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**OSCILLATION AND NONOSCILLATION CRITERIA FOR SOME  
SELF-ADJOINT EVEN ORDER LINEAR DIFFERENTIAL  
OPERATORS**

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## OSCILLATION AND NONOSCILLATION CRITERIA FOR SOME SELF-ADJOINT EVEN ORDER LINEAR DIFFERENTIAL OPERATORS

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**Oscillation and nonoscillation results are presented for the operator**

$$L_{2n}y = \sum_{k=0}^n (-1)^{n-k} (p_k y^{(n-k)})^{(n-k)}$$

where  $p_0(x) > 0$  on  $(0, \infty)$  and for  $k = 0, 1, \dots, n$ ,  $p_k$  is a real-valued,  $n - k$  times differentiable function on  $(0, \infty)$ . Also,  $y$  is an element of the set of all real-valued,  $2n - \text{fold}$  continuously differentiable, finite functions on  $(0, \infty)$ .

In particular, a nonoscillation result is given for  $L_{2n}$  without sign restrictions on the coefficients. Oscillation results are given for  $L_4$  without the requirement that  $p_1$  be negative for large  $x$ . Finally, the oscillation of

$$L_{2n}y = (-1)^n (ry^{(n)})^{(n)} + py$$

is considered for  $r(x)$  not necessarily bounded.

The oscillatory behavior of  $L_4$  has been considered by Leighton and Nehari [8], Barrett [1], and Hinton [4]. In general,  $L_{2n}$  has been considered by Glazman [2], Hinton [5], Hunt [6], and Hunt and Namboodiri [7].

**DEFINITION 0.1.** The operator  $L_{2n}$  is called *oscillatory on*  $[a, b]$  provided there is a function  $y$ ,  $y \not\equiv 0$ , and numbers  $c$  and  $d$  for which  $a \leq c < d \leq b$  such that  $L_{2n}y = 0$  and

$$y^{(k)}(c) = 0 = y^{(k)}(d) \text{ for } k = 0, 1, \dots, n - 1.$$

Otherwise,  $L_{2n}$  is called *nonoscillatory on*  $[a, b]$ . The operator  $L_{2n}$  is called *oscillatory on*  $[a, \infty)$  if for any given  $c \geq a$  there is a  $d > c$  such that  $L_{2n}$  is oscillatory on  $[c, d]$ .

**DEFINITION 0.2.** Given a positive integer  $n$  and a number  $a$  define  $\mathfrak{D}_n(b)$  for all  $b > a$  to be the set of all real-valued functions  $y$  with the following properties:

- (a)  $y^{(k)}$  is absolutely continuous on  $[a, b]$  for  $k = 0, 1, \dots, n - 1$ ,
- (b)  $y^{(n)}$  is essentially bounded on  $[a, b]$ , and
- (c)  $y^{(k)}(a) = 0 = y^{(k)}(b)$  for  $k = 0, 1, \dots, n - 1$ .

For  $y \in \mathfrak{D}_n(b)$  define

$$I_b(y) = \int_a^b \sum_{k=0}^n p_k(x) (y^{(n-k)}(x))^2 dx$$

which is called the *quadratic functional* for  $L_{2n}$ .

The following theorem has provided the primary motivation for the results which are to follow.

**THEOREM 0.1** (Reid [9]). *The following two statements are equivalent.*

- (i) *The operator  $L_{2n}$  is nonoscillatory on  $[a, b]$ .*
- (ii) *If  $y \in \mathfrak{D}_n(b)$  and  $y \not\equiv 0$  then  $I_b(y) > 0$ .*

Consequently, in order to show that  $L_{2n}$  is oscillatory on  $(0, \infty)$ , given any  $a > 0$ , it will suffice to construct a  $y \in \mathfrak{D}_n(b)$  for some  $b > a$  for which  $I_b(y)$  is not positive and  $y \not\equiv 0$ . This is the technique of proof for all of the oscillation theorems which follow.

This method of proof is especially conducive to oscillation theorems which require that integral conditions be met by the coefficients of  $L_{2n}$ . For example, Glazman [2, p. 104] showed that  $(-1)^n y^{(2n)} + py$  is oscillatory on  $(0, \infty)$  when  $\int_0^\infty p = -\infty$  (see Theorem 3.2).

Initially, the construction of  $y$  is suggested by the conditions of the hypothesis on the coefficients of  $L_{2n}$  and the corresponding quadratic formula. For example, to establish the above result, Glazman let  $y \equiv 1$  over the major portion of the interval  $[a, b]$ . To show that  $y^{iv} - (qy)'$  is oscillatory when  $\int_0^\infty q = -\infty$  (see Theorem 2.2) the author let  $y(x) = x - a$  over a portion of  $[a, b]$ . Next, we construct  $y$  over the remaining portion of  $[a, b]$  to insure that  $y \in \mathfrak{D}_n(b)$  and the integral of  $p_{n-k} \cdot y^{(n-k)^2}$  is bounded above for  $k = 0, 1, \dots, n$  independent of  $b$ .

For other proofs using this method the reader should consult Glazman [2, pp. 95-105] and Hinton [4].

