A SIMPLE PROOF OF THE MOY-PEREZ GENERALIZATION OF THE SHANNON-MCMILLAN THEOREM

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The Shannon-McMillan Theorem of Information Theory has been generalized by Moy and Perez. The purpose of this paper is to give a simple proof of this generalization.

1. Introduction. Let $T$ be either the semigroup of nonnegative integers $N$ or nonnegative real numbers $R^+$. Let $\mathcal{U} = \{U^t: t \in T\}$ be a semigroup of measurable mappings from a given measurable space $(\Omega, \mathcal{F})$ to itself. (We suppose $U^0$ is the identity map.) If $X_0$ is a measurable mapping from the space $(\Omega, \mathcal{F})$ to another space, let $(X_t: t \in T)$ be the process generated by $\mathcal{U}$; that is, $X_t = X_0 \cdot U^t$, $t \in T$. If $a, b \in T$, $a \leq b$, let $\mathcal{F}_{ab}$ denote the sub-sigmafield of $\mathcal{F}$ generated by the mappings $\{X_t: t \in T, a \leq t \leq b\}$.

Let $P, Q$ be probability measures on $\mathcal{F}$; let $P_{ab}(Q_{ab})$ be the restriction of $P(Q)$ to $\mathcal{F}_{ab}$. We suppose that $P_{ot}$ is absolutely continuous with respect to $Q_{ot}$, $t \in T$. Then the Radon-Nikodym derivatives $f_{st} = dP_{st}/dQ_{st}$ exist, $s \leq t$. We assume that the entropies $H_{st} = \int_o \log f_{st}dP$, $s \leq t$, are all finite. (We use natural logarithms.) It is known that

(1) $H_{ot}$ is a nonnegative, nondecreasing function of $t$ ([6], p. 54); and

(2) If $\| \cdot \|$ denotes the $L'(P)$ norm, then

$$\| \log (f_{ru}/f_{st}) \| \leq H_{ru} - H_{st} + 1, \quad r \leq s \leq t \leq u,$$

([6], inequality (2.4.10), and p. 54).

The Moy-Perez result. The following generalization of the Shannon-McMillan Theorem was proved by Moy [4] for the case $T = N$, and by Perez [5], for the case $T = R^+$.

THEOREM. Let $(X_t: t \in T)$ be a stationary process with respect to $P$ and a Markov process with stationary transition probabilities with respect to $Q$. If the sequence $\{n^{-1}H_{ot}: n = 1, 2, \ldots\}$ is bounded above, then the functions $\{t^{-1}\log f_{st}: t > 0, t \in T\}$ converge as $t \to \infty$ in $L'(P)$ to a function $h$ which is invariant with respect to $\mathcal{U}$; that is, $h \cdot U^t = h$, $t \in T$.

To prove this theorem, Moy and Perez embedded the process $(X_t)$ in a bilateral process $(X_t: -\infty < t < \infty)$, stationary with respect to $P$ and Markov with respect to $Q$; Doob's Martingale Convergence
Theorem was then used. We present a simple proof which requires no such embedding and no martingale theory. The method of proof is a generalization of the method used by Gallager ([2], p. 60) to prove the Shannon-McMillan Theorem, and uses the \( L^1 \) version of the Mean Ergodic Theorem.

**Proof of the Moy-Perez result.** The assumptions made on \( P \) and \( Q \) imply that:

1. The sequence \( \{H_{on}: n = 1, 2, \ldots\} \) is convex ([4], Theorem 2); (A sequence \( c_1, c_2, \ldots \) is convex if \( c_{n+2} - 2c_{n+1} + c_n \geq 0, n = 1, 2, \ldots \).
2. \( f_{ot} \cdot U^s = f_{s,s+t} \) a.e. \([P]\), and therefore \( H_{ot} = H_{s,s+t} \) ([4], Theorem 1); and
3. \( E_\varphi(f_{rt} \mid \mathcal{F}_0) = f_{rs}, r \leq s \leq t. \)

Because of (3), \( H_{on} - H_{o,n-1} \) is an increasing sequence and so has a limit \( H \), possibly infinite. Since

\[
n^{-1}H_{on} = n^{-1} \sum_{i=1}^{n} (H_{oi} - H_{o,i-1}) + n^{-1}H_{00}, \quad \text{and} \quad \left\{ \frac{1}{n} H_{on} \right\}
\]

is bounded,

\[
\lim_{n \to \infty} n^{-1}H_{on} = \lim_{n \to \infty} (H_{on} - H_{o,n-1}) = H < \infty .
\]

From (1), we have

\[
\frac{1}{t} H_{0[1]} [t] t^{-1} \leq t^{-1} H_{ot} \leq (\lfloor t \rfloor + 1)^{-1} H_{0[1]} + ((\lfloor t \rfloor + 1) t^{-1} ,
\]

which implies that \( \lim_{t \to \infty} t^{-1} H_{ot} = H. \)

Also, since

\[
\| t^{-1} \log f_{ot} - [t]^{-1} f_{0[t]} \| \leq \| t^{-1} \log f_{0[t]} - [t]^{-1} \log f_{0[t]} \| + \| t^{-1} \log (f_{ot}/f_{0[t]}) \| \,
\]

and by (2)

\[
\| t^{-1} \log (f_{ot}/f_{0[t]}) \| \leq t^{-1} (H_{ot} - H_{0[1]} + 1),
\]

we see that the convergence of \( n^{-1} \log f_{on} \) in \( L'(P) \) as \( n \to \infty \) would imply the convergence of \( t^{-1} \log f_{ot} \) in \( L'(P) \) as \( t \to \infty \) to the same limit.

