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**ON THE EXISTENCE OF SUPPORT POINTS OF SOLID
CONVEX SETS**

JÜRGEN THOMAS MARTI

ON THE EXISTENCE OF SUPPORT POINTS OF SOLID CONVEX SETS

J. T. MARTI

Let E be a separable Fréchet lattice. It is shown that a solid convex set X with void interior in E is supported at each of its boundary points if and only if the span of X is not dense in E . This result then is applied to the case of solid convex sets with void interior in real Fréchet spaces with an unconditional Schauder basis and in the real Banach lattice $C(S)$, S compact Hausdorff.

1. Introduction. If E is a real Hausdorff topological vector space and X is a convex subset of E with nonempty interior and boundary ∂X then, by a known theorem, every point of ∂X supports X , that is, for every $x \in \partial X$ there exists a continuous nontrivial linear functional f on E such that $\sup f(X) = f(x)$. However, if X has void interior, there are examples of compact convex sets, e.g., the Hilbert cube in l_2 [1, p. 160], which have boundary points that are not support points.

The object of this note is to investigate conditions on convex sets X with void interior in a separable real Fréchet lattice E , such that every point of ∂X is a support point of X . A theorem obtained is that for such sets X which are also solid, X is supported at each boundary point if and only if the span $\text{sp } X$ of X is not dense in E . Moreover, if E is a real Fréchet space with an unconditional basis $\{x_n, f_n\}$ (the sequence space s , the Banach spaces c_0 and l_p ($1 \leq p < \infty$)) and so all separable real Hilbert spaces are examples of such spaces) and if E is equipped with the ordering induced by the basis $\{x_n, f_n\}$, then a solid convex set X with void interior in E is supported at each of its boundary points if and only if \bar{X} does not contain a weak order unit of E . On the other hand, if E is the Banach lattice $C(S)$, S compact Hausdorff, all solid convex subsets X with void interior in E have the property that the boundary points and the support points of X coincide.

2. Support properties of solid convex sets with void interior. A set X in a Fréchet lattice E is said to be *solid* if y is in X whenever x is in X and $|y| \leq |x|$. An element x in the positive cone of E is said to be a *weak order unit* of E if $y = 0$ whenever y is in E and $x \wedge |y| = 0$. For the terminology see also H. H. Schaefer [5] or A. L. Peressini [4]. The topological boundary of X is denoted by ∂X .

THEOREM 1. *Let X be a solid convex set with void interior in a separable real Fréchet lattice E . Then every $x \in \partial X$ supports X if and only if $\text{sp } X$ is not dense in E .*

Proof. (Sufficiency) Let E' be the topological dual of E . If $\text{sp } X$ is not dense in E there is an $f \in E' \setminus \{0\}$ such that $f(X) = \{0\}$. In this case f obviously is a supporting functional of X for every $x \in \partial X$.

(Necessity) Let K be the positive cone of E and $K' = \{f \in E' : f(x) \geq 0, x \in K\}$ the dual cone in E' . We define the sets $S_x, x \in X \cap K$, by

$$S_x = \{f \in K' : f(x) = 0\}.$$

It is clear that each S_x is a $\sigma(E', E)$ -closed set in E' which contains 0. Moreover, let

$$S = \bigcap_{x \in X \cap K} S_x.$$

Since E is a Fréchet space there exists a countable base $\{U_n\}$ of neighborhoods of 0 in E , and since E is separable there is a sequence $\{V_n\}$ of open $\sigma(E', E)$ -neighborhoods of 0 in E' satisfying $\bigcap_{n=1}^{\infty} V_n = \{0\}$. (The sequence $\{V_n\}$ can, for instance, be constructed in the following way: If $\{x_n\}$ is a dense set in E , let W_{mn} be defined by $W_{mn} = \{f \in E' : |f(x_n)| < 1/m\}$. It then follows that $\bigcap_{m,n=1}^{\infty} W_{mn} = \{0\}$ since for each f in this last intersection one has $f(\{x_n\}) = \{0\}$ and hence $f = 0$.) We assume now that $S = \{0\}$. Then $E' \setminus \{0\} = \mathcal{C} S$ and so for all $m, n \in N$ one has

$$U_n^0 \subset \bigcup_{k=1}^{\infty} U_k^0 = E' = V_m \cup \bigcup_{x \in X \cap K} \mathcal{C} S_x.$$