### 1. The nonoscillation of $L_{2n}$ .

**LEMMA 1.1** (Glazman [2, p. 83]).

- (i) *If  $g(a) = 0$  for some  $a > 0$  and  $g'$  is continuous on  $[a, b]$ , then*

$$\int_a^b x^{-2m}(g(x))^2 dx \leq \left(\frac{2}{2m-1}\right)^2 \int_a^b x^{-2m+2}(g'(x))^2 dx$$

*for  $m$  a positive integer. Moreover, if  $g \not\equiv 0$  on  $[a, b]$ , the above inequality is strict.*

- (ii) *If  $g \not\equiv 0$ ,  $g^{(m)}$  is continuous on  $[a, b]$ , and  $g(a) = \dots = g^{(m-1)}(a) = 0$ , then*

$$\int_a^b x^{-2m}(g(x))^2 dx < \left(\frac{2^m}{1 \cdot 3 \cdot \dots \cdot (2m-1)}\right)^2 \int_a^b (g^{(m)})^2 dx.$$

A well known result in oscillation theory is a sufficient condition for the nonoscillation of  $L_2$  due to Hille [3]. A generalization of this result for  $L_{2n}$  is given in the next theorem.

**THEOREM 1.1.** *For  $L_{2n}$  defined above with  $p_0(x) \equiv 1$  let  $P_k^0(x) = p_k(x)$  and*

$$P_k^m(x) = \int_x^\infty P_k^{m-1}(t)dt$$

for  $m$  an integer greater than or equal to one when

$$-\infty < \int_x^\infty P_k^{m-1}(t)dt < \infty .$$

If for  $k = 1, \dots, n$  and  $x \geq a$  we have  $-\infty < \int_a^\infty P_k^m < \infty$  for  $m = 0, 1, \dots, k - 1$ ,  $x^k |P_k^k| \leq a_k$ , and  $\sum_{k=1}^n a_k M_k \leq 1$  where  $M_k = k! 2^{4k-1}/(2k)!$ , then  $L_{2n}$  is nonoscillatory on  $[a, b]$  for all  $b > a$ .

*Proof.* The proof is given only for  $n > 1$ . Suppose  $L_{2n}$  is oscillatory on  $[a, b]$ . Then, there are numbers  $c$  and  $d$  and a function  $y$  which is not identically zero such that  $L_{2n}y = 0$  and  $y^{(k)}(c) = 0 = y^{(k)}(d)$  for  $k = 0, 1, \dots, n - 1$ . Since  $(L_{2n}y)y = 0$  then

$$-\sum_{k=1}^n (-1)^{n-k} \int_c^d (p_k y^{(n-k)})^{(n-k)} y = (-1)^n \int_c^d y^{(2n)} y = \int_c^d [y^{(n)}]^2 ,$$

by integrating by parts  $n$  times. By integrating by parts  $n - k$  times we find that

$$-\int_c^d (-1)^{n-k} (p_k y^{(n-k)})^{(n-k)} y = -\int_c^d p_k (y^{(n-k)})^2 .$$

However, by integrating by parts  $k$  times and using Leibniz's rule we obtain

$$\begin{aligned} -\int_c^d p_k (y^{(n-k)})^2 &= -\int_c^d P_k^k [(y^{(n-k)})^2]^{(k)} = -\int_c^d P_k^k \cdot \sum_{i=0}^k \binom{k}{i} (y^{(n-k)})^{(k-i)} (y^{(n-k)})^{(i)} \\ &\leq \sum_{i=0}^k \binom{k}{i} \int_c^d \frac{|y^{(n-i)}|}{t^i} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \cdot t^k |P_k^k| \\ &\leq a_k \sum_{i=0}^k \binom{k}{i} \cdot \int_c^d \frac{|y^{(n-i)}|}{t^k} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \\ &= a_k \left[ 2 \int_c^d |y^{(n)}| \cdot \frac{|y^{(n-k)}|}{t^k} + \sum_{i=1}^{k-1} \binom{k}{i} \int_c^d \frac{|y^{(n-i)}|}{t^i} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \right] \\ &\leq a_k \left[ 2 \|y^{(n)}\|_2 \left\| \frac{y^{(n-k)}}{t^k} \right\|_2 + \sum_{i=1}^{k-1} \binom{k}{i} \left\| \frac{y^{(n-i)}}{t^i} \right\|_2 \left\| \frac{y^{(n-k+i)}}{t^{k-i}} \right\|_2 \right] \end{aligned}$$

$$\begin{aligned}
&< a_k \left[ \|y^{(n)}\|_2^2 \left( \frac{2^{k+1}}{1 \cdot 3 \cdot \dots \cdot (2k-1)} \right) \right. \\
&\quad \left. + \sum_{i=1}^{k-1} \binom{k}{i} \|y^{(n)}\|_2^2 \left( \frac{2^i}{1 \cdot 3 \cdot \dots \cdot (2i-1)} \right) \left( \frac{2^{k-i}}{1 \cdot 3 \cdot \dots \cdot (2(k-i)-1)} \right) \right] \\
&= a_k C_k \int_c^d [y^{(n)}]^2
\end{aligned}$$

by Lemma 1.1 and the Cauchy inequality where

$$\begin{aligned}
C_k &= \frac{2^{k+1}}{1 \cdot 3 \cdot \dots \cdot (2k-1)} \\
&\quad + \sum_{i=1}^{k-1} \binom{k}{i} \cdot \left( \frac{2^k}{[1 \cdot 3 \cdot \dots \cdot (2i-1)][1 \cdot 3 \cdot \dots \cdot (2(k-i)-1)]} \right).
\end{aligned}$$

A simplification shows that

$$C_k = [2^{2k} k! / (2k)!] \sum_{i=0}^k \binom{2k}{2i}.$$

Since

$$0 = (1-1)^{2k} = \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} = \sum_{i=0}^k \binom{2k}{2i} - \sum_{i=1}^k \binom{2k}{2i-1},$$

then

$$2^{2k-1} = \frac{1}{2} (1+1)^{2k} = \frac{1}{2} \left[ \sum_{i=0}^k \binom{2k}{2i} + \sum_{i=1}^k \binom{2k}{2i-1} \right] = \sum_{i=0}^k \binom{2k}{2i}.$$

Therefore,

$$C_k = [2^{4k-1} k! / (2k)!] = M_k.$$

Consequently,

$$\begin{aligned}
\int_c^d [y^{(n)}]^2 &= - \sum_{k=1}^n (-1)^{n-k} \int_c^d (p_k y^{(n-k)})^{(n-k)} y \\
&= - \sum_{k=1}^n \int_c^d p_k (y^{(n-k)})^2 < \sum_{k=1}^n a_k M_k \int_c^d [y^{(n)}]^2 \leq \int_c^d [y^{(n)}]^2
\end{aligned}$$

which is a contradiction. Therefore,  $L_{2n}$  is nonoscillatory on  $[a, b]$ .

