OSCILLATION AND NONOSCILLATION CRITERIA FOR SOME SELF-ADJOINT EVEN ORDER LINEAR DIFFERENTIAL OPERATORS

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Oscillation and nonoscillation results are presented for the operator
\[ L_{2n}y = \sum_{k=0}^{n} (-1)^{n-k}(p_k y^{(n-k)})^{(n-k)} \]
where \( p_0(x) > 0 \) on \((0, \infty)\) and for \( k = 0, 1, \ldots, n, p_k \) is a real-valued, \( n-k \) times differentiable function on \((0, \infty)\). Also, \( y \) is an element of the set of all real-valued, \( 2n \) fold continuously differentiable, finite functions on \((0, \infty)\).

In particular, a nonoscillation result is given for \( L_{2n} \) without sign restrictions on the coefficients. Oscillation results are given for \( L_4 \) without the requirement that \( p_1 \) be negative for large \( x \). Finally, the oscillation of
\[ L_{2n}y = (-1)^n (ry^{(n)})^{(n)} + py \]
is considered for \( r(x) \) not necessarily bounded.

The oscillatory behavior of \( L_4 \) has been considered by Leighton and Nehari [8], Barrett [1], and Hinton [4]. In general, \( L_{2n} \) has been considered by Glazman [2], Hinton [5], Hunt [6], and Hunt and Namboodiri [7].

DEFINITION 0.1. The operator \( L_{2n} \) is called oscillatory on \([a, b]\) provided there is a function \( y, y \neq 0 \), and numbers \( c \) and \( d \) for which \( a \leq c < d \leq b \) such that \( L_{2n}y = 0 \) and
\[ y^{(k)}(c) = 0 = y^{(k)}(d) \quad \text{for} \quad k = 0, 1, \ldots, n - 1. \]
Otherwise, \( L_{2n} \) is called nonoscillatory on \([a, b]\). The operator \( L_{2n} \) is called oscillatory on \([a, \infty)\) if for any given \( c \geq a \) there is a \( d > c \) such that \( L_{2n} \) is oscillatory on \([c, d]\).

DEFINITION 0.2. Given a positive integer \( n \) and a number \( a \) define \( \mathcal{D}_n(a) \) for all \( b > a \) to be the set of all real-valued functions \( y \) with the following properties:
(a) \( y^{(k)} \) is absolutely continuous on \([a, b]\) for \( k = 0, 1, \ldots, n - 1 \),
(b) \( y^{(n)} \) is essentially bounded on \([a, b]\), and
(c) \( y^{(k)}(a) = 0 = y^{(k)}(b) \) for \( k = 0, 1, \ldots, n - 1 \).

For \( y \in \mathcal{D}_n(b) \) define
\[ I_b(y) = \int_a^b \sum_{k=0}^{n} p_k(x)(y^{(n-k)}(x))^2 \, dx. \]
which is called the quadratic functional for $L_{2n}$.

The following theorem has provided the primary motivation for the results which are to follow.

**THEOREM 0.1 (Reid [9]).** The following two statements are equivalent.

(i) The operator $L_{2n}$ is nonoscillatory on $[a, b]$.
(ii) If $y \in D_n(b)$ and $y \not\equiv 0$ then $I_s(y) > 0$.

Consequently, in order to show that $L_{2n}$ is oscillatory on $(0, \infty)$, given any $a > 0$, it will suffice to construct a $y \in D_n(b)$ for some $b > a$ for which $I_s(y)$ is not positive and $y \not\equiv 0$. This is the technique of proof for all of the oscillation theorems which follow.

This method of proof is especially conducive to oscillation theorems which require that integral conditions be met by the coefficients of $L_{2n}$. For example, Glazman [2, p. 104] showed that $(-1)^n y^{(2m)} + py$ is oscillatory on $(0, \infty)$ when $\int_{-\infty}^{\infty} p = -\infty$ (see Theorem 3.2).

Initially, the construction of $y$ is suggested by the conditions of the hypothesis on the coefficients of $L_{2n}$ and the corresponding quadratic formula. For example, to establish the above result, Glazman let $y = 1$ over the major portion of the interval $[a, b]$. To show that $y^{(n)} - (qy')'$ is oscillatory when $\int_{a}^{b} q = -\infty$ (see Theorem 2.2) the author let $y(x) = x - a$ over a portion of $[a, b]$. Next, we construct $y$ over the remaining portion of $[a, b]$ to insure that $y \in D_n(b)$ and the integral of $p_n y^{(n-k)^2}$ is bounded above for $k = 0, 1, \ldots, n$ independent of $b$.

For other proofs using this method the reader should consult Glazman [2, pp. 95-105] and Hinton [4].

