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**ON THE EXISTENCE OF SUPPORT POINTS OF SOLID  
CONVEX SETS**

JÜRGEN THOMAS MARTI

## ON THE EXISTENCE OF SUPPORT POINTS OF SOLID CONVEX SETS

J. T. MARTI

Let  $E$  be a separable Fréchet lattice. It is shown that a solid convex set  $X$  with void interior in  $E$  is supported at each of its boundary points if and only if the span of  $X$  is not dense in  $E$ . This result then is applied to the case of solid convex sets with void interior in real Fréchet spaces with an unconditional Schauder basis and in the real Banach lattice  $C(S)$ ,  $S$  compact Hausdorff.

1. Introduction. If  $E$  is a real Hausdorff topological vector space and  $X$  is a convex subset of  $E$  with nonempty interior and boundary  $\partial X$  then, by a known theorem, every point of  $\partial X$  supports  $X$ , that is, for every  $x \in \partial X$  there exists a continuous nontrivial linear functional  $f$  on  $E$  such that  $\sup f(X) = f(x)$ . However, if  $X$  has void interior, there are examples of compact convex sets, e.g., the Hilbert cube in  $l_2$  [1, p. 160], which have boundary points that are not support points.

The object of this note is to investigate conditions on convex sets  $X$  with void interior in a separable real Fréchet lattice  $E$ , such that every point of  $\partial X$  is a support point of  $X$ . A theorem obtained is that for such sets  $X$  which are also solid,  $X$  is supported at each boundary point if and only if the span  $\text{sp } X$  of  $X$  is not dense in  $E$ . Moreover, if  $E$  is a real Fréchet space with an unconditional basis  $\{x_n, f_n\}$  (the sequence space  $s$ , the Banach spaces  $c_0$  and  $l_p$  ( $1 \leq p < \infty$ )) and so all separable real Hilbert spaces are examples of such spaces) and if  $E$  is equipped with the ordering induced by the basis  $\{x_n, f_n\}$ , then a solid convex set  $X$  with void interior in  $E$  is supported at each of its boundary points if and only if  $\bar{X}$  does not contain a weak order unit of  $E$ . On the other hand, if  $E$  is the Banach lattice  $C(S)$ ,  $S$  compact Hausdorff, all solid convex subsets  $X$  with void interior in  $E$  have the property that the boundary points and the support points of  $X$  coincide.

2. Support properties of solid convex sets with void interior. A set  $X$  in a Fréchet lattice  $E$  is said to be *solid* if  $y$  is in  $X$  whenever  $x$  is in  $X$  and  $|y| \leq |x|$ . An element  $x$  in the positive cone of  $E$  is said to be a *weak order unit* of  $E$  if  $y = 0$  whenever  $y$  is in  $E$  and  $x \wedge |y| = 0$ . For the terminology see also H. H. Schaefer [5] or A. L. Peressini [4]. The topological boundary of  $X$  is denoted by  $\partial X$ .

**THEOREM 1.** *Let  $X$  be a solid convex set with void interior in a separable real Fréchet lattice  $E$ . Then every  $x \in \partial X$  supports  $X$  if and only if  $\text{sp } X$  is not dense in  $E$ .*

*Proof.* (Sufficiency) Let  $E'$  be the topological dual of  $E$ . If  $\text{sp } X$  is not dense in  $E$  there is an  $f \in E' \setminus \{0\}$  such that  $f(X) = \{0\}$ . In this case  $f$  obviously is a supporting functional of  $X$  for every  $x \in \partial X$ .

(Necessity) Let  $K$  be the positive cone of  $E$  and  $K' = \{f \in E': f(x) \geq 0, x \in K\}$  the dual cone in  $E'$ . We define the sets  $S_x, x \in X \cap K$ , by

$$S_x = \{f \in K': f(x) = 0\}.$$

It is clear that each  $S_x$  is a  $\sigma(E', E)$ -closed set in  $E'$  which contains 0. Moreover, let

$$S = \bigcap_{x \in X \cap K} S_x.$$

Since  $E$  is a Fréchet space there exists a countable base  $\{U_n\}$  of neighborhoods of 0 in  $E$ , and since  $E$  is separable there is a sequence  $\{V_n\}$  of open  $\sigma(E', E)$ -neighborhoods of 0 in  $E'$  satisfying  $\bigcap_{n=1}^{\infty} V_n = \{0\}$ . (The sequence  $\{V_n\}$  can, for instance, be constructed in the following way: If  $\{x_n\}$  is a dense set in  $E$ , let  $W_{mn}$  be defined by  $W_{mn} = \{f \in E': |f(x_n)| < 1/m\}$ . It then follows that  $\bigcap_{m,n=1}^{\infty} W_{mn} = \{0\}$  since for each  $f$  in this last intersection one has  $f(\{x_n\}) = \{0\}$  and hence  $f = 0$ .) We assume now that  $S = \{0\}$ . Then  $E' \setminus \{0\} = \bigcup S$  and so for all  $m, n \in N$  one has

$$U_n^0 \subset \bigcup_{k=1}^{\infty} U_k^0 = E' = V_m \cup \bigcup_{x \in X \cap K} \bigcup S_x.$$