It will be useful in applying Theorem 1.1 to note that  $M_{k+1} = 8M_k / (2k+1)$ .

For the remainder of this paper we will assume that  $p_k(x)$  is identically zero for  $k=1$  to  $n-2$  and will denote  $p_0(x)$ ,  $p_{n-1}(x)$ , and  $p_n(x)$  by  $r(x)$ ,  $q(x)$ , and  $p(x)$  respectively. Similarly,  $P_n^k(x)$  and  $P_{n-1}^k(x)$  will be denoted by  $P_k(x)$  and  $Q_k(x)$  respectively.

If  $p(x) = kx^{-4}$ ,  $r \equiv 1$ , and  $q \equiv 0$ , then  $L_4 y = 0$  is the familiar Euler equation. In this case,  $L_4$  is oscillatory if and only if  $k < -9/16$ .

Also,  $k < -9/16$  and  $p(x) = kx^{-4}$  implies  $x^2P_2(x) < -3/32$ . Theorem 1.1 shows that  $L_4y = y^{iv} + py$  is nonoscillatory when  $x^2|P_2(x)| \leq 3/32$ .

2. The oscillation of  $L_4$ . Using Theorem 0.1, Hinton [4] showed that  $L_4$  is oscillatory when  $\int^\infty 1/r = \infty, q \leq 0$ , and  $\int^\infty p = -\infty$ . The same technique yields the following results.

**THEOREM 2.1.** *Suppose that  $r(x) \leq N, q(x) \leq M$  and  $\int^\infty p = -\infty$  for  $x > 0$ , then  $L_4y = (ry'')'' - (qy')' + py$  is oscillatory on  $(0, \infty)$ .*

**THEOREM 2.2.** *If  $0 < r(x) \leq M, \int^\infty q = -\infty$ , and  $\int^\infty x^2|p(x)| < \infty$  then  $L_4y = (ry'')'' - (qy')' + py$  is oscillatory on  $(0, \infty)$ .*

*Proof.* Let  $\xi(x) = x^2/2$ . Define  $y(x)$  as follows:

$$y(x) = \begin{cases} 0 & x < a \\ \xi(x - a) & a \leq x < a + 1 \\ x - a - 1/2 & a + 1 \leq x < b_1 \\ -\xi(x - b_2) + b_1 - a & b_1 \leq x < b_2 = b_1 + 1 \\ b_1 - a & b_2 \leq x < b_3 \\ -\xi(x - b_3) + b_1 - a & b_3 \leq x < b_4 \\ -\xi'(b_4 - b_3)(x - b_4) + b_1 - a - \xi(b_4 - b_3) & b_4 \leq x < b_5 \\ \xi(x - b) & b_5 \leq x < b \\ 0 & b \leq x \end{cases}$$

It is easy to show that

$$\int_a^d r(y'')^2 + py^2 < 16M + \int_a^\infty x^2|p|$$

if we require that  $b_4 - b_3 = b - b_5 \leq 1$ . There is a number  $c$  such that

$$1 + 16M + \int_a^\infty x^2|p| + \int_a^{a+1} q(y')^2 + \int_{a+1}^x q \leq 0$$

for all  $x \geq c$ .

Let  $Y_1(x) = \int_c^x q(t)dt$ . Since  $Y_1(x)$  tends to  $-\infty$  as  $x$  tends to  $\infty$  there is a number  $b_1$  which is the last zero of  $Y_1(x)$ . Hence,

$$\int_{b_1}^{b_2} q(y')^2 = Y_1(x)(y')^2|_{b_1}^{b_2} - 2\int_{b_1}^{b_2} y'y'' Y_1 < 0$$

since  $Y_1(b_1) = 0 = y'(b_2), y'' = -1, y' \geq 0$ , and  $Y_1 < 0$  on  $(b_1, b_2]$ .

Let  $Y_2(x) = \int_{b_2}^x q(t)dt$  and let  $b_3$  be the last zero of  $Y_2$ . Pick  $b_4$  so

that  $-1/2 \leq Y_2(x) \leq 0$  on  $[b_3, b_4]$  and  $b_4 - b_3 \leq 1$ . Since  $y' = 0$  on  $[b_2, b_3]$ ,  $-1 \leq y' \leq 0$  on  $[b_3, b]$ ,  $Y_2 \leq 0$  on  $[b_3, \infty)$ ,  $\int_{b_1}^{b_2} q(y')^2 < 0$  we have that

$$\begin{aligned} \int_{b_1}^b q(y')^2 &< \int_{b_3}^b q(y')^2 = Y_2(x)(y')^2 \Big|_{b_3}^b - 2 \int_{b_3}^b y' y'' Y_2 \\ &= -2 \int_{b_3}^{b_4} y'(-1) Y_2 - 2 \int_{b_5}^b y'(1) Y_2 \\ &< 2 \int_{b_3}^{b_4} y' Y_2 < 2 \int_{b_3}^{b_4} |y' Y_2| < 1. \end{aligned}$$

Consequently,

$$\begin{aligned} I_4(y) &= \int_a^b r(y'')^2 + q(y')^2 + p y^2 \\ &< 16M + \int_a^\infty x^2 |p| + \int_a^{a+1} q(y')^2 + \int_{a+1}^{b_1} q + 1 \leq 0 \end{aligned}$$

which completes the proof.