1. The nonoscillation of $L_{2n}$.

**LEMMA 1.1 (Glazman [2, p. 83]).**

(i) If $g(a) = 0$ for some $a > 0$ and $g'$ is continuous on $[a, b]$, then

$$\int_{a}^{b} x^{-2m}(g(x))^2 dx \leq \left(\frac{2}{2m - 1}\right)^2 \int_{a}^{b} x^{-2m+2}(g'(x))^2 dx$$

for $m$ a positive integer. Moreover, if $g \not\equiv 0$ on $[a, b]$, the above inequality is strict.

(ii) If $g \not\equiv 0$, $g^{(m)}$ is continuous on $[a, b]$, and $g(a) = \cdots = g^{(m-1)}(a) = 0$, then

$$\int_{a}^{b} x^{-2m}(g(x))^2 dx < \left(\frac{2^m}{1 \cdot 3 \cdot \cdots \cdot (2m - 1)}\right)^2 \int_{a}^{b} (g^{(m)})^2 dx.$$
A well known result in oscillation theory is a sufficient condition for the nonoscillation of $L_2$ due to Hille [3]. A generalization of this result for $L_{2n}$ is given in the next theorem.

**Theorem 1.1.** For $L_{2n}$ defined above with $p_0(x) \equiv 1$ let $P_k(x) = p_k(x)$ and

$$P_k^m(x) = \int_{-\infty}^{\infty} P_{k-1}^{m-1}(t)dt$$

for $m$ an integer greater than or equal to one when

$$-\infty < \int_{a}^{b} P_{k-1}^{m-1}(t)dt < \infty.$$

If for $k = 1, \cdots , n$ and $x \geq a$ we have $-\infty < \int_{a}^{b} P_{k-1}^{m-1}(t)dt < \infty$ for $m = 0, 1, \cdots , k - 1$, $x^k | P_k^k | \leq a_k$, and $\sum_{k=1}^{n} a_k M_k \leq 1$ where $M_k = k! 2^{k-1}/(2k)!$, then $L_{2n}$ is nonoscillatory on $[a, b]$ for all $b > a$.

**Proof.** The proof is given only for $n > 1$. Suppose $L_{2n}$ is oscillatory on $[a, b]$. Then, there are numbers $c$ and $d$ and a function $y$ which is not identically zero such that $L_{2n}y = 0$ and $y^{(k)}(c) = 0 = y^{(k)}(d)$ for $k = 0, 1, \cdots , n - 1$. Since $(L_{2n}y)y = 0$ then

$$-\sum_{k=1}^{n} (-1)^{n-k} \int_{c}^{d} (p_k y^{(n-k)})^{(n-k)}y = (-1)^{n} \int_{c}^{d} y^{(2n)}y = \int_{c}^{d} [y^{(n)}]^2,$$

by integrating by parts $n$ times. By integrating by parts $n - k$ times we find that

$$-\int_{c}^{d} (-1)^{n-k}(p_k y^{(n-k)})^{(n-k)}y = -\int_{c}^{d} p_k(y^{(n-k)})^2.$$

However, by integrating by parts $k$ times and using Leibniz's rule we obtain

$$-\int_{c}^{d} p_k(y^{(n-k)})^2 = -\int_{c}^{d} P_k [(y^{(n-k)})^{(k)}] = -\int_{c}^{d} P_k \cdot \sum_{i=0}^{k} \binom{k}{i} (y^{(n-k)})^{(k-i)} (y^{(n-k)})^{(i)}$$

$$\leq \sum_{k=0}^{k} \binom{k}{i} \int_{c}^{d} \frac{|y^{(n-i)}|}{t^i} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \cdot P_k^i$$

$$\leq a_k \sum_{i=0}^{k} \binom{k}{i} \int_{c}^{d} \frac{|y^{(n-i)}|}{t^i} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}}$$

$$= a_k \left[ 2 \int_{c}^{d} |y^{(n)}| \cdot \frac{|y^{(n-k)}|}{t^k} + \sum_{i=1}^{k-1} \binom{k}{i} \int_{c}^{d} \frac{|y^{(n-i)}|}{t^i} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \right]$$

$$\leq a_k \left[ 2 \|y^{(n)}\|_2 \|y^{(n-k)}\|_2 \frac{1}{t^k} + \sum_{i=1}^{k-1} \binom{k}{i} \|y^{(n-i)}\|_2 \|y^{(n-k+i)}\|_2 \frac{1}{t^{k-i}} \right].$$
by Lemma 1.1 and the Cauchy inequality where

\[
C_k = \frac{2^{k+1}}{1 \cdot 3 \cdots (2k-1)} + \frac{k}{k} \sum_{i=1}^{k-1} \binom{k}{i} \left( \frac{2^{k-i}}{1 \cdot 3 \cdots (2i-1)} \right) \left( \frac{2^{k-i}}{1 \cdot 3 \cdots (2(k-i)-1)} \right).
\]

A simplification shows that

\[
C_k = \left[ \frac{2^k k!}{(2k)!} \right] \sum_{i=0}^{k} \binom{2k}{2i}.
\]