Since the polars U_n^0 of U_n are $\sigma(E', E)$ -compact there is for each m and each n in N a finite set A_{mn} in $X \cap K$ such that

$$U_n^0 \subset V_m \cup \bigcup_{x \in A_{mn}} \mathcal{C} S_x.$$

If $\{x_k\}$ is a sequence in $X \cap K$ such that $\{x_k\} = \bigcup_{m,n=1}^{\infty} A_{mn}$ we get for all $m \in N$,

$$E' = \bigcup_{n=1}^{\infty} U_n^0 = V_m \cup \bigcup_{k=1}^{\infty} \mathcal{C} S_{x_k}.$$

Whence

$$E' = \bigcap_{m=1}^{\infty} \left(V_m \cup \bigcup_{k=1}^{\infty} \mathcal{C} S_{x_k} \right) = \bigcup_{k=1}^{\infty} \mathcal{C} S_{x_k} \cup \{0\}$$

and

$$(1) \quad \bigcap_{k=1}^{\infty} S_{x_k} = \mathcal{C} \left(\bigcup_{k=1}^{\infty} \mathcal{C} S_{x_k} \right) = \mathcal{C} (E' \setminus \{0\}) = \{0\}.$$

Next, if d is a translation invariant metric generating the topology of E , we define the real sequence $\{a_k\}$ by

$$a_k = \inf \{2^{-k}, \sup \{t > 0: d(0, sx_k) \leq 2^{-k}, s \in [0, t]\} \} .$$

Since X is solid we have $(2^n - 1)2^{-n+k} a_k x_k \in X$ for all $k, n \in N$ and since X is convex,

$$\sum_{k=1}^n a_k x_k = (2^n - 1)^{-1} 2^n \sum_{k=1}^n 2^{-k} (2^n - 1) 2^{-n+k} a_k x_k \in X$$

for all $n \in N$. Since \bar{X} is complete and since for $n < m$

$$d\left(\sum_{k=1}^n a_k x_k, \sum_{k=1}^m a_k x_k\right) \leq \sum_{k=n+1}^m d(0, a_k x_k) \leq \sum_{k=n+1}^m 2^{-k} < n^{-1} ,$$

$\lim_n \sum_{k=1}^n a_k x_k$ exists in \bar{X} and this limit is denoted by x . Since $\text{int } X = \emptyset$ and \bar{X} is solid [4, Proposition 2.4.8] it follows that $1/2x \in \partial X$ and thus is a support point of X . If f is a corresponding support functional we have $f \neq 0$ and $0 = f(0) \leq f(1/2x) = f(x) - f(1/2x)$. If $\{y_n\} \subset X$ is a sequence that converges to x in E one obtains $f(x) \leq \sup_n f(y_n) \leq f(1/2x)$, and hence $f(x) = 0$. Now, since E is a locally convex lattice and again since \bar{X} is solid, it follows that

$$\begin{aligned} 0 &\leq |f|(x) \\ &\leq \sup \{f(y): y \in E, |y| \leq x\} \leq \sup f(\bar{X}) = \sup f(X) = f\left(\frac{1}{2}x\right) = 0 . \end{aligned}$$

Therefore, $|f| \in K' \setminus \{0\}$ and $|f|(x) = 0$. This shows that

$$S_x \setminus \{0\} \neq \emptyset .$$

Since $0 \leq a_k x_k \leq x$, one has $0 \leq g(x_k) \leq a_k^{-1} g(x) = 0, g \in S_x, k \in N$. In view of (1) one thus obtains

$$\{0\} \subset S_x \subset \bigcap_{k=1}^{\infty} S_{x_k} = \{0\} ,$$

and this contradiction shows that

$$S \setminus \{0\} \neq \emptyset .$$

If f is a nonzero element of S then $f(X \cap K) = \{0\}$. Thus for any $x \in X$ we have $f(x) = f(x^+) - f(x^-) = 0$ because $x^\pm \in X$. Hence $f(X) = \{0\}$, showing that $\text{sp } X$ cannot be dense in E .

