POWER INVARIANT RINGS

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A ring $A$ is called power invariant if whenever $B$ is a ring such that the formal power series rings $A[[X]]$ and $B[[X]]$ are isomorphic, then $A$ and $B$ are isomorphic. A ring $A$ is said to be strongly power invariant if whenever $B$ is a ring and $\phi$ is an isomorphism of $A[[X]]$ onto $B[[X]]$, then there exists a $B$-automorphism $\psi$ of $B[[X]]$ such that $\psi(X) = \phi(X)$. Strongly power invariant rings are power invariant. For any commutative ring $A$, $A/J(A)^n$ is strongly power invariant, where $J(A)$ is the Jacobson radical of $A$, and $n$ is any positive integer. A left or right Artinian ring is strongly power invariant. If $A$ is a left or right Noetherian ring, then $A[t]$, the polynomial ring in an indeterminate $t$ over $A$, is strongly power invariant.

Introduction. Coleman and Enochs [2] raised the following question: Can there be nonisomorphic rings $A$ and $B$ whose polynomial rings $A[X]$ and $B[X]$ are isomorphic? Recently Hochster [4] answered this question in the affirmative. The analogous question about a commutative formal power series ring was raised by O'Malley [7]: If $A[[X]] \cong B[[X]]$, must $A \cong B$? We know no counterexamples.

In this paper all rings are assumed to have identity elements. The Jacobson radical and the prime radical (the intersection of all prime ideals) of a ring $A$ will be denoted by $J(A)$ and $\text{rad}(A)$, respectively. Let $A[[X]]$ be the formal power series ring in a commutative indeterminate $X$ over a ring $A$, and let $\beta$ be a central element of $A[[X]]$. Then $(\beta^n)$ will denote the ideal of $A[[X]]$ generated by $\beta^n$ for a nonnegative integer $n$, and $(A[[X]], (\beta))$ denotes the topological ring $A[[X]]$ with the $(\beta)$-adic topology. It is well known that $(A[[X]], (\beta))$ is Hausdorff if and only if $\bigcap_{n=1}^{\infty} (\beta^n) = (0)$. The $(\beta)$-adic topology is metrizable in the obvious way, and we say that $(A[[X]], (\beta))$ is complete if each Cauchy sequence of $A[[X]]$ converges in $A[[X]]$. Then clearly $(A[[X]], (X))$ is a complete Hausdorff space.

Extending the terminology used in [2], O'Malley [7] defined "power invariant ring" and "strongly power invariant ring" as follows: A ring $A$ is power invariant whenever $B$ is a ring such that $A[[X]] \cong B[[X]]$, then $A \cong B$. A ring $A$ is said to be strongly power invariant if whenever $B$ is a ring and $\phi$ is an isomorphism of $A[[X]]$ onto $B[[X]]$, then there exists a $B$-automorphism $\psi$ of $B[[X]]$ such that $\psi(X) = \phi(X)$.

Let $A$ be a strongly power invariant ring and let $\phi$ be an isomorphism of $A[[X]]$ onto $B[[X]]$. Then there exists a $B$-automorphism
ψ of $B[[X]]$ such that $\psi(X) = \phi(X)$. Then $\psi^{-1}\phi$ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $(\psi^{-1}\phi)(X) = X$. Therefore, $A \cong A[[X]]/(X) \cong B[[X]]/(X) \cong B$. Thus a strongly power invariant ring is power invariant.

In this paper we attempt to impose conditions on a ring $A$ so that $A[X] \cong B[[X]]$ implies $A \cong B$.

1. Strongly power invariant rings. The following theorem extends Theorem (4.5) in [8].

**Theorem 1.1.** Let $B$ be a ring and $\beta = \sum_{i=0}^{\infty} b_i X^i$, an element of $B[[X]]$. Then the following statements are equivalent:

(1) $b_i$ is central for each $i$, $b_i$ is a unit, and $(B[[X]], (\beta))$ is a complete Hausdorff space.

(2) There exists a $B$-automorphism of $\psi$ of $B[[X]]$ such that $\psi(X) = \beta$.