Since

\[
0 = (1 - 1)^{2k} = \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} = \sum_{i=0}^{k} \binom{2k}{2i} - \sum_{i=1}^{k} \binom{2k}{2i-1},
\]

then

\[
2^{2k-1} = \frac{1}{2} (1 + 1)^{2k} = \frac{1}{2} \left[ \sum_{i=0}^{k} \binom{2k}{2i} + \sum_{i=1}^{k} \binom{2k}{2i-1} \right] = \sum_{i=0}^{k} \binom{2k}{2i}.
\]

Therefore,

\[
C_k = \left[ \frac{2^{4k-3k} k!}{(2k)!} \right] = M_k.
\]

Consequently,

\[
\int_{c}^{d} [y^{(n)}]^2 \leq \sum_{k=1}^{n} \frac{1}{k} \int_{c}^{d} \left( p_k y^{(n-k)} \right)^2 y^2 
\]

which is a contradiction. Therefore, \( L_k \) is nonoscillatory on \([a, b]\).

It will be useful in applying Theorem 1.1 to note that \( M_{k+1} = 8M_{k}/(2k+1) \).

For the remainder of this paper we will assume that \( p(x) \) is identically zero for \( k = 1 \) to \( n - 2 \) and will denote \( p_0(x) \), \( p_{n-1}(x) \), and \( p_n(x) \) by \( r(x) \), \( q(x) \), and \( p(x) \) respectively. Similarly, \( P^k_n(x) \) and \( P^k_{n-1}(x) \) will be denoted by \( P_k(x) \) and \( Q_k(x) \) respectively.

If \( p(x) = kx^{-r} \), \( r = 1 \), and \( q = 0 \), then \( L_4 y = 0 \) is the familiar Euler equation. In this case, \( L_4 \) is oscillatory if and only if \( k < -9/16 \).
Also, \( k < -9/16 \) and \( p(x) = kx^{-4} \) implies \( x^2P_4(x) < -3/32 \). Theorem 1.1 shows that \( L_4y = y^{iv} + py \) is nonoscillatory when \( x^2|P_4(x)| \leq 3/32 \).

2. The oscillation of \( L_4 \). Using Theorem 0.1, Hinton [4] showed that \( L_4 \) is oscillatory when \( \int \frac{1}{r} = \infty, q \leq 0, \) and \( \int p = -\infty \). The same technique yields the following results.

**Theorem 2.1.** Suppose that \( r(x) \leq N, q(x) \leq M \) and \( \int p = -\infty \) for \( x > 0 \), then \( L_4y = (ry'')' - (qy')' + py \) is oscillatory on \((0, \infty)\).

**Theorem 2.2.** If \( 0 < r(x) \leq M, \int q = -\infty, \) and \( \int x^2|p(x)| < \infty \) then \( L_4y = (ry'')' - (qy')' + py \) is oscillatory on \((0, \infty)\).

**Proof.** Let \( \xi(x) = x^2/2 \). Define \( y(x) \) as follows:

\[
y(x) = \begin{cases} 
0 & x < a \\
\xi(x - a) & a \leq x < a + 1 \\
x - a - 1/2 & a + 1 \leq x < b_1 \\
-\xi(x - b_2) + b_1 - a & b_1 \leq x < b_2 = b_1 + 1 \\
(b_1 - a) & b_2 \leq x < b_3 \\
-\xi(x - b_3) + b_1 - a & b_3 \leq x < b_4 \\
-\xi'(b_4 - b_3)(x - b_4) + b_1 - a - \xi(b_4 - b_3) & b_4 \leq x < b_5 \\
\xi(x - b) & b_5 \leq x < b \\
0 & b \leq x.
\end{cases}
\]

It is easy to show that

\[
\int_a^x r(y'')^2 + py^2 < 16M + \int_a^\infty x^2|p| \leq 0
\]

if we require that \( b_4 - b_3 = b - b_5 \leq 1 \). There is a number \( c \) such that

\[
1 + 16M + \int a^{a+1} x^2|p| + \int a^{a+1} q(y')^2 + \int a^{a+1} q \leq 0
\]

for all \( x \geq c \).

Let \( Y_1(x) = \int_x^c q(t)dt \). Since \( Y_1(x) \) tends to \(-\infty\) as \( x \) tends to \( \infty \) there is a number \( b_1 \) which is the last zero of \( Y_1(x) \). Hence,

\[
\int_{b_1}^{b_2} q(y')^2 = Y_1(x)(y')^2|_{b_1}^{b_2} - 2\int_{b_1}^{b_2} y'y'' Y_1 < 0
\]

since \( Y_1(b_1) = 0 = y'(b_2), y'' = -1, y' \geq 0, \) and \( Y_1 < 0 \) on \((b_1, b_2)\).

