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A ring A is called power invariant if whenever B is a ring such that the formal power series rings $A[[X]]$ and $B[[X]]$ are isomorphic, then A and B are isomorphic. A ring A is said to be strongly power invariant if whenever B is a ring and ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$, then there exists a B-automorphism ψ of $B[[X]]$ such that $\psi(X) = \phi(X)$. Strongly power invariant rings are power invariant. For any commutative ring A, $A/J(A)^n$ is strongly power invariant, where $J(A)$ is the Jacobson radical of A, and n is any positive integer. A left or right Artinian ring is strongly power invariant. If A is a left or right Noetherian ring, then $A[t]$, the polynomial ring in an indeterminate t over A, is strongly power invariant.

Introduction. Coleman and Enochs [2] raised the following question: Can there be nonisomorphic rings A and B whose polynomial rings $A[X]$ and $B[X]$ are isomorphic? Recently Hochster [4] answered this question in the affirmative. The analogous question about a commutative formal power series ring was raised by O'Malley [7]: If $A[[X]] \cong B[[X]]$, must $A \cong B$? We know no counterexamples.

In this paper all rings are assumed to have identity elements. The Jacobson radical and the prime radical (the intersection of all prime ideals) of a ring A will be denoted by $J(A)$ and $\mathrm{rad}(A)$, respectively. Let $A[[X]]$ be the formal power series ring in a commutative indeterminate X over a ring A, and let β be a central element of $A[[X]]$. Then (β^n) will denote the ideal of $A[[X]]$ generated by β^n for a nonnegative integer n, and $(A[[X]], (\beta))$ denotes the topological ring $A[[X]]$ with the (β)-adic topology. It is well known that $(A[[X]], (\beta))$ is Hausdorff if and only if $\bigcap_{n=1}^{\infty} (\beta^n) = (0)$. The (β)-adic topology is metrizable in the obvious way, and we say that $(A[[X]], (\beta))$ is complete if each Cauchy sequence of $A[[X]]$ converges in $A[[X]]$. Then clearly $(A[[X]], (X))$ is a complete Hausdorff space.

Extending the terminology used in [2], O'Malley [7] defined "power invariant ring" and "strongly power invariant ring" as follows: A ring A is power invariant if whenever B is a ring such that $A[[X]] \cong B[[X]]$, then $A \cong B$. A ring A is said to be strongly power invariant if whenever B is a ring and ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$, then there exists a B-automorphism ψ of $B[[X]]$ such that $\psi(X) = \phi(X)$.

Let A be a strongly power invariant ring and let ϕ be an isomorphism of $A[[X]]$ onto $B[[X]]$. Then there exists a B-automorphism

ψ of $B[[X]]$ such that $\psi(X) = \phi(X)$. Then $\psi^{-1}\phi$ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $(\psi^{-1}\phi)(X) = X$. Therefore, $A \cong A[[X]]/(X) \cong B[[X]]/(X) \cong B$. Thus a strongly power invariant ring is power invariant.

In this paper we attempt to impose conditions on a ring A so that $A[X] \cong B[[X]]$ implies $A \cong B$.

1. Strongly power invariant rings. The following theorem extends Theorem (4.5) in [8].

THEOREM 1.1. *Let B be a ring and $\beta = \sum_{i=0}^{\infty} b_i X^i$, an element of $B[[X]]$. Then the following statements are equivalent:*

(1) *b_i is central for each i, b_1 is a unit, and $(B[[X]], (\beta))$ is a complete Hausdorff space.*

(2) *There exists a B-automorphism of ψ of $B[[X]]$ such that $\psi(X) = \beta$.*

Proof. Suppose that (2) holds. Since $(B[[X]], (X))$ is a complete Hausdorff space and ψ is a uniformly bicontinuous mapping of $(B[[X]], (X))$ onto $(B[[X]], (\beta))$, $(B[[X]], (\beta))$ is a complete Hausdorff space. Since X commutes with every element of B, β commutes with any element of B and therefore b_i is central for each i. Let C be the center of B. Then $C[[X]]$ is the center of $B[[X]]$ and hence $\phi(C[[X]]) = C[[X]]$. Then ψ induces the C-automorphism of $C[[X]]$ which maps X onto β. Therefore, by Theorem (4.5) in [8], b_1 is a unit. Thus (2) implies (1).

Suppose that (1) holds. Since $(B[[X]], (\beta))$ is a complete Hausdorff space, there is a B-endomorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$. This comes from the same argument as the commutative case; namely (2.2) in [8]. Since b_i is central for each i, that ψ is a B-automorphism, also follows from the commutative argument; namely Lemma (4.2) and Corollary (4.4) in [8]. This completes the proof.