Let now E be a real Fréchet space with an unconditional basis $\{x_n, f_n\}$. It is known that the set $K = \{x \in E: f_n(x) \geq 0, n \in N\}$ is a closed, normal, generating cone in E and equipped with K, E becomes an order complete locally convex lattice [3, Theorem 5]. Obviously, $\{x_n\} \subset K$ and the coefficient functionals f_n are positive with respect to K . Therefore, the basis is a positive Schauder basis for E [2].

REMARK. A slight modification of the above argument shows that in Theorem 1 the separability of E can be replaced by the (weaker) condition: There exists a sequence $\{u_n\}$ in the positive cone of E such that $\text{sp } \bigcup_{n=1}^{\infty} [0, u_n]$ is dense in E .

THEOREM 2. *If X is a solid convex set with void interior in E (E being specified above), then every point of ∂X supports X if and only if \bar{X} does not contain a weak order unit of E .*

Proof. (Necessity) Let every point of ∂X support X and let us assume that \bar{X} contains a weak order unit x of E . Then from [3, Proposition 11] it follows that the span of $[0, x]$ is dense in E . Since $[0, x] \subset \bar{X}$ this contradicts Theorem 1. Hence \bar{X} does not contain a weak order unit.

(Sufficiency) If \bar{X} does not contain a weak order unit of E , suppose that $\sup f_n(X) > 0$, $n \in N$. Then for every n there is a $y_n \in X$ such that $\sup f_n(X) \leq 2f_n(y_n)$. Since X is solid this yields for all n that $\sup f_n(X)x_n \leq 2|y_n|$; whence $1/2 \sup f_n(X)x_n \in X$. In the same way as in the proof of the necessity part of the preceding theorem we can now construct an element $x \in \bar{X} \cap K$ such that $x = \lim_n \sum_{i=1}^n a_i x_i$, where $a_i > 0$, $i \in N$. If $y \in K \setminus \{0\}$ then there must be a positive integer n such that $f_n(y) > 0$. If $z \in K$ is given by $z = \inf \{a_n, f_n(y)\}x_n$ it follows that $z \neq 0$ and $z = x \wedge y$, i.e., x is a weak order unit of E in \bar{X} . By this contradiction to our assumption there is an $n \in N$ such that $\sup f_n(X) = 0$. Therefore, $\text{sp } X$ cannot be dense in E and an application of Theorem 1 finally completes the proof.

Concerning the real Banach lattice $C(S)$, S compact Hausdorff, it is clear that there can exist solid subsets X of $C(S)$ with void interior containing a weak order unit of $C(S)$ and such that every boundary point of X is a support point of X . For instance, take $X = \{y \in C[0, 1] : |y| \leq x\}$, where x , given by $x(s) = s$, $s \in [0, 1]$, is a weak order unit of $C[0, 1]$. Therefore, that \bar{X} contains no weak order unit of $C(S)$ is not a necessary condition for X to be supported at each boundary point, as is also seen by the following theorem:

THEOREM 3. *If X is a convex solid set with void interior in $C(S)$ then every boundary point of X supports X .*

Proof. We assume that $\text{sp } X$ is dense in $C(S)$. If f_s is the point evaluation functional of a general point s of S , this implies that $\sup f_s(X) > 0$, $s \in S$. In this case, since X is solid, there is for every $s \in S$ an $x_s \geq 0$ in X such that $x_s(s) > 0$. Hence for every $s \in S$

there is an open neighborhood V_s of s in S such that $\inf x_s(V_s) > 0$. Since S is compact and $\{V_s: s \in S\}$ is an open covering of S there is a finite subcovering $\{V_{s(1)}, \dots, V_{s(m)}\}$ for S . Taking $x = m^{-1} \sum_{n=1}^m x_{s(n)}$ it is clear that x is in X since X is convex, and that

$$\inf x(S) \geq m^{-1} \inf_{n \leq m} \inf x_{s(n)}(V_{s(n)}) > 0 .$$

If U is the unit ball of $C(S)$ we obtain $(\inf x(S)) |y| \leq x, y \in U$, which, since \overline{X} is solid, implies that $(\inf x(S))U \subset X$. This contradiction shows that $\overline{\text{sp}} X \neq C(S)$ and the result follows in the same way as in the sufficiency part of the proof of Theorem 1.

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