Let \( Y_2(x) = \int_{b_2}^c q(t)dt \) and let \( b_3 \) be the last zero of \( Y_2 \). Pick \( b_3 \) so
that \(-1/2 \leq Y_2(x) \leq 0\) on \([b_3, b_4]\) and \(\delta_4 - b_3 \leq 1\). Since \(y' = 0\) on \([b_2, b_3]\), \(-1 \leq y' \leq 0\) on \([b_3, b]\), \(Y_2 \leq 0\) on \([b_4, \infty)\), \(\int_{b_2}^{b_4} q(y')^2 < 0\) we have that

\[
\int_{b_2}^{b} q(y')^2 < \int_{b_3}^{b} q(y')^2 = Y_2(x)(y')^2 |_{b_3}^{b} - 2 \int_{b_3}^{b} y'y'' Y_2 = -2 \int_{b_3}^{b} y'(-1) Y_2 - 2 \int_{b_3}^{b} y'(1) Y_2 < 2 \int_{b_3}^{b} y' Y_2 < 2 \int_{b_3}^{b} |y' Y_2| < 1.
\]

Consequently,

\[
I_4(y) = \int_{a}^{b} r(y'')^2 + q(y')^2 + py^2 < 16M + \int_{a}^{\infty} p^q + \int_{a}^{a+1} q(y')^2 + \int_{a+1}^{b_1} q + 1 \leq 0
\]

which completes the proof.

We now know that \(L_4\) is oscillatory on \((0, \infty)\) is for \(r\) bounded either \(\int_{0}^{\infty} p = -\infty\) and \(q \leq 0\) or \(\int_{0}^{\infty} q = -\infty\) and \(p \leq 0\). These facts suggest the results of the following theorem.

**Theorem 2.3.** If \(\int_{0}^{\infty} p = -\infty\), \(\int_{0}^{\infty} q = -\infty\), and \(0 < r(x) \leq M\) then \(L_4\) is oscillatory on \((0, \infty)\).

**Proof.** Except for some changes in the parameters we may define \(y(x)\) as in the proof of Theorem 2.2. As before, if \(b_4 - b_3 = b_5 - b_6 \leq 1\) then \(\int_{a}^{b} r(y'')^2 \leq 16M\).

There is a number \(c\) such that

\[
1 + 16M + \int_{a}^{a+1} q(y')^2 + \int_{a+1}^{x} q < 0
\]

for all \(x \geq c\). Let \(Y(x) = \int_{c}^{x} q(t)dt\) and let \(b_1\) be the last zero of \(Y(x)\). Integrating by parts we obtain the fact that

\[
\int_{b_1}^{b} q(y')^2 = -2 \int_{b_1}^{b} y'y'' Y
\]

since \(Y(b_1) = 0 = y'(b)\). Since \(y' \geq 0\), \(y'' = -1\), and \(Y \leq 0\) on \([b_1, b_2]\) then

\[
-2 \int_{b_1}^{b} y'y'' Y < -2 \int_{b_2}^{b} y'y'' Y.
\]

Since \(y'' = 0\) on \([b_2, b_3]\) and \([b_4, b_5]\), \(y' \leq 0\) on \([b_5, b]\), \(y'' = 1\) on \([b_5, b]\), and \(Y < 0\) on \([b_6, b]\), then
\[-2\int_{b_2}^{b} y'y''Y < -2\int_{b_2}^{b} y'y''Y = 2\int_{b_3}^{b} y'Y.\]

But, on \([b_3, b_4]\), \(y' \leq 1\). Consequently,

\[\int_{b_1}^{b} q(y')^2 < 2\int_{b_3}^{b} y'Y \leq 2\int_{b_3}^{b} |Y|.
\]

Since \(\int_{b_1}^{b} p = -\infty\), there is a number \(d > b_5\) such that for \(x \geq d\)

\[\int_{b_2}^{b} py^2 + (b_i - a)^2 \int_{b_2}^{x} p < 0.
\]

Let \(W(x) = \int_{a}^{x} p\) and \(b_5 \geq d\) be the last zero of \(W\). Hence,

\[\int_{b_2}^{b} py^2 = -2\int_{b_3}^{b} y'W(t)dt < 0.
\]

Let \(N = \max \{|Y(x)|: x \in [b_3, b_3 + 1]\}\) which we may assume is greater than or equal to one. Pick \(b_i\) so that \(b_i - b_3 = 1/(2N)\). Consequently,

\[\int_{b_1}^{b} q(y')^2 < 1.
\]

Pick \(b_5\) so that \(\lim_{x \to b_5-} y(x) = (b_i - b_3)/2\) and pick \(b\) so that \(b - b_5 = b_i - b_3\). We now have that

\[I_b(y) < 16M + \int_{a}^{a+1} q(y')^2 \int_{a}^{x} q + 1 + \int_{a}^{b_2} py^2 + \int_{b_2}^{b_3} (b_i - a)^2 p < 0.
\]

**Theorem 2.4.** If \(0 < r(x) \leq M\), \(-\infty < \int_{a}^{b} p < \infty\), \(\int_{b}^{c} P_1 = -\infty\), \(\int_{b}^{c} q|x^{-1} < \infty\), and \(q(x) \to 0\) as \(x \to \infty\) then \(L_i\) is oscillatory on \((0, \infty)\).