Let ϕ be an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta = \sum_{i=0}^{\infty} b_i X^i$. By similar argument as in the proof of Theorem 1.1, we see that b_i is central in B for each i and $(B[[X]], (\beta))$ is a complete Hausdorff space. Therefore, by Theorem 1.1, we see that a ring A is strongly power invariant if and only if whenever B is a ring and ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$, then b_1 is a unit.

The following lemma has appeared as Result 4.3 in [7] for the commutative case.

LEMMA 1.2. *For any ring A, $A/J(A)$ is strongly power invariant. In particular, if A is a semisimple ring then A is strongly power invariant.*

Proof. Let A be a semisimple. To prove this lemma, it suffices to show that A is strongly power invariant. Let B be a ring such that there is an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$. Let $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$. Since $J(A) = (0)$, it follows that $J(A[[X]]) = (X)$, and

$$\phi(J(A[[X]])) = \phi((X)) = (\phi(X)) = \phi(X)\cdot B[[X]] = J(B[[X]]) .$$

Clearly $X \in J(B[[X]])$, and so there exists $\sum_{i=6}^{\infty} c_i X^i \in B[[X]]$ such that $\phi(X)\cdot\sum_{i=0}^{\infty} c_i X^i = X$; i.e., $(\sum_{i=0}^{\infty} b_i X^i)\cdot(\sum_{i=0}^{\infty} c_i X^i) = X$. Then $b_0 c_1 + b_1 c_0 = 1$. But $b_0 \in J(B)$, so $1 - b_0 c_1$ is a unit. Therefore, $b_1 c_0$ is a unit, and so b_1 is a unit. Hence A is strongly power invariant.

THEOREM 1.3. *If A is a commutative ring, then for any positive integer n, $A/J(A)^n$ is strongly power invariant.*

Proof. Let A be a commutative ring such that $J(A)$ is nilpotent. To prove this theorem, it suffices to show that A is strongly power invariant. Let B be a ring such that there is an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$, and let $\phi(X) = \beta = \sum_{i=0}^{\infty} b_i X^i$. Then clearly B is commutative. Let N be the ideal of nilpotent elements of B, and let $\{P_\gamma\}$ be the collection of prime ideals of B. Then $N = \bigcap_\gamma P_\gamma$, and for each γ, $P_\gamma[[X]]$ is a prime ideal of $B[[X]]$. Therefore, the ideal of nilpotent elements of $B[[X]]$ is a subset of $N[[X]]$. Note that $N[[X]]$ is not necessarily the ideal of nilpotent elements of $B[[X]]$. Since $J(A)$ is nilpotent, $J(A)[[X]]$ is the ideal of nilpotent elements of $A[[X]]$. Therefore, $\phi(J(A)[[X]]) \subseteqq N[[X]]$. In order to show the opposite inclusion, let $g = \sum_{i=0}^{\infty} g_i X^i \in N[[X]]$; $g_i \in N$ for each i, and let $\phi^{-1}(X) = \alpha = \sum_{i=0}^{\infty} a_i X^i$, $a_i \in A$. Then $\phi^{-1}(g) = \sum_{i=0}^{\infty} \phi^{-1}(g_i)\alpha^i$, and $\phi^{-1}(g_i)$ is a nilpotent element of $A[[X]]$ for each i. Note that $a_0 \in J(A)$ i.e., a_0 is nilpotent, and $\phi^{-1}(g_i) \in J(A)[[X]]$. Expanding $\sum_{i=0}^{\infty} \phi^{-1}(g_i)\alpha^i$ in powers of X, we see that the coefficient of X^i is an element of $J(A)$ for each i since a_0 is nilpotent. Thus $\phi^{-1}(g) \in J(A)[[X]]$. Therefore, we get $\phi(J(A))[[X]] = N[[X]]$. Consider the isomorphism $\bar{\phi}$: $(A/J(A))[[X]] \to (B/N)[[X]]$ given by

$$(A/J(A))[[X]] \longrightarrow A[[X]]/J(A)[[X]] \longrightarrow B[[X]]/N[[X]] \longrightarrow (B/N)[[X]]$$

where the middle isomorphism is induced by ϕ and others are the obvious ones. Then it follows that $\bar{\phi}(X) = \sum_{i=0}^{\infty} \bar{b}_i X^i$, where $\bar{b}_i$ denotes the coset $b_i + N$ in B/N. Since $A/J(A)$ is strongly power invariant, $\bar{b}_1$ is a unit in B/N. But $N \subseteqq J(B)$ so b_1 is a unit in B. Thus A is strongly power invariant. This completes the proof.