**Proof.** Suppose that (2) holds. Since $(B[[X]], (\beta))$ is a complete Hausdorff space and $\psi$ is a uniformly bicontinuous mapping of $(B[[X]], (X))$ onto $(B[[X]], (\beta))$, $(B[[X]], (\beta))$ is a complete Hausdorff space. Since $X$ commutes with every element of $B$, $\beta$ commutes with any element of $B$ and therefore $b_i$ is central for each $i$. Let $C$ be the center of $B$. Then $C[[X]]$ is the center of $B[[X]]$ and hence $\phi(C[[X]]) = C[[X]]$. Then $\psi$ induces the $C$-automorphism of $C[[X]]$ which maps $X$ onto $\beta$. Therefore, by Theorem (4.5) in [8], $b_i$ is a unit. Thus (2) implies (1).

Suppose that (1) holds. Since $(B[[X]], (\beta))$ is a complete Hausdorff space, there is a $B$-endomorphism $\psi$ of $B[[X]]$ such that $\psi(X) = \beta$. This comes from the same argument as the commutative case; namely (2.2) in [8]. Since $b_i$ is central for each $i$, that $\psi$ is a $B$-automorphism, also follows from the commutative argument; namely Lemma (4.2) and Corollary (4.4) in [8]. This completes the proof.

Let $\phi$ be an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta = \sum_{i=0}^{\infty} b_i X^i$. By similar argument as in the proof of Theorem 1.1, we see that $b_i$ is central in $B$ for each $i$ and $(B[[X]], (\beta))$ is a complete Hausdorff space. Therefore, by Theorem 1.1, we see that a ring $A$ is strongly power invariant if and only if whenever $B$ is a ring and $\phi$ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$, then $b_i$ is a unit.

The following lemma has appeared as Result 4.3 in [7] for the commutative case.

**Lemma 1.2.** For any ring $A$, $A/J(A)$ is strongly power invariant. In particular, if $A$ is a semisimple ring then $A$ is strongly power invariant.
Proof. Let $A$ be a semisimple. To prove this lemma, it suffices to show that $A$ is strongly power invariant. Let $B$ be a ring such that there is an isomorphism $\phi$ of $A[[X]]$ onto $B[[X]]$. Let $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$. Since $J(A) = (0)$, it follows that $J(A[[X]]) = (X)$, and

$$\phi(J(A[[X]])) = \phi((X)) = (\phi(X)) \cdot B[[X]] = J(B[[X]]) .$$

Clearly $X \in J(B[[X]])$, and so there exists $\sum_{i=0}^{\infty} c_i X^i \in B[[X]]$ such that $\phi(X) \cdot \sum_{i=0}^{\infty} c_i X^i = X$; i.e., $(\sum_{i=0}^{\infty} b_i X^i) \cdot (\sum_{i=0}^{\infty} c_i X^i) = X$. Then $b_0 c_i + b_i c_0 = 1$. But $b_0 \in J(B)$, so $1 - b_0 c_1$ is a unit. Therefore, $b_i c_0$ is a unit, and so $b_i$ is a unit. Hence $A$ is strongly power invariant.

**Theorem 1.3.** If $A$ is a commutative ring, then for any positive integer $n$, $A/J(A)^n$ is strongly power invariant.

**Proof.** Let $A$ be a commutative ring such that $J(A)$ is nilpotent. To prove this theorem, it suffices to show that $A$ is strongly power invariant. Let $B$ be a ring such that there is an isomorphism $\phi$ of $A[[X]]$ onto $B[[X]]$, and let $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$. Then clearly $B$ is commutative. Let $N$ be the ideal of nilpotent elements of $B$, and let $\{P_i\}$ be the collection of prime ideals of $B$. Then $N = \cap_i P_i$, and for each $\gamma$, $P_i[[X]]$ is a prime ideal of $B[[X]]$. Therefore, the ideal of nilpotent elements of $B[[X]]$ is a subset of $N[[X]]$. Note that $N[[X]]$ is not necessarily the ideal of nilpotent elements of $B[[X]]$. Since $J(A)$ is nilpotent, $J(A[[X]])$ is the ideal of nilpotent elements of $A[[X]]$. Therefore, $\phi(J(A[[X]])) \subseteq N[[X]]$. In order to show the opposite inclusion, let $g = \sum_{i=0}^{\infty} g_i X^i \in N[[X]]$; $g_i \in N$ for each $i$, and let $\phi^{-1}(X) = \alpha = \sum_{i=0}^{\infty} a_i X^i$, $a_i \in A$. Then $\phi^{-1}(g) = \sum_{i=0}^{\infty} \phi^{-1}(g_i) \alpha^i$, and $\phi^{-1}(g_i)$ is a nilpotent element of $A[[X]]$ for each $i$. Note that $a_0 \in J(A)$ i.e., $a_0$ is nilpotent, and $\phi^{-1}(g_i) \in J(A)[[X]]$. Expanding $\sum_{i=0}^{\infty} \phi^{-1}(g_i) \alpha^i$ in powers of $X$, we see that the coefficient of $X^i$ is an element of $J(A)$ for each $i$ since $a_0$ is nilpotent. Thus $\phi^{-1}(g) \in J(A)[[X]]$. Therefore, we get $\phi(J(A))[[X]] = N[[X]]$. Consider the isomorphism $\tilde{\phi}: (A/J(A))[[X]] \rightarrow (B/N)[[X]]$ given by