We now know that  $L_4$  is oscillatory on  $(0, \infty)$  is for  $r$  bounded either  $\int_a^\infty p = -\infty$  and  $q \leq 0$  or  $\int_a^\infty q = -\infty$  and  $p \leq 0$ . These facts suggest the results of the following theorem.

**THEOREM 2.3.** *If  $\int_a^\infty p = -\infty$ ,  $\int_a^\infty q = -\infty$ , and  $0 < r(x) \leq M$  then  $L_4$  is oscillatory on  $(0, \infty)$ .*

*Proof.* Except for some changes in the parameters we may define  $y(x)$  as in the proof of Theorem 2.2. As before, if  $b_4 - b_3 = b_6 - b_5 \leq 1$  then  $\int_a^b r(y'')^2 \leq 16M$ .

There is a number  $c$  such that

$$1 + 16M + \int_a^{a+1} q(y')^2 + \int_{a+1}^x q < 0$$

for all  $x \geq c$ . Let  $Y(x) = \int_c^x q(t)dt$  and let  $b_1$  be the last zero of  $Y(x)$ . Integrating by parts we obtain the fact that

$$\int_{b_1}^b q(y')^2 = -2 \int_{b_1}^b y' y'' Y$$

since  $Y(b_1) = 0 = y'(b_1)$ . Since  $y' \geq 0$ ,  $y'' = -1$ , and  $Y \leq 0$  on  $[b_1, b_2]$  then

$$-2 \int_{b_1}^b y' y'' Y < -2 \int_{b_2}^b y' y'' Y.$$

Since  $y'' = 0$  on  $[b_2, b_3]$  and  $[b_4, b_5]$ ,  $y' \leq 0$  on  $[b_5, b]$ ,  $y'' = 1$  on  $[b_5, b]$ , and  $Y < 0$  on  $[b_5, b]$ , then

$$-2 \int_{b_2}^b y' y'' Y < -2 \int_{b_3}^{b_4} y' y'' Y = 2 \int_{b_3}^{b_4} y' Y.$$

But, on  $[b_3, b_4]$ ,  $y' \leq 1$ . Consequently,

$$\int_{b_1}^b q(y')^2 < 2 \int_{b_3}^{b_4} y' Y \leq 2 \int_{b_3}^{b_4} |Y|.$$

Since  $\int_a^\infty p = -\infty$ , there is a number  $d > b_2$  such that for  $x \geq d$

$$\int_a^{b_2} p y^2 + (b_1 - a)^2 \int_{b_2}^x p < 0.$$

Let  $W(x) = \int_a^x p$  and  $b_3 \geq d$  be the last zero of  $W$ . Hence,

$$\int_{b_3}^b p y^2 = -2 \int_{b_3}^b y y' W(t) dt < 0.$$

Let  $N = \max \{ |Y(x)| : x \in [b_3, b_3 + 1] \}$  which we may assume is greater than or equal to one. Pick  $b_4$  so that  $b_4 - b_3 = 1/(2N)$ . Consequently,

$$\int_{b_1}^b q(y')^2 < 1.$$

Pick  $b_5$  so that  $\lim_{x \rightarrow b_5^-} y(x) = (b_4 - b_3)^2/2$  and pick  $b$  so that  $b - b_5 = b_4 - b_3$ . We now have that

$$I_b(y) < 16M + \int_a^{a+1} q(y')^2 + \int_{a+1}^x q + 1 + \int_a^{b_2} p y^2 + \int_{b_2}^{b_3} (b_1 - a)^2 p < 0.$$

**THEOREM 2.4.** *If  $0 < r(x) \leq M$ ,  $-\infty < \int_a^\infty p < \infty$ ,  $\int_a^\infty P_1 = -\infty$ ,  $\int_a^\infty |q| x^{-1} < \infty$ , and  $q(x) \rightarrow 0$  as  $x \rightarrow \infty$  then  $L_4$  is oscillatory on  $(0, \infty)$ .*

*Proof.* Let  $\xi(x) = -(3x^3 - 5x^2)/2$ ,  $\alpha(x) = \sqrt{x}$ , and  $\beta(x) = x^2$ . Let

$$y(x) = \begin{cases} 0 & x < a \\ \xi(x - a) & a \leq x < a + 1 \\ \alpha(x - a) & a + 1 \leq x < b_1 \\ -\beta(x - b_2) + \alpha(b_1 - a) + \beta(b_1 - b_2) & b_1 \leq x < b_2 \\ \alpha(b_1 - a) + \beta(b_1 - b_2) & b_2 \leq x < b_3 \\ -\beta(x - b_3) + y(b_2) & b_3 \leq x < b_4 \\ \alpha(b - x) & b_4 \leq x < b_5 \\ \xi(b - x) & b_5 \leq x < b \\ 0 & b \leq x. \end{cases}$$



Given  $b_1$  and  $b_3$  we choose  $b_4$ ,  $b_5$ , and  $b$  so that  $b_4 - b_3 = b_2 - b_1$ ,  $b_5 - b_4 = b_1 - (a + 1)$ , and  $b = b_5 + 1$ . Actually, only  $b_1$  and  $b_3$  will be chosen for reasons other than symmetry and the continuity of  $y$  and  $y'$ .