**Proof.** Let \(\xi(x) = -(3x^3 - 5x^5)/2\), \(\alpha(x) = \sqrt{x}\), and \(\beta(x) = x^2\). Let

\[y(x) = \begin{cases} 
0 & x < a \\
\xi(x - a) & a \leq x < a + 1 \\
\alpha(x - a) & a + 1 \leq x < b_1 \\
-\beta(x - b_2) + \alpha(b_i - a) + \beta(b_1 - b_2) & b_1 \leq x < b_2 \\
\alpha(b_i - a) + \beta(b_i - b_2) & b_2 \leq x < b_3 \\
-\beta(x - b_3) + \alpha(b_3) & b_3 \leq x < b_4 \\
\alpha(b - x) & b_4 \leq x < b_5 \\
\xi(b - x) & b_5 \leq x < b \\
0 & b \leq x.
\end{cases}
\]
Given \( b_i \) and \( b_s \) we choose \( b_\ast, b_\circ, \) and \( b \) so that
\[
\begin{align*}
&b_4 - b_3 = b_s - b_i, \quad b_s - b_4 = b_i - (a + 1), \\
&b = b_s + 1. 
\end{align*}
\]
Actually, only \( b_i \) and \( b_s \) will be chosen for reasons other than symmetry and the continuity of \( y \) and \( y' \).

First, note that since we are going to pick \( b_i, \cdots, b_s \) so that \( y \in \mathcal{D}_s(b) \) then
\[
\int_a^b py^2 = -P_i y^2|^b_a + \int_a^b P_i(y')' = \int_a^b P_i(y')'.
\]
Hence,
\[
I_i(y) \leq \int_a^b M(y'')^2 + q(y')^2 + P_i(y')'.
\]
Calculations show that
\[
\int_a^b M(y'')^2 \leq M \left[ 2 \int_0^1 (5 - 9x)^2 + 2 + \frac{1}{2} \int_1^\infty x^{-3}dx \right] = M_i
\]
since \( y' \) being continuous requires that \( 0 < b_s - b_i = b_i - b_4 \leq 1/4 \).

Since \( \lim_{x \to -\infty} q(x) = 0 \) and \( q \) is continuous then \( q \) is bounded by some number, \( B \), on \([a, \infty)\). Let
\[
A = 4B \left[ \int_0^1 u^2 + B \left( -\frac{9}{2} u^2 + 5u \right)^2 du + 1 \right].
\]
There is a number \( c \) so that
\[
\begin{align*}
M_i + 2 + \int_a^{a+1} [q(y')^2 + P_i(y')'] \\
&+ A + B + (a + 1) \int_{a+1}^\infty x^{-1} q(x) | \leq - \int_{a+1}^z P_i(t) dt
\end{align*}
\]
and \( |P_i(x)| \leq 1 \) for all \( x \geq c \) since \( P_i \to 0 \) as \( x \to \infty \).

Let \( R(x) = \int_x^z P_i(t) dt \) and \( b_1 \) be the last zero of \( R(x) \). Pick \( b_2 \) so that \( 1/(2\sqrt{b_1 - a}) = -2(b_1 - b_2) \) which insures that \( y' \) is continuous at \( b_1 \). We now have that
\[
\int_{a+1}^{b_1} q(y')^2 \leq \int_{a+1}^{b_1} (x - a)^{-1} q | = \int_{a+1}^{b_1} x(x - a)^{-1} x^{-1} | q |
\]
\[
< (a + 1) \int_{a+1}^{b_1} x^{-1} q | < (a + 1) \int_{a+1}^\infty x^{-1} q |
\]
and
\[
\int_{b_1}^{b_2} q(y')^2 \leq B \int_{b_1}^{b_2} 4(x - b_s)^2 < B
\]
since \( b_2 - b_s \leq 1/4 \). Also,
\[
\int_{b_1}^{b_2} P_1(y')' = 2yy'R \left| \int_{b_1}^{b_2} [(y')^2 + yy'']R \right| \leq -2 \int_{b_1}^{b_2} (y')^2 R
\]
\[
= -2 \left[ R(t) \left| \int_{b_2}^{t} (y')^2 \right| \right] - \int_{b_1}^{t} \left( P_1(t) \int_{b_1}^{t} (y')^2 \right)
\]
\[
\leq 2 \int_{b_1}^{b_2} \left| P_1(t) \right| (y')^2 < 1
\]
since \( y'(b_2) = 0 = R(b_1), y'' \leq 0, y \geq 0, \) and \( R \leq 0 \) on \([b_1, b_2]\). Pick \( b_3 \) so that \( |P_1(x)| \leq [6y(b_2)(b_2 - a)]^{-1} \) and \( |q(x)| \leq 4(b_1 - a - 1)^{-1} \) for \( x \geq b_2 \). Consequently,
\[
\int_{b_2}^{b} P_1(y')' = \int_{b_3}^{b} P_1(y')' \leq 2 \int_{b_3}^{b} |P_1| |y| |y'| \leq 1
\]
since \( |y| \leq y(b_3), |y'| \leq 3, \) and \( b - b_3 = b_2 - a \). Also,
\[
\int_{b_2}^{b} q(y')^2 \leq B \int_{b_3}^{b} 4(x - b_3)^2 + (1/4) \int_{b_4}^{b_5} (b_6 - x)^{-1} |q(x)|
\]
\[
+ B \int_{b_5}^{b} [5(b_6 - x) - 9(b_6 - x)^2/2]^2
\]
\[
< (A - 1) + (1/4) \int_{b_4}^{b_6} |q| \leq A .
\]
In conclusion,
\[
I_3(y) \leq M_1 + \int_{a}^{a+1} q(y')^2 + (a + 1) \int_{a+1}^{\infty} x^{-1} |q| + B + A
\]
\[
+ \int_{a}^{a+1} P_1(y')' + \int_{a+1}^{b_1} P_1 + 1 + 1 \leq 0 .
\]