$$(A/J(A))[[X]] \longrightarrow A[[X]]/J(A)[[X]] \longrightarrow B[[X]]/N[[X]] \longrightarrow (B/N)[[X]].$$

where the middle isomorphism is induced by $\phi$ and others are the obvious ones. Then it follows that $\tilde{\phi}(X) = \sum_{i=0}^{\infty} \tilde{b}_i X^i$, where $\tilde{b}_i$ denotes the coset $b_i + N$ in $B/N$. Since $A/J(A)$ is strongly power invariant, $\tilde{b}_i$ is a unit in $B/N$. But $N \subseteq J(B)$ so $b_i$ is a unit in $B$. Thus $A$ is strongly power invariant. This completes the proof.

**Corollary 1.4.** Let $A$ be a ring and $C$, the center of $A$. If $J(C)$ is nilpotent, then $A$ is strongly power invariant. In particular, if $C$ is a Artinian ring, then $A$ is strongly power invariant.
Proof. Let $B$ be a ring such that there is an isomorphism $\phi$ of $A[[X]]$ onto $B[[X]]$, and let $\phi(X) = \sum_{i=0}^{\infty} b_iX^i$. If $D$ denotes the center of $B$, $\phi(C[[X]]) = D[[X]]$. But by Theorem 1.3, $C$ is strongly power invariant. Therefore, $b_i$ is a unit and so $A$ is strongly power invariant.

It is well known that the prime radical of a ring $A$, denoted by $\text{rad}(A)$, is the intersection of all prime ideals of $A$, and also it is the ideal of all strongly nilpotent elements of $A$. (P. 55–56 in [6].) Clearly, every strongly nilpotent element is nilpotent. In particular, if $A$ is commutative, then every nilpotent element is strongly nilpotent. Note that if $A$ is a commutative Noetherian ring, and $N$ is the ideal of nilpotent elements of $A$, then $N[[X]]$ is the ideal of nilpotent elements of $A[[X]]$ [3]. The following lemma extends this statement to the noncommutative case.