First, note that since we are going to pick  $b_1, \dots, b_5$  so that  $y \in \mathfrak{D}_2(b)$  then

$$\int_a^b py^2 = -P_1 y^2 \Big|_a^b + \int_a^b P_1 (y^2)' = \int_a^b P_1 (y^2)' .$$

Hence,

$$I_b(y) \leq \int_a^b M(y'')^2 + q(y')^2 + P_1(y^2)' .$$

Calculations show that

$$\int_a^b M(y'')^2 \leq M \left[ 2 \int_0^1 (5 - 9x)^2 + 2 + \frac{1}{2} \int_1^\infty x^{-3} dx \right] \equiv M_1$$

since  $y'$  being continuous requires that  $0 < b_2 - b_1 = b_4 - b_3 \leq 1/4$ .

Since  $\lim_{x \rightarrow \infty} q(x) = 0$  and  $q$  is continuous then  $q$  is bounded by some number,  $B$ , on  $[a, \infty)$ . Let

$$A = 4B \int_0^1 u^2 + B \int_0^1 \left( -\frac{9}{2} u^2 + 5u \right)^2 du + 1 .$$

There is a number  $c$  so that

$$\begin{aligned} M_1 + 2 + \int_a^{a+1} [q(y')^2 + P_1(y^2)'] \\ + A + B + (a + 1) \int_{a+1}^\infty x^{-1} |q(x)| \leq - \int_{a+1}^x P_1(t) dt \end{aligned}$$

and  $|P_1(x)| \leq 1$  for all  $x \geq c$  since  $P_1 \rightarrow 0$  as  $x \rightarrow \infty$ .

Let  $R(x) = \int_a^x P_1(t) dt$  and  $b_1$  be the last zero of  $R(x)$ . Pick  $b_2$  so that  $1/(2\sqrt{b_1 - a}) = -2(b_1 - b_2)$  which insures that  $y'$  is continuous at  $b_1$ . We now have that

$$\begin{aligned} \int_{a+1}^{b_1} q(y')^2 &\leq \int_{a+1}^{b_1} (x - a)^{-1} |q| = \int_{a+1}^{b_1} x(x - a)^{-1} x^{-1} |q| \\ &< (a + 1) \int_{a+1}^{b_1} x^{-1} |q| < (a + 1) \int_{a+1}^\infty x^{-1} |q| \end{aligned}$$

and

$$\int_{b_1}^{b_2} q(y')^2 \leq B \int_{b_1}^{b_2} 4(x - b_2)^2 < B$$

since  $b_2 - b_1 \leq 1/4$ . Also,

$$\begin{aligned} \int_{b_1}^{b_2} P_1(y^2)' &= 2yy'R \Big|_{b_1}^{b_2} - 2 \int_{b_1}^{b_2} [(y')^2 + yy'']R \leq -2 \int_{b_1}^{b_2} (y')^2 R \\ &= -2 \left[ R(t) \int_{b_2}^t (y')^2 \Big|_{b_1}^{b_2} - \int_{b_1}^{b_2} \left( P_1(t) \int_{b_2}^t (y')^2 \right) \right] \\ &\leq 2 \int_{b_1}^{b_2} |P_1(t)| \int_{b_1}^{b_2} (y')^2 < 1 \end{aligned}$$

since  $y'(b_2) = 0 = R(b_1)$ ,  $y'' \leq 0$ ,  $y \geq 0$ , and  $R \leq 0$  on  $[b_1, b_2]$ . Pick  $b_3$  so that  $|P_1(x)| \leq [6y(b_2)(b_2 - a)]^{-1}$  and  $|q(x)| \leq 4(b_1 - a - 1)^{-1}$  for  $x \geq b_3$ . Consequently,

$$\int_{b_2}^b P_1(y^2)' = \int_{b_3}^b P_1(y^2)' \leq 2 \int_{b_3}^b |P_1| |y| |y'| \leq 1$$

since  $|y| \leq y(b_2)$ ,  $|y'| \leq 3$ , and  $b - b_3 = b_2 - a$ . Also,

$$\begin{aligned} \int_{b_2}^b q(y')^2 &\leq B \int_{b_3}^{b_4} 4(x - b_3)^2 + (1/4) \int_{b_4}^{b_5} (b_6 - x)^{-1} |q(x)| \\ &\quad + B \int_{b_5}^b [5(b_6 - x) - 9(b_6 - x)^2/2]^2 \\ &< (A - 1) + (1/4) \int_{b_4}^{b_5} |q| \leq A . \end{aligned}$$

In conclusion,

$$\begin{aligned} I_b(y) &\leq M_1 + \int_a^{a+1} q(y')^2 + (a + 1) \int_{a+1}^\infty x^{-1} |q| + B + A \\ &\quad + \int_a^{a+1} P_1(y^2)' + \int_{a+1}^{b_1} P_1 + 1 + 1 \leq 0 . \end{aligned}$$

The conditions of Theorem 2.4,  $\int^\infty |q|/x < \infty$  and  $\lim_{x \rightarrow \infty} q(x) = 0$ , could be replaced by the conditions,  $\int^\infty |q| < \infty$  and  $q$  bounded, to obtain the same result with a similar proof.