The conditions of Theorem 2.4, \( \int_{0}^{\infty} q/|x| < \infty \) and \( \lim_{x \to \infty} q(x) = 0 \), could be replaced by the conditions, \( \int_{0}^{\infty} |q| < \infty \) and \( q \) bounded, to obtain the same result with a similar proof.

**Theorem 2.5.** Suppose \( 0 < r(x) \leq M, \int_{0}^{\infty} p < \infty, \int_{0}^{\infty} P_1 < \infty, \) and \( P_1(x) \leq Cx^{-4} \) for \( x > 0 \). If \( \lim_{x \to \infty} \inf x^2 P_2(x) < -7/32 M \) then \( L_1 y = (ry'')' + py \) is oscillatory on \((0, \infty)\).

**Proof.** We will use the fact that for \( a > 0 \)
\[
I_a(y) = \int_{a}^{b} [r(y')^2 + 2yy'P_1]
\]
for \( y \) given below. Let \( \xi(x) = -(3x^3 - 5x^5)/2, \alpha(x) = \sqrt{x}, \) and \( \beta(x) = x^2 \). For \( 0 < \mu < 1, 0 < \sigma \leq 1, \rho > 0, \) and \( 0 < \gamma \leq 1 \) define \( y \) as follows:
\[
\begin{align*}
y(x) = & \begin{cases} 
0 & x < \mu \rho \\
\xi \left( \frac{x - \mu \rho}{\rho[1 - \mu]} \right) & \mu \rho \leq x < \rho \\
\alpha \left( \frac{x - \mu \rho}{\rho[1 - \mu]} \right) & \rho \leq x < R \\
-\beta(x - (R + \sigma)) + \beta(\sigma) + \alpha \left( \frac{R - \mu \rho}{\rho[1 - \mu]} \right) & R \leq x < R + \sigma \\
\beta(\sigma) + \alpha \left( \frac{R - \mu \rho}{\rho[1 - \mu]} \right) & R + \sigma \leq x < N \\
-\beta(x - N) + y(R + \sigma) & N \leq x < N + \gamma \\
-2\gamma(x - N - \gamma) + y(R + \sigma) - \beta(\gamma) & N + \gamma \leq x < b - \gamma \\
\beta(x - b) & b - \gamma \leq x < b \\
0 & b \leq x.
\end{cases}
\end{align*}
\]

Calculations show that
\[
\int_{\rho^*}^\delta \tau(y'')^2 < (1/\rho^*(1 - \mu^*)) \left[ 7 \frac{1}{32} M(1 - \mu)^{-2} + 4\sigma M\rho^*(1 - \mu) + 8\gamma M\rho^*(1 - \mu) \right].
\]

Since
\[
\lim inf x^2 \int_x^\infty P_i \leq -7 \frac{1}{32} M,
\]
there is a \( \delta > 0 \) and a sequence \( \langle \rho_k \rangle \to \infty \) for which
\[
\lim \rho_k^i \int_{\rho_k}^\infty P_i \leq - \left( 7 \frac{1}{32} M + 2\delta \right).
\]

Pick \( \mu^* \) so close to zero that \( 7(1/32)M(1 - \mu)^{-2} = 7(1/32)M + 7\delta/8 \).

There is a positive integer \( N \) so large that \( \mu \rho_k > a, C(\mu^3 - 1)/\rho_k < \delta/8 \), and
\[
\rho_k^i \int_{\rho_k}^\infty P_i \leq - \left( 7 \frac{1}{32} M + 13\delta/8 \right)
\]
for all \( k \geq N \). Let \( \rho = \rho_N \).