**Lemma 1.5.** If $A$ is a left or right Noetherian ring, then $\text{rad}(A[[X]]) = \text{rad}(A)[[X]]$.

**Proof.** We show that if $P$ is a prime ideal of $A$, then $P[[X]]$ is a prime ideal of $A[[X]]$. Suppose that $P$ is a prime ideal of $A$ and $P[[X]]$ is not a prime ideal of $A[[X]]$. Then there exist $f = \sum_{i=0}^{\infty} f_iX^i$ and $g = \sum_{i=0}^{\infty} g_iX^i$ in $A[[X]]$ such that $f \cdot A[[X]] \cdot g \subseteq P[[X]]$ but $f \not\in P[[X]]$ and $g \not\in P[[X]]$. Let $m$ be the smallest integer such that $f_m \in P$, and let $n$ be the smallest integer such that $g_n \in P$. Since $f \cdot A[[X]] \cdot g \subseteq P[[X]]$, $f \cdot a \cdot g$ belongs to $P[[X]]$ for any element $a$ of $A$. Expanding $f \cdot a \cdot g$ in powers of $X$, we see that the coefficient of $X^{m+n}$ in $\sum_{i+j=0}^{m+n} f_i a_j g_j$ which is in $P$. But $\sum_{i+j=0}^{m+n} f_i a_j g_j - f_m a_n g_n \subseteq P$, so $f_m a_n g_n$ must be in $P$. Therefore, $f_m a_n g_n \subseteq P$, but $P$ is a prime ideal of $A$; so $f_m \in P$ or $g_n \in P$. This is a contradiction to our choice of $m$ and $n$. Hence $P[[X]]$ is a prime ideal of $A[[X]]$. Therefore, it follows that $\text{rad}(A[[X]]) \subseteq \text{rad}(A)[[X]]$. To show the opposite inclusion, we let $\sum_{i=0}^{\infty} a_iX^i \in \text{rad}(A)[[X]]$. Then each $a_i$ is strongly nilpotent. Let $A$ be the ideal of $A$ generated by the set of all $a_i$'s. Then clearly $A \subseteq \text{rad}(A)$; therefore, $A$ is a nil ideal of $A$. But since $A$ is left or right Noetherian, $A$ is nilpotent. Thus $\sum_{i=0}^{\infty} a_iX^i \in \text{rad}(A[[X]])$. Therefore, $\text{rad}(A[[X]]) = \text{rad}(A)[[X]]$.

**Theorem 1.6.** Let $A$ be a left or right Noetherian ring and let $N = \text{rad}(A)$. Then $A$ is strongly power invariant if $A/N$ is strongly power invariant.

**Proof.** Let $B$ be a ring such that there is an isomorphism $\phi$ of $A[[X]]$ onto $B[[X]]$, and let $M = \text{rad}(B)$. Since $A$ is left (or right) Noetherian, $A[[X]]$ is left (or right) Noetherian. Then $B[[X]]$ is left
(or right) Noetherian, and therefore, $B$ is left (or right) Noetherian. So $\text{rad}(B[[X]]) = M[[X]]$ (by Lemma 1.5). From the invariance of the prime radical under isomorphism, we have that $\phi(N[[X]]) = M[[X]]$. Write $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$; $b_i \in B$. Consider the isomorphism, $\tilde{\phi}: (A/N)[[X]] \rightarrow (B/M)[[X]]$ given by

$$
(A/N)[[X]] \longrightarrow A[[X]]/N[[X]] \longrightarrow B[[X]]/M[[X]] \longrightarrow (B/M)[[X]],
$$

where the middle isomorphism is induced by $\phi$ and the others are the obvious ones. Since $A/N$ is strongly power invariant, we can show that $b_i$ is a unit of $B$ by the same argument as in the proof of Theorem 1.3. Thus $A$ is a strongly power invariant ring.

**Corollary 1.7.** If $A$ is a left or right Noetherian ring such that $J(A)$ is nil, then $A$ is strongly power invariant.

**Proof.** Clearly $J(A)$ is nilpotent. So every element of $J(A)$ is strongly nilpotent. Therefore, $J(A) = \text{rad}(A)$. By Lemma 1.2 and Theorem 1.6, $A$ is strongly power invariant.

**Corollary 1.8.** A left or right Artinian ring is strongly power invariant.

**Corollary 1.9.** If $A$ is a left or right Noetherian ring and if $A[t]$ is the polynomial ring in a commutative indeterminate $t$ over $A$, then $A[t]$ is strongly power invariant.

**Proof.** It is well known that for any ring $A$, $J(A[t]) = N[t]$ holds, where $N = J(A[t]) \cap A$ and $N$ is a nil ideal in $A$ [1]. Since $A$ is left (or right) Noetherian, $N$ is nilpotent and $A[t]$ is left (or right) Noetherian. Thus $J(A[t]) = N[t]$ is a nilpotent ideal in $A[t]$. Therefore, by Corollary 1.7, $A[t]$ is strongly power invariant.

2. Perfect power invariant rings. The following proposition extends Theorem 3.1 in [7].

**Proposition 2.1.** Let $A$ and $B$ be rings and suppose that $\phi$ is an isomorphism of $A[[X]]$ onto $B[[X]]$. If $\phi(A) \subseteq B$, then $\phi(A) = B$.