COROLLARY 1.4. *Let A be a ring and C, the center of A. If $J(C)$ is nilpotent, then A is strongly power invariant. In particular, if C is a Artinian ring, then A is strongly power invariant.*

Proof. Let B be a ring such that there is an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$, and let $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$. If D denotes the center of B, $\phi(C[[X]]) = D[[X]]$. But by Theorem 1.3, C is strongly power invariant. Therefore, b_1 is a unit and so A is strongly power invariant.

It is well known that the prime radical of a ring A, denoted by rad (A), is the intersection of all prime ideals of A, and also it is the ideal of all strongly nilpotent elements of A. (P. 55-56 in [6].) Clearly, every strongly nilpotent element is nilpotent. In particular, if A is commutative, then every nilpotent element is strongly nilpotent. Note that if A is a commutative Noetherian ring, and N is the ideal of nilpotent elements of A, then $N[[X]]$ is the ideal of nilpotent elements of $A[[X]]$ [3]. The following lemma extends this statement to the noncommutative case.

LEMMA 1.5. *If A is a left or right Noetherian ring, then* $\operatorname{rad}(A[[X]]) = \operatorname{rad}(A)[[X]]$.

Proof. We show that if P is a prime ideal of A, then $P[[X]]$ is a prime ideal of $A[[X]]$. Suppose that P is a prime ideal of A and $P[[X]]$ is not a prime ideal of $A[[X]]$. Then there exist $f = \sum_{i=0}^{\infty} f_i X^i$ and $g = \sum_{i=0}^{\infty} g_i X^i$ in $A[[X]]$ such that $f \cdot A[[X]] \cdot g \subseteqq P[[X]]$ but $f \notin P[[X]]$ and $g \notin P[[X]]$. Let m be the smallest integer such that $f_m \notin P$, and let n be the smallest integer such that $g_n \notin P$. Since $f \cdot A[[X]] \cdot g \subseteqq P[[X]]$, $f \cdot a \cdot g$ belongs to $P[[X]]$ for any element a of A. Expanding $f \cdot a \cdot g$ in powers of X, we see that the coefficient of X^{m+n} is $\sum_{i+j=0}^{m+n} f_i a g_j$ which is in P. But $\sum_{i+j=0}^{m+n} f_i a g_j - f_m a g_n \in P$, so $f_m a g_n$ must be in P. Therefore, $f_m A g_n \subseteqq P$, but P is a prime ideal of A; so $f_m \in P$ or $g_n \in P$. This is a contradiction to our choice of m and n. Hence $P[[X]]$ is a prime ideal of $A[[X]]$. Therefore, it follows that $\operatorname{rad}(A[[X]]) \subseteqq \operatorname{rad}(A)[[X]]$. To show the opposite inclusion, we let $\sum_{i=0}^{\infty} a_i X^i \in \operatorname{rad}(A)[[X]]$. Then each a_i is strongly nilpotent. Let $\mathfrak{A}$ be the ideal of A generated by the set of all a_i's. Then clearly $\mathfrak{A} \subseteqq \operatorname{rad}(A)$; therefore, $\mathfrak{A}$ is a nil ideal of A. But since A is left or right Noetherian, $\mathfrak{A}$ is nilpotent. Thus $\sum_{i=0}^{\infty} a_i X^i \in \operatorname{rad}(A[[X]])$. Therefore, $\operatorname{rad}(A[[X]]) = \operatorname{rad}(A)[[X]]$.

THEOREM 1.6. *Let A be a left or right Noetherian ring and let $N = \operatorname{rad}(A)$. Then A is strongly power invariant if A/N is strongly power invariant.*

Proof. Let B be a ring such that there is an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$, and let $M = \operatorname{rad}(B)$. Since A is left (or right) Noetherian, $A[[X]]$ is left (or right) Noetherian. Then $B[[X]]$ is left (or right) Noetherian, and therefore, B is left (or right) Noetherian. So $\operatorname{rad}(B[[X]]) = M[[X]]$ (by Lemma 1.5). From the invariance of the prime radical under isomorphism, we have that $\phi(N[[X]]) = M[[X]]$. Write $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$; $b_i \in B$. Consider the isomorphism, $\bar{\phi}: (A/N)[[X]] \to (B/M)[[X]]$ given by

$$(A/N)[[X]] \longrightarrow A[[X]]/N[[X]] \longrightarrow B[[X]]/M[[X]] \longrightarrow (B/M)[[X]]\,,$$

where the middle isomorphism is induced by ϕ and the others are the obvious ones. Since A/N is strongly power invariant, we can show that b_1 is a unit of B by the same argument as in the proof of Theorem 1.3. Thus A is a strongly power invariant ring.

COROLLARY 1.7. *If A is a left or right Noetherian ring such that $J(A)$ is nil, then A is strongly power invariant.*

Proof. Clearly $J(A)$ is nilpotent. So every element of $J(A)$ is strongly nilpotent. Therefore, $J(A) = \operatorname{rad}(A)$. By Lemma 1.2 and Theorem 1.6, A is strongly power invariant.