Given \( R \), we will pick \( \sigma \) so that \( y'(x) \) is continuous at \( x = R \). Therefore,
\[
\sigma = 1/(4\sqrt{\rho[1 - \mu](R - \mu \rho)}) \cdot
\]
Since \( \sigma \to 0 \) as \( R \to \infty \) and \( P_i \) is bounded on \( [a, \infty) \) pick \( R \) so large that
\[
\rho^i \int_{\rho}^R P_i \leq - \left( 7 \frac{1}{32} M + 13\delta/8 \right),
\]
\[
\sigma < \frac{(\delta/8)}{[4M\rho^4(1 - \mu)]},
\]
and \(\sigma | P_1 | < \frac{\delta}{(8\rho^3)}\) for all \(x \geq R\). On \([R, R + \sigma]\),
\[
0 < y(x) \leq \alpha((x - \mu\rho)/(\rho[1 - \mu]))
\]
and
\[
0 \leq y'(x) \leq \alpha'((x - \mu\rho)/(\rho[1 - \mu])) \cdot 1/(\rho[1 - \mu])
\]
which implies that \(0 \leq 2yy' \leq 1/(\rho[1 - \mu])\) on \([R, R + \sigma]\).
On \([\mu\rho, \rho]\) \(0 \leq 2yy' < 3/(\rho[1 - \mu])\). Hence,
\[
2\int_R^b yy' P_1 < 3\rho[1 - \mu]^{-1}\int_{\rho}^b P_1 + (\rho[1 - \mu])^{-1}\int_{\rho}^b P_1
+ (\rho[1 - \mu])^{-1}\int_{\rho}^{R + \sigma} |P_1| + 2\int_{N}^b |yy'||P_1|
< 3C\rho[1 - \mu]^{-1}\int_{\rho}^b x^{-4}dx + (\rho[1 - \mu])^{-1}\int_{\rho}^b P_1
+ \frac{\delta}{8}/(\rho^3[1 - \mu])^{-1} + 2\int_{N}^b |yy'||P_1|
\]
where \(P_1^+(x) = P_1(x)\) when \(P_1(x) \geq 0\) and zero otherwise.

On \([N, b]\) \(0 \leq y(x) \leq y(R + \sigma)\) and \(|y'| \leq 2\gamma\). Since \(y\) is linear on \([N + \gamma, b - \gamma]\) we have that
\[
[y(R + \sigma) - 2\gamma] /[b - N - 2\gamma] = 2\gamma
\]
or
\[
b - N = [y(R + \sigma)]/(2\gamma) + \gamma.
\]
Since \(P_1(x) \to 0\) as \(x \to \infty\) we can pick \(N\) so large that
\[
|P_1| \leq \frac{\delta}{8}/[2y(R + \sigma)]^2\rho^3[1 - \mu]
\]
for all \(x \geq N\). Pick \(\gamma\) so small that \(2\gamma^2[y(R + \sigma)]^{-1} < 1\) and
\[
8M\gamma\rho^3[1 - \mu] < \delta/8.
\]
Pick \(b\) so that
\[
\lim_{x \to (b - \gamma)} y(x) = \gamma^2.
\]
We now have that
\[
2\int_{N}^b |yy'||P_1| \leq 4\gamma y(R + \sigma)\int_{N}^b |P_1|
\leq 2\gamma(b - N) \cdot (\delta/8)/(y(R + \sigma)\rho^3[1 - \mu])
= 2\gamma([y(R + \sigma)/(2\gamma)] + \gamma) \cdot (\delta/8)/(y(R + \sigma)\rho^3[1 - \mu])
= (\delta/8)/(\rho^3[1 - \mu]) + 2\gamma^3(\delta/8)/(y(R + \sigma)\rho^3[1 - \mu])
< (\delta/4)/(\rho^3[1 - \mu]).
\]
Since
"Rr{ywγ S Mpλ" {ywY + M(vR)a[B(y<»y
μp Jμp JR
= MAi/ip**-1-^! - μf1"1

which completes the proof.

3. The oscillation of \( L_{2n} y = (-1)^n (ry^{(n)})^{(n)} + py \).

**Theorem 3.1.** If \( p(x) \leq 0 \), \( 0 < r(x) \leq Mx^a \) for \( \alpha < 2n - 1 \), and

\[
\limsup_{x \to \infty} x^{2n-1-a} \int_x^\infty |p(t)| dt > MA_n^2
\]

where

\[
A_n^{-1} = \sqrt{2n - 1} / [(n - 1)!] \sum_{k=1}^n (-1)^{k-1} \binom{n - 1}{k - 1} (2n - k)^{-1}
\]

then \( L_{2n} y = (-1)^n (ry^{(n)})^{(n)} + p(x)y \) is oscillatory on \((0, \infty)\).

**Proof.** Let \( \xi(x) \) be the polynomial of degree \( 2n - 1 \) such that

\[
\xi(0) = \xi^{(k)}(0) = \xi^{(k)}(1) = 0 \quad \text{for} \quad k = 1, 2, \ldots, n - 1 \quad \text{and} \quad \xi(1) = 1.
\]

Given \( a > 0 \), define \( y(x) \) as follows:

\[
y(x) = \begin{cases} 
0 & x < \mu \rho \\
\xi([x - \mu \rho] / [\rho(1 - \mu)]) & \mu \rho \leq x < \rho \\
1 & \rho \leq x < \nu R \\
\xi([\nu R - x] / [R(\nu - 1)]) & R \leq x < \nu R \\
0 & \nu R \leq x.
\end{cases}
\]

It can be shown that \( \int_0^1 (\xi^{(n)}(x))^2 dx = A_n^2 \).