**Proof.** Let $\phi(X) = \beta = \sum_{i=0}^{\infty} b_i X^i$; $b_i \in B$. Then $b_i$ is central for each $i$ and $(B[[X]], (\beta))$ is a complete Hausdorff space. Then there exists a $B$-endomorphism $\psi$ of $B[[X]]$ into $B[[X]]$ such that $\psi(X) = \beta$. Then by hypothesis, we have

$$
B[[X]] = \phi(A)[[\beta]] \subseteq B[[\beta]] \subseteq B[[X]].
$$
Therefore, \( B[[\beta]] = B[[X]] \), which implies \( \psi \) is onto. Now let \( \bar{B} \) be \( B/(b_i) \) and let \( \bar{b} = b + (b_i) \) for \( b \in B \). Then \( X \to \sum_{i=0}^{\infty} \bar{b}_i X^i \) induces a surjective \( \bar{B} \)-endomorphism of \( \bar{B}[[X]] \). But \( \bar{b}_i \) is 0, so this impossible unless \( (b_i) = B \); i.e., \( b_i \) is a unit. Therefore, by Theorem 1.1, \( \psi \) is a \( B \)-automorphism of \( B[[X]] \). Then \( \psi^{-1} \phi \) is an isomorphism of \( A[[X]] \) onto \( B[[X]] \) such that \( \psi^{-1} \phi(A) \subseteq B \) and \( \psi^{-1} \phi(X) = X \). So \( \psi^{-1} \phi(A) = B \); but \( \psi^{-1}(B) = B \); therefore \( \phi(A) = B \).

**DEFINITION.** A ring \( A \) is said to be perfectly power invariant if whenever \( B \) is a ring and \( \phi \) is an isomorphism of \( A[[X]] \) onto \( B[[X]] \), then \( \phi(A) \subseteq B \).

Let \( A \) be a perfectly power invariant ring, and let \( B \) be a ring such that there is an isomorphism \( \phi \) of \( A[[X]] \) onto \( B[[X]] \). In the proof of Proposition 2.1, we have shown that there exists a \( B \)-automorphism \( \psi \) of \( B[[X]] \) such that \( \psi(X) = \phi(X) \). So a perfectly power invariant ring is strongly power invariant. But a strongly power invariant ring is not necessarily perfectly power invariant.

**EXAMPLE.** Let \( K \) be a field and let \( K[t] \) be the polynomial ring in an indeterminate \( t \) over \( K \) then \( K[t] \) is strongly power invariant (by Corollary 1.9). But, by Corollary 2.8 in [5], we see that there is an automorphism \( \phi \) of \( K[[X]] \) such that \( \phi(K[t]) \not\subseteq K[t] \). Therefore, \( K[t] \) is not perfectly power invariant.

**PROPOSITION 2.2.** If a ring \( A \) is generated by its central idempotents, then \( A \) is perfectly power invariant. In particular a Boolean ring is perfectly power invariant.

**Proof.** Let \( B \) be a ring such that there is an isomorphism \( \phi \) of \( A[[X]] \) onto \( B[[X]] \). It is straightforward to show that the only central idempotents of \( B[[X]] \) are those of \( B \), therefore \( \phi(A) \subseteq B \). Thus \( B \) is perfectly power invariant.

**PROPOSITION 2.3.** Let \( K \) be a field and let \( \Pi \) be the prime field of \( K \). If \( K \) is algebraic over \( \Pi \), then \( K \) is perfectly power invariant.

**Proof.** Let \( B \) be a ring such that there is an isomorphism \( \phi \) of \( K[[X]] \) onto \( B[[X]] \). Since \( K \) is strongly power invariant, we have \( K \cong B \). Therefore, \( B \) is a field. Clearly, \( \phi(\Pi) \) is the prime field of \( B \). It is straightforward to show that any element \( f \in B[[X]]; f \notin B \), is not algebraic over a field \( B \). So \( f \) is not algebraic over \( \phi(\Pi) \). But \( \phi(K) \) is algebraic over \( \phi(\Pi) \), therefore \( \phi(K) \subseteq B \). Thus \( K \) is perfectly power invariant.
**COROLLARY 2.4.** Let $D$ be an integral domain and let $\Pi$ be the prime ring of $D$ (that is, $\Pi$ is the subring of $D$ generated by the identity element of $D$). If $D$ is integral over $\Pi$, then $D$ is perfectly power invariant.

**COROLLARY 2.5.** An algebraic number field is perfectly power invariant, and the ring of algebraic integers is perfectly power invariant.

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