Now, for fixed \( s \in T, \) we have for \( t \geq s, \)

\[
\| t^{-1} \log f_{ot} - t^{-1} \log f_{s,s+t} \| \leq \| t^{-1} \log (f_{ot}/f_{st}) \|
\]

\[
+ \| t^{-1} \log (f_{s,s+t}/f_{st}) \| \leq \frac{2}{t} (H_{ot} - H_{0,t-s} + 1),
\]

using (2) and (4). Consequently if \( \lim_{t \to \infty} t^{-1} \log f_{ot} = h, \) then

\[
\lim_{t \to \infty} t^{-1} \log f_{s,t+s} = h .
\]
It follows then that \( h = h \cdot U^s \) because
\[
\lim_{t \to \infty} t^{-1} \log f_{x, s+t} = \lim_{t \to \infty} (t^{-1} \log f_{0,t}) \cdot U^s = h \cdot U^s ,
\]
where we used (4).

These considerations show that it suffices to prove the \( L^1(P) \) convergence as \( n \to \infty \) of \( \{n^{-1} \log f_{0,n} : n = 1, 2, \cdots \} \). This we now do.

Given \( \varepsilon > 0 \), pick \( N \) to be a positive integer so large that
\[
|N^{-1} H_{0N} - H| < \varepsilon, \text{ and } |H_{0N+1} - H_{0N} - H| < \varepsilon.
\]
Define the sequence of functions \( h_n, n = N + 1, N + 2, \cdots \), as follows:
\[
h_n = f_{0N} \prod_{k=0}^{n-N-1} (f_{4k,N+t+1}/f_{4k,N+t})I(f_{4k,N+t}) ,
\]
where for a given function \( f, I(f) \) we define to be the function such that \( I(f) = 1 \) if \( f > 0 \), and \( I(f) = 0 \), otherwise.

Now, using (5), we have
\[
E_Q(h_n \mid F_{0,n-1}) = h_{n-1}[I(f_{n-N-1,n-1})/f_{n-N-1,n-1}]E_Q(f_{n-N-1,n} \mid F_{0,n-1}) \leq h_{n-1} .
\]
Since \( h_n \) is \( F_{0n} \)-measurable, it follows that
\[
E_P(h_n/f_{0n}) \leq E_Q(h_n) \leq E_Q(h_{N+1}) \leq E_Q(f_{0,N+1}) = 1 .
\]

Now
\[
| \log x | = 2 \log +x - \log x \leq 2x - \log x ;
\]
therefore,
\[
|n^{-1} \log (h_n/f_{0n})| \leq 2n^{-1}(h_n/f_{0n}) - n^{-1} \log (h_n/f_{0n}) ,
\]
a.e. \([P]\). Integrating with respect to \( P \), we obtain
\[
|| n^{-1} \log f_{0n} - n^{-1} \log h_n || \leq 2n^{-1} - n^{-1}E_P [\log (h_n/f_{0n})] .
\]
However,
\[
-E_P[\log (h_n/f_{0n})] = -H_{0N} - (n - N)(H_{0N+1} - H_{0N}) + H_{0n} \\
\leq -N(H - \varepsilon) - (n - N)(H - \varepsilon) \\
+ H_{0n} = -n(H - \varepsilon) + H_{0n} ,
\]
and so \( \lim_{n \to \infty} || n^{-1} \log f_{0n} - n^{-1} \log h_n || \leq \varepsilon .
\]

Using (4), we have, a.e. \([P]\),
\[
n^{-1} \log h_n = n^{-1} \log f_{0N} + n^{-1} \sum_{k=0}^{N-1} \log (f_{0,N+1}/f_{0N}) \cdot U^i ,
\]
which converges as \( n \to \infty \) in \( L^1(P) \) to a function \( h_\varepsilon \) by the Mean Ergodic Theorem ([1], p. 667). This gives
\[
\lim_{n \to \infty} || n^{-1} \log f_{0n} - h_\varepsilon || \leq \varepsilon , \text{ for every } \varepsilon > 0 ,
\]
which makes \( n^{-1} \log f_n \) a Cauchy sequence in \( L^1(P) \), and therefore a convergent sequence.

**Final Remark.** For the reader who may wish to consult [5], we point out that the proof of the Moy-Perez Theorem given in [5] is erroneous. The Theorem 2.3 of [5] states that the Moy-Perez result holds as well for the case when \( (X_t: t \in T) \) is stationary with respect to \( P \) and \( Q \), with no Markov assumption made. This is false; a counterexample is given in [3].

**REFERENCES**


Received October 3, 1972.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan
Pacific Journal of Mathematics
Vol. 51, No. 1 November, 1974

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