A result due to Glazman [2, p. 100] considers the case when \( \alpha \leq 0 \). Consequently, we will consider here only the case in which \( \alpha > 0 \). Since

\[
\int_{\mu \rho}^{\nu R} r(y^{(n)})^2 \leq M\rho^a \int_{\mu \rho}^{\rho} (y^{(n)})^2 + M(\nu R)^a \int_{\nu R}^{\infty} (y^{(n)})^2 \\
= MA_n^2/[\rho^{2n-1-a}(1 - \mu)^{2n-1}] \\
+ MA_n^2/[R^{2n-1-a}(\nu^{-a/(2n-1)} - \nu^{-a/(2n-1)})^{2n-1}] 
\]
and \( p(x) \leq 0 \), then

\[
I_{n}(y) = \int_{\rho_{n}}^{R} [r(y^{(n)})^2 + py^2]
\]

\[
\leq \frac{1}{\rho^{2n-1-a}} \left( \frac{MA^{2}_{n}}{(1 - \mu)^{2n-1}} + \frac{MA^{2}_{n} \rho^{2n-1-a}}{R^{2n-1-a}(\nu^{1-a/(2n-1)} - \nu^{-a/(2n-1)})^{2n-1}} - \rho^{2n-1-a} \int_{\rho}^{R} |p| \right).
\]

There is a sequence \( \langle \rho_{k} \rangle \to \infty \) and a number \( \delta > 0 \) such that

\[
\lim_{k \to \infty} \rho^{2n-1-a}_{k} \int_{\rho_{k}}^{\infty} |p| \geq MA^{2}_{n} + \delta.
\]

Choose \( \mu > 0 \) so small that

\[
MA^{2}_{n}/[(1 - \mu)^{2n-1}] < MA^{2}_{n} + \delta/4.
\]

There is a number \( K \) so that \( \mu \rho_{k} > a \) and

\[
\rho^{2n-1-a}_{k} \int_{\rho_{k}}^{\infty} |p| > MA^{2}_{n} + 3\delta/4
\]

for all \( k \geq K \). Set \( \rho = \rho_{K} \). Choose \( R \) so large that

\[
\rho^{2n-1-a} \int_{\rho}^{R} |p| > MA^{2}_{n} + \delta/2.
\]

Choose \( \nu > 1 \) so large that

\[
MA^{2}_{n} \rho^{2n-1-a}/[R^{2n-1-a}(\nu^{1-a/(2n-1)} - \nu^{-a/(2n-1)})^{2n-1}] < \delta/4.
\]

We now have that \( I_{n}(y) < 0 \) which implies that \( L_{2n} \) is oscillatory on \((0, \infty)\).

**Theorem 3.2.** If there are numbers \( M \) and \( \alpha \) such that \( 0 < r(x) \leq Mx^{\alpha} \) and if for some \( \nu > 1 \) and \( A_{n} \) as in Theorem 3.1

\[
\lim_{x \to \infty} \left( Kx^{\alpha-2n+1} + \int_{a}^{x} p \right) = -\infty
\]

where \( K = MA^{2}_{n} \nu^{\alpha}/(\nu - 1)^{2n-1} \) then \( L_{2n}y = (-1)^{n}(ry^{(n)}) + py \) is oscillatory on \((0, \infty)\).

**Proof.** For \( \mu, \rho, R, \) and \( \nu \) below, let \( y(x) \) be as in the proof of Theorem 3.1. Pick \( \mu \) and \( \nu \) so that \( 0 < \mu < 1 \) and \( \nu > 1 \). Pick \( \rho \) so large that \( \mu \rho \geq a \). As in the proof of Theorem 3.1

\[
\int_{\rho}^{R} r(y^{(n)})^2 \leq MA^{2}_{n} \left( \frac{\rho^{\alpha-2n+1}}{(1 - \mu)^{2n-1}} + \frac{R^{\alpha-2n+1}\nu^{\alpha}}{(\nu - 1)^{2n-1}} \right).
\]

There is a number \( c \) such that
Since $T(x) \to -\infty$ as $x \to \infty$, there is a last zero of $T(x)$. Let $\bar{R}$ be the last zero of $T(x)$. This implies that

$$\int_{\rho}^{\bar{R}} p y^2 = -2\int_{\rho}^{\bar{R}} y y'T(x) < 0.$$ 

Since

$$\int_{\rho}^{\bar{R}} p y^2 < \int_{\rho}^{\bar{R}} p y^2 + \int_{\rho}^{\bar{R}} p$$

then $I_\rho(y) < 0$.

REFERENCES


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