COROLLARY 1.8. *A left or right Artinian ring is strongly power invariant.*

COROLLARY 1.9. *If A is a left or right Noetherian ring and if $A[t]$ is the polynomial ring in a commutative indeterminate t over A, then $A[t]$ is strongly power invariant.*

Proof. It is well known that for any ring A, $J(A[t]) = N[t]$ holds, where $N = J(A[t]) \cap A$ and N is a nil ideal in A [1]. Since A is left (or right) Noetherian, N is nilpotent and $A[t]$ is left (or right) Noetherian. Thus $J(A[t]) = N[t]$ is a nilpotent ideal in $A[t]$. Therefore, by Corollary 1.7, $A[t]$ is strongly power invariant.

2. Perfect power invariant rings. The following proposition extends Theorem 3.1 in [7].

PROPOSITION 2.1. *Let A and B be rings and suppose that ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$. If $\phi(A) \subseteqq B$, then $\phi(A) = B$.*

Proof. Let $\phi(X) = \beta = \sum_{i=0}^{\infty} b_i X^i$; $b_i \in B$. Then b_i is central for each i and $(B[[X]], (\beta))$ is a complete Hausdorff space. Then there exists a B-endomorphism ψ of $B[[X]]$ into $B[[X]]$ such that $\psi(X) = \beta$. Then by hypothesis, we have

$$B[[X]] = \phi(A)[[\beta]] \subseteqq B[[\beta]] \subseteqq B[[X]]\,.$$

Therefore, $B[[\beta]] = B[[X]]$, which implies ψ is onto. Now let $\bar{B}$ be $B/(b_1)$ and let $\bar{b} = b + (b_1)$ for $b \in B$. Then $X \rightarrow \sum_{i=0}^{\infty} \bar{b}_i X^i$ induces a surjective $\bar{B}$-endomorphism of $\bar{B}[[X]]$. But $\bar{b}_1$ is 0, so this impossible unless $(b_1) = B$; i.e., b_1 is a unit. Therefore, by Theorem 1.1, ψ is a B-automorphism of $B[[X]]$. Then $\psi^{-1}\phi$ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\psi^{-1}\phi(A) \subseteqq B$ and $\psi^{-1}\phi(X) = X$. So $\psi^{-1}\phi(A) = B$; but $\psi^{-1}(B) = B$; therefore $\phi(A) = B$.

DEFINITION. A ring A is said to be perfectly power invariant if whenever B is a ring and ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$, then $\phi(A) \subseteqq B$.

Let A be a perfectly power invariant ring, and let B be a ring such that there is an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$. In the proof of Proposition 2.1, we have shown that there exists a B-automorphism ψ of $B[[X]]$ such that $\psi(X) = \phi(X)$. So a perfectly power invariant ring is strongly power invariant. But a strongly power invariant ring is not necessarily perfectly power invariant.

EXAMPLE. Let K be a field and let $K[t]$ be the polynomial ring in an indeterminate t over K then $K[t]$ is strongly power invariant (by Corollary 1.9). But, by Corollary 2.8 in [5], we see that there is an automorphism ϕ of $K[t][[X]]$ such that $\phi(K[t]) \not\subseteqq K[t]$. Therefore, $K[t]$ is not perfectly power invariant.

PROPOSITION 2.2. *If a ring A is generated by its central idempotents, then A is perfectly power invariant. In particular a Boolean ring is perfectly power invariant.*

Proof. Let B be a ring such that there is an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$. It is straightforward to show that the only central idempotents of $B[[X]]$ are those of B, therefore $\phi(A) \subseteqq B$. Thus B is perfectly power invariant.

PROPOSITION 2.3. *Let K be a field and let Π be the prime field of K. If K is algebraic over Π, then K is perfectly power invariant.*

Proof. Let B be a ring such that there is an isomorphism ϕ of $K[[X]]$ onto $B[[X]]$. Since K is strongly power invariant, we have $K \cong B$. Therefore, B is a field. Clearly, $\phi(\Pi)$ is the prime field of B. It is straightforward to show that any element $f \in B[[X]]$; $f \notin B$, is not algebraic over a field B. So f is not algebraic over $\phi(\Pi)$. But $\phi(K)$ is algebraic over $\phi(\Pi)$, therefore $\phi(K) \subseteqq B$. Thus K is perfectly power invariant.

COROLLARY 2.4. *Let D be an integral domain and let Π be the prime ring of D (that is, Π is the subring of D generated by the identity element of D). If D is integral over Π, then D is perfectly power invariant.*

COROLLARY 2.5. *An algebraic number field is perfectly power invariant, and the ring of algebraic integers is perfectly power invariant.*

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