**THEOREM 2.5.** *Suppose  $0 < r(x) \leq M$ ,  $\int^\infty p < \infty$ ,  $\int^\infty P_1 < \infty$ , and  $P_1(x) \leq Cx^{-4}$  for  $x > 0$ . If  $\lim_{x \rightarrow \infty} \inf x^2 P_2(x) < -7\frac{1}{32}M$  then  $L_\mu y = (ry'')'' + py$  is oscillatory on  $(0, \infty)$ .*

*Proof.* We will use the fact that for  $a > 0$

$$I_b(y) = \int_a^b [r(y'')^2 + 2yy'P_1]$$

for  $y$  given below. Let  $\xi(x) = -(3x^3 - 5x^2)/2$ ,  $\alpha(x) = \sqrt{x}$ , and  $\beta(x) = x^2$ . For  $0 < \mu < 1$ ,  $0 < \sigma \leq 1$ ,  $\rho > 0$ , and  $0 < \gamma \leq 1$  define  $y$  as follows:

$$y(x) = \begin{cases} 0 & x < \mu\rho \\ \xi\left(\frac{x - \mu\rho}{\rho[1 - \mu]}\right) & \mu\rho \leq x < \rho \\ \alpha\left(\frac{x - \mu\rho}{\rho[1 - \mu]}\right) & \rho \leq x < R \\ -\beta(x - (R + \sigma)) + \beta(\sigma) + \alpha\left(\frac{R - \mu\rho}{\rho[1 - \mu]}\right) & R \leq x < R + \sigma \\ \beta(\sigma) + \alpha\left(\frac{R - \mu\rho}{\rho[1 - \mu]}\right) & R + \sigma \leq x < N \\ -\beta(x - N) + y(R + \sigma) & N \leq x < N + \gamma \\ -2\gamma(x - N - \gamma) + y(R + \sigma) - \beta(\gamma) & N + \gamma \leq x < b - \gamma \\ \beta(x - b) & b - \gamma \leq x < b \\ 0 & b \leq x \end{cases}$$

Calculations show that

$$\int_{\mu\rho}^b r(y'')^2 < (1/[1 - \mu]) \left[ 7 \frac{1}{32} M(1 - \mu)^{-2} + 4\sigma M\rho^3(1 - \mu) + 8\gamma M\rho^3(1 - \mu) \right].$$

Since

$$\liminf x^2 \int_x^\infty P_1 < -7 \frac{1}{32} M,$$

there is a  $\delta > 0$  and a sequence  $\langle \rho_k \rangle \rightarrow \infty$  for which

$$\lim_{k \rightarrow \infty} \rho_k^2 \int_{\rho_k}^\infty P_1 \leq -\left(7 \frac{1}{32} M + 2\delta\right).$$

Pick  $\mu$  so close to zero that  $7(1/32)M(1 - \mu)^{-2} = 7(1/32)M + 7\delta/8$ . There is a positive integer  $N$  so large that  $\mu\rho_k > a$ ,  $C(\mu^{-3} - 1)/\rho_k < \delta/8$ , and

$$\rho_k^2 \int_{\rho_k}^\infty P_1 \leq -\left(7 \frac{1}{32} M + 7\delta/4\right)$$

for all  $k \geq N$ . Let  $\rho = \rho_N$ .

Given  $R$ , we will pick  $\sigma$  so that  $y'(x)$  is continuous at  $x = R$ . Therefore,

$$\sigma = 1/(4\sqrt{\rho[1 - \mu](R - \mu\rho)}).$$

Since  $\sigma \rightarrow 0$  as  $R \rightarrow \infty$  and  $P_1$  is bounded on  $[a, \infty)$  pick  $R$  so large that

$$\rho^2 \int_\rho^R P_1 \leq -\left(7 \frac{1}{32} M + 13\delta/8\right),$$

$$\sigma < (\delta/8)/[4M\rho^3(1 - \mu)] ,$$

and  $\sigma|P_1| < \delta/(8\rho^2)$  for all  $x \geq R$ . On  $[R, R + \sigma]$ ,

$$0 < y(x) \leq \alpha(x - \mu\rho)/(\rho[1 - \mu])$$

and

$$0 \leq y'(x) \leq \alpha'((x - \mu\rho)/(\rho[1 - \mu])) \cdot 1/(\rho[1 - \mu])$$

which implies that  $0 \leq 2yy' \leq 1/(\rho[1 - \mu])$  on  $[R, R + \sigma]$ .

On  $[\mu\rho, \rho]$   $0 \leq 2yy' < 3/(\rho[1 - \mu])$ . Hence,

$$\begin{aligned} 2 \int_a^b yy' P_1 &< 3(\rho[1 - \mu])^{-1} \int_{\mu\rho}^{\rho} P_1^+ + (\rho[1 - \mu])^{-1} \int_{\rho}^R P_1 \\ &+ (\rho[1 - \mu])^{-1} \int_R^{R+\sigma} |P_1| + 2 \int_N^b |yy'| |P_1| \\ &< 3C(\rho[1 - \mu])^{-1} \int_{\mu\rho}^{\rho} x^{-4} dx + (\rho[1 - \mu])^{-1} \int_{\rho}^R P_1 \\ &+ (\delta/8)/(\rho^3[1 - \mu])^{-1} + 2 \int_N^b |yy'| |P_1| \end{aligned}$$

where  $P_1^+(x) = P_1(x)$  when  $P_1(x) \geq 0$  and zero otherwise.

On  $[N, b]$   $0 \leq y(x) \leq y(R + \sigma)$  and  $|y'| \leq 2\gamma$ . Since  $y$  is linear on  $[N + \gamma, b - \gamma]$  we have that

$$[y(R + \sigma) - 2\gamma^2]/[b - N - 2\gamma] = 2\gamma$$

or

$$b - N = [y(R + \sigma)]/(2\gamma) + \gamma .$$

Since  $P_1(x) \rightarrow 0$  as  $x \rightarrow \infty$  we can pick  $N$  so large that

$$|P_1| \leq (\delta/8)/(2[y(R + \sigma)]^2 \rho^3 [1 - \mu])$$

for all  $x \geq N$ . Pick  $\gamma$  so small that  $2\gamma^2[y(R + \sigma)]^{-1} < 1$  and

