1}(h_n/f_{0n}) - n^{-1} \log (h_n/f_{0n}) ,$$

a.e. $[P]$. Integrating with respect to P , we obtain

$$\|n^{-1} \log f_{0n} - n^{-1} \log h_n\| \leq 2n^{-1} - n^{-1} E_P [\log (h_n/f_{0n})] .$$

However,

$$\begin{aligned} -E_P[\log (h_n/f_{0n})] &= -H_{0N} - (n - N)(H_{0, N+1} - H_{0N}) + H_{0n} \\ &\leq -N(H - \varepsilon) - (n - N)(H - \varepsilon) \\ &\quad + H_{0n} = -n(H - \varepsilon) + H_{0n} , \end{aligned}$$

and so $\overline{\lim}_{n \rightarrow \infty} \|n^{-1} \log f_{0n} - n^{-1} \log h_n\| \leq \varepsilon$.

Using (4), we have, a.e. $[P]$,

$$n^{-1} \log h_n = n^{-1} \log f_{0N} + n^{-1} \sum_{i=0}^{N-n-1} \log (f_{0, N+1}/f_{0N}) \cdot U^i ,$$

which converges as $n \rightarrow \infty$ in $L^1(P)$ to a function h_ε by the Mean Ergodic Theorem ([1], p. 667). This gives

$$\overline{\lim}_{n \rightarrow \infty} \|n^{-1} \log f_{0n} - h_\varepsilon\| \leq \varepsilon , \text{ for every } \varepsilon > 0 ,$$

which makes $n^{-1} \log f_{0n}$ a Cauchy sequence in $L^1(P)$, and therefore a convergent sequence.

Final Remark. For the reader who may wish to consult [5], we point out that the proof of the Moy-Perez Theorem given in [5] is erroneous. The Theorem 2.3 of [5] states that the Moy-Perez result holds as well for the case when $(X_t; t \in T)$ is stationary with respect to P and Q , with no Markov assumption made. This is false; a counterexample is given in [3].

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear Operators Part I*, 1958.
2. R. G. Gallager, *Information Theory and Reliable Communication*, Wiley, 1968.
3. J. C. Kieffer, *A counterexample to Perez's generalization of the Shannon-McMillan theorem*, *Annals of Probability*, Vol. 1, No. 2, April, 1973.
4. Shu-Teh C. Moy, *Generalizations of Shannon-McMillan theorem*, *Pacific J. Math.*, **11** (1961), 705-714.
5. A. Perez, *Extensions of Shannon-McMillan's Limit Theorem to More General Stochastic Processes*, *Transactions of the Third Prague Conference on Information Theory*, Prague, (1964), 545-574.
6. M. S. Pinsker, *Information and Information Stability of Random Variables and Processes*, Holden-Day, 1964.

Received October 3, 1972.

UNIVERSITY OF MISSOURI-ROLLA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Zvi Arad, π -homogeneity and π' -closure of finite groups	1
Ivan Baggs, <i>A connected Hausdorff space which is not contained in a maximal connected space</i>	11
Eric Bedford, <i>The Dirichlet problem for some overdetermined systems on the unit ball in C^n</i>	19
R. H. Bing, Woodrow Wilson Bledsoe and R. Daniel Mauldin, <i>Sets generated by rectangles</i>	27
Carlo Cecchini and Alessandro Figà-Talamanca, <i>Projections of uniqueness for $L^p(G)$</i>	37
Gokulananda Das and Ram N. Mohapatra, <i>The non absolute Nörlund summability of Fourier series</i>	49
Frank Rimi DeMeyer, <i>On separable polynomials over a commutative ring</i>	57
Richard Detmer, <i>Sets which are tame in arcs in E^3</i>	67
William Erb Dietrich, <i>Ideals in convolution algebras on Abelian groups</i>	75
Bryce L. Elkins, <i>A Galois theory for linear topological rings</i>	89
William Alan Feldman, <i>A characterization of the topology of compact convergence on $C(X)$</i>	109
Hillel Halkin Gershenson, <i>A problem in compact Lie groups and framed cobordism</i>	121
Samuel R. Gordon, <i>Associators in simple algebras</i>	131
Marvin J. Greenberg, <i>Strictly local solutions of Diophantine equations</i>	143
Jon Craig Helton, <i>Product integrals and inverses in normed rings</i>	155
Domingo Antonio Herrero, <i>Inner functions under uniform topology</i>	167
Jerry Alan Johnson, <i>Lipschitz spaces</i>	177
Marvin Stanford Keener, <i>Oscillatory solutions and multi-point boundary value functions for certain nth-order linear ordinary differential equations</i>	187
John Cronan Kieffer, <i>A simple proof of the Moy-Perez generalization of the Shannon-McMillan theorem</i>	203
Joong Ho Kim, <i>Power invariant rings</i>	207
Gangaram S. Ladde and V. Lakshmikantham, <i>On flow-invariant sets</i>	215
Roger T. Lewis, <i>Oscillation and nonoscillation criteria for some self-adjoint even order linear differential operators</i>	221
Jürg Thomas Marti, <i>On the existence of support points of solid convex sets</i>	235
John Rowlay Martin, <i>Determining knot types from diagrams of knots</i>	241
James Jerome Metzger, <i>Local ideals in a topological algebra of entire functions characterized by a non-radial rate of growth</i>	251
K. C. O'Meara, <i>Intrinsic extensions of prime rings</i>	257
Stanley Poreda, <i>A note on the continuity of best polynomial approximations</i>	271
Robert John Sacker, <i>Asymptotic approach to periodic orbits and local prolongations of maps</i>	273
Eric Peter Smith, <i>The Garabedian function of an arbitrary compact set</i>	289
Arne Stray, <i>Pointwise bounded approximation by functions satisfying a side condition</i>	301
John St. Clair Werth, Jr., <i>Maximal pure subgroups of torsion complete abelian p-groups</i>	307
Robert S. Wilson, <i>On the structure of finite rings. II</i>	317
Kari Ylinen, <i>The multiplier algebra of a convolution measure algebra</i>	327