$$8M\gamma\rho^3[1 - \mu] < \delta/8 .$$

Pick  $b$  so that

$$\lim_{x \rightarrow (b-\gamma)^-} y(x) = \gamma^2 .$$

We now have that

$$\begin{aligned} 2 \int_N^b |yy'| |P_1| &\leq 4\gamma y(R + \sigma) \int_N^b |P_1| \\ &\leq 2\gamma(b - N) \cdot (\delta/8)/(y(R + \sigma)\rho^3[1 - \mu]) \\ &= 2\gamma([y(R + \sigma)]/(2\gamma) + \gamma) \cdot (\delta/8)/(y(R + \sigma)\rho^3[1 - \mu]) \\ &= (\delta/8)/(\rho^3[1 - \mu]) + 2\gamma^2(\delta/8)/(y(R + \sigma)\rho^3[1 - \mu]) \\ &< (\delta/4)/(\rho^3[1 - \mu]) . \end{aligned}$$

Consequently,

$$2 \int_a^b yy' P_1 < (\rho^3[1 - \mu])^{-1} (C(\mu^{-3} - 1)\rho^{-1} + \rho^2 \int_\rho^R P_1 + 3\delta/8) < (\rho^3[1 - \mu])^{-1} (\delta/2 + \rho^2 \int_\rho^R P_1).$$

Hence,

$$I_b(y) = \int_a^b r(y'')^2 + 2yy' P_1 < (\rho^3[1 - \mu])^{-1} (7 \frac{1}{32} M + 7\delta/8 + \delta/8 + \delta/8 + \delta/2 + \rho^2 \int_\rho^R P_1) \leq 0$$

which completes the proof.

3. The oscillation of  $L_{2n}y = (-1)^n (ry^{(n)})^{(n)} + py$ .

**THEOREM 3.1.** *If  $p(x) \leq 0$ ,  $0 < r(x) \leq Mx^\alpha$  for  $\alpha < 2n - 1$ , and*

$$\limsup_{x \rightarrow \infty} x^{2n-1-\alpha} \int_x^\infty |p(t)| dt > MA_n^2$$

where

$$A_n^{-1} = \sqrt{2n - 1} / [(n - 1)!] \sum_{k=1}^n (-1)^{k-1} \binom{n-1}{k-1} (2n - k)^{-1}$$

then  $L_{2n}y = (-1)^n (ry^{(n)})^{(n)} + p(x)y$  is oscillatory on  $(0, \infty)$ .

*Proof.* Let  $\xi(x)$  be the polynomial of degree  $2n - 1$  such that  $\xi(0) = \xi^{(k)}(0) = \xi^{(k)}(1) = 0$  for  $k = 1, 2, \dots, n - 1$  and  $\xi(1) = 1$ . Given  $a > 0$ , define  $y(x)$  as follows:

$$y(x) = \begin{cases} 0 & x < \mu\rho \\ \xi([x - \mu\rho]/[\rho(1 - \mu)]) & \mu\rho \leq x < \rho \\ 1 & \rho \leq x < R \\ \xi([\nu R - x]/[R(\nu - 1)]) & R \leq x < \nu R \\ 0 & \nu R \leq x. \end{cases}$$

It can be shown that  $\int_0^1 (\xi^{(n)}(x))^2 dx = A_n^2$ .

A result due to Glazman [2, p. 100] considers the case when  $\alpha \leq$

0. Consequently, we will consider here only the case in which  $\alpha > 0$ .

Since

$$\begin{aligned} \int_{\mu\rho}^{\nu R} r(y^{(n)})^2 &\leq M\rho^\alpha \int_{\mu\rho}^\rho (y^{(n)})^2 + M(\nu R)^\alpha \int_R^{\nu R} (y^{(n)})^2 \\ &= MA_n^2 / [\rho^{2n-1-\alpha}(1 - \mu)^{2n-1}] \\ &\quad + MA_n^2 / [R^{2n-1-\alpha}(\nu^{1-\alpha/(2n-1)} - \nu^{-\alpha/(2n-1)})^{2n-1}] \end{aligned}$$

and  $p(x) \leq 0$ , then

$$I_{\nu R}(y) = \int_{\mu\rho}^R [r(y^{(n)})^2 + py^2] \\ \leq \frac{1}{\rho^{2n-1-\alpha}} \left( \frac{MA_n^2}{(1-\mu)^{2n-1}} + \frac{MA_n^2 \rho^{2n-1-\alpha}}{R^{2n-1-\alpha}(\nu^{1-\alpha/(2n-1)} - \nu^{-\alpha/(2n-1)})^{2n-1}} - \rho^{2n-1-\alpha} \int_{\rho}^R |p| \right).$$

There is a sequence  $\langle \rho_k \rangle \rightarrow \infty$  and a number  $\delta > 0$  such that

$$\lim_{k \rightarrow \infty} \rho_k^{2n-1-\alpha} \int_{\rho_k}^{\infty} |p| \geq MA_n^2 + \delta.$$

Choose  $\mu > 0$  so small that

$$MA_n^2 / [(1-\mu)^{2n-1}] < MA_n^2 + \delta/4.$$

There is a number  $K$  so that  $\mu\rho_k > a$  and

$$\rho_k^{2n-1-\alpha} \int_{\rho_k}^{\infty} |p| > MA_n^2 + 3\delta/4$$

for all  $k \geq K$ . Set  $\rho = \rho_K$ . Choose  $R$  so large that

$$\rho^{2n-1-\alpha} \int_{\rho}^R |p| > MA_n^2 + \delta/2.$$

Choose  $\nu > 1$  so large that

$$MA_n^2 \rho^{2n-1-\alpha} / [R^{2n-1-\alpha}(\nu^{1-\alpha/(2n-1)} - \nu^{-\alpha/(2n-1)})^{2n-1}] < \delta/4.$$

We now have that  $I_{\nu R}(y) < 0$  which implies that  $L_{2n}$  is oscillatory on  $(0, \infty)$ .

**THEOREM 3.2.** *If there are numbers  $M$  and  $\alpha$  such that  $0 < r(x) \leq Mx^\alpha$  and if for some  $\nu > 1$  and  $A_n$  as in Theorem 3.1*

$$\lim_{x \rightarrow \infty} \left( Kx^{\alpha-2n+1} + \int_a^x p \right) = -\infty$$

where  $K = MA_n^2 \nu^\alpha / (\nu - 1)^{2n-1}$  then  $L_{2n}y = (-1)^n (ry^{(n)}) + py$  is oscillatory on  $(0, \infty)$ .

*Proof.* For  $\mu, \rho, R$ , and  $\nu$  below, let  $y(x)$  be as in the proof of Theorem 3.1. Pick  $\mu$  and  $\nu$  so that  $0 < \mu < 1$  and  $\nu > 1$ . Pick  $\rho$  so large that  $\mu\rho \geq a$ . As in the proof of Theorem 3.1

$$\int_{\mu\rho}^{\nu R} r(y^{(n)})^2 \leq MA_n^2 \left( \frac{\rho^{\alpha-2n+1}}{(1-\mu)^{2n-1}} + \frac{R^{\alpha-2n+1} \nu^\alpha}{(\nu-1)^{2n-1}} \right).$$

There is a number  $c$  such that

$$MA_n^2 \left( \frac{\rho^{\alpha-2n+1}}{(1-\mu)^{2n-1}} + \frac{x^{\alpha-2n+1}\nu^\alpha}{(\nu-1)^{2n-1}} \right) + \int_{\mu\rho}^{\rho} py^2 + \int_{\rho}^x p < 0$$

for all  $x \geq c$ . Let  $T(x) = \int_c^x p$ . Since  $T(x) \rightarrow -\infty$  as  $x \rightarrow \infty$ , there is a last zero of  $T(x)$ . Let  $R$  be the last zero of  $T(x)$ . This implies that

$$\int_R^{\nu R} py^2 = -2 \int_R^{\nu R} yy' T(x) < 0.$$

Since

$$\int_{\mu\rho}^{\nu R} py^2 < \int_{\mu\rho}^{\rho} py^2 + \int_{\rho}^R p$$

then  $I_R(y) < 0$ .

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Zvi Arad, $\pi$ -homogeneity and $\pi'$ -closure of finite groups . . . . .	1
Ivan Baggs, A connected Hausdorff space which is not contained in a maximal connected space . . . . .	11
Eric Bedford, The Dirichlet problem for some overdetermined systems on the unit ball in $C^n$ . . . . .	19
R. H. Bing, Woodrow Wilson Bledsoe and R. Daniel Mauldin, Sets generated by rectangles . . . . .	27
Carlo Cecchini and Alessandro Figà-Talamanca, Projections of uniqueness for $L^p(G)$ . . . . .	37
Gokulananda Das and Ram N. Mohapatra, The non absolute Nörlund summability of Fourier series . . . . .	49
Frank Rimi DeMeyer, On separable polynomials over a commutative ring . . . . .	57
Richard Detmer, Sets which are tame in arcs in $E^3$ . . . . .	67
William Erb Dietrich, Ideals in convolution algebras on Abelian groups . . . . .	75
Bryce L. Elkins, A Galois theory for linear topological rings . . . . .	89
William Alan Feldman, A characterization of the topology of compact convergence on $C(X)$ . . . . .	109
Hillel Halkin Gershenson, A problem in compact Lie groups and framed cobordism . . . . .	121
Samuel R. Gordon, Associators in simple algebras . . . . .	131
Marvin J. Greenberg, Strictly local solutions of Diophantine equations . . . . .	143
Jon Craig Helton, Product integrals and inverses in normed rings . . . . .	155
Domingo Antonio Herrero, Inner functions under uniform topology . . . . .	167
Jerry Alan Johnson, Lipschitz spaces . . . . .	177
Marvin Stanford Keener, Oscillatory solutions and multi-point boundary value functions for certain $n$ th-order linear ordinary differential equations . . . . .	187
John Cronan Kieffer, A simple proof of the Moy-Perez generalization of the Shannon-McMillan theorem . . . . .	203
Joong Ho Kim, Power invariant rings . . . . .	207
Gangaram S. Ladde and V. Lakshmikantham, On flow-invariant sets . . . . .	215
Roger T. Lewis, Oscillation and nonoscillation criteria for some self-adjoint even order linear differential operators . . . . .	221
Jürg Thomas Marti, On the existence of support points of solid convex sets . . . . .	235
John Rowlay Martin, Determining knot types from diagrams of knots . . . . .	241
James Jerome Metzger, Local ideals in a topological algebra of entire functions characterized by a non-radial rate of growth . . . . .	251
K. C. O'Meara, Intrinsic extensions of prime rings . . . . .	257
Stanley Poreda, A note on the continuity of best polynomial approximations . . . . .	271
Robert John Sacker, Asymptotic approach to periodic orbits and local prolongations of maps . . . . .	273
Eric Peter Smith, The Garabedian function of an arbitrary compact set . . . . .	289
Arne Stray, Pointwise bounded approximation by functions satisfying a side condition . . . . .	301
John St. Clair Werth, Jr., Maximal pure subgroups of torsion complete abelian $p$ -groups . . . . .	307
Robert S. Wilson, On the structure of finite rings. II . . . . .	317
Kari Ylinen, The multiplier algebra of a convolution measure algebra . . . . .	327