Since the polars  $U_n^0$  of  $U_n$  are  $\sigma(E', E)$ -compact there is for each  $m$  and each  $n$  in  $N$  a finite set  $A_{mn}$  in  $X \cap K$  such that

$$U_n^0 \subset V_m \cup \bigcup_{x \in A_{mn}} \bigcup S_x.$$

If  $\{x_k\}$  is a sequence in  $X \cap K$  such that  $\{x_k\} = \bigcup_{m,n=1}^{\infty} A_{mn}$  we get for all  $m \in N$ ,

$$E' = \bigcup_{n=1}^{\infty} U_n^0 = V_m \cup \bigcup_{k=1}^{\infty} \bigcup S_{x_k}.$$

Whence

$$E' = \bigcap_{m=1}^{\infty} \left( V_m \cup \bigcup_{k=1}^{\infty} \bigcup S_{x_k} \right) = \bigcup_{k=1}^{\infty} \bigcup S_{x_k} \cup \{0\}$$

and

$$(1) \quad \bigcap_{k=1}^{\infty} \bigcup S_{x_k} = \bigcup \left( \bigcup_{k=1}^{\infty} \bigcup S_{x_k} \right) = \bigcup (E' \setminus \{0\}) = \{0\}.$$

Next, if  $d$  is a translation invariant metric generating the topology of  $E$ , we define the real sequence  $\{a_k\}$  by

$$a_k = \inf \{2^{-k}, \sup \{t > 0: d(0, sx_k) \leq 2^{-k}, s \in [0, t]\}\} .$$

Since  $X$  is solid we have  $(2^n - 1)2^{-n+k} a_k x_k \in X$  for all  $k, n \in N$  and since  $X$  is convex,

$$\sum_{k=1}^n a_k x_k = (2^n - 1)^{-1} 2^n \sum_{k=1}^n 2^{-k} (2^n - 1) 2^{-n+k} a_k x_k \in X$$

for all  $n \in N$ . Since  $\bar{X}$  is complete and since for  $n < m$

$$d\left(\sum_{k=1}^n a_k x_k, \sum_{k=1}^m a_k x_k\right) \leq \sum_{k=n+1}^m d(0, a_k x_k) \leq \sum_{k=n+1}^m 2^{-k} < n^{-1} ,$$

$\lim_n \sum_{k=1}^n a_k x_k$  exists in  $\bar{X}$  and this limit is denoted by  $x$ . Since  $\text{int } X = \emptyset$  and  $\bar{X}$  is solid [4, Proposition 2.4.8] it follows that  $1/2x \in \partial X$  and thus is a support point of  $X$ . If  $f$  is a corresponding support functional we have  $f \neq 0$  and  $0 = f(0) \leq f(1/2x) = f(x) - f(1/2x)$ . If  $\{y_n\} \subset X$  is a sequence that converges to  $x$  in  $E$  one obtains  $f(x) \leq \sup_n f(y_n) \leq f(1/2x)$ , and hence  $f(x) = 0$ . Now, since  $E$  is a locally convex lattice and again since  $\bar{X}$  is solid, it follows that

$$\begin{aligned} 0 &\leq |f|(x) \\ &\leq \sup \{f(y): y \in E, |y| \leq x\} \leq \sup f(\bar{X}) = \sup f(X) = f\left(\frac{1}{2}x\right) = 0 . \end{aligned}$$

Therefore,  $|f| \in K \setminus \{0\}$  and  $|f|(x) = 0$ . This shows that

$$S_x \setminus \{0\} \neq \emptyset .$$

Since  $0 \leq a_k x_k \leq x$ , one has  $0 \leq g(x_k) \leq a_k^{-1} g(x) = 0, g \in S_x, k \in N$ . In view of (1) one thus obtains

$$\{0\} \subset S_x \subset \bigcap_{k=1}^{\infty} S_{x_k} = \{0\} ,$$

and this contradiction shows that

$$S \setminus \{0\} \neq \emptyset .$$

If  $f$  is a nonzero element of  $S$  then  $f(X \cap K) = \{0\}$ . Thus for any  $x \in X$  we have  $f(x) = f(x^+) - f(x^-) = 0$  because  $x^\pm \in X$ . Hence  $f(X) = \{0\}$ , showing that  $\text{sp } X$  cannot be dense in  $E$ .

Let now  $E$  be a real Fréchet space with an unconditional basis  $\{x_n, f_n\}$ . It is known that the set  $K = \{x \in E: f_n(x) \geq 0, n \in N\}$  is a closed, normal, generating cone in  $E$  and equipped with  $K, E$  becomes an order complete locally convex lattice [3, Theorem 5]. Obviously,  $\{x_n\} \subset K$  and the coefficient functionals  $f_n$  are positive with respect to  $K$ . Therefore, the basis is a positive Schauder basis for  $E$  [2].

REMARK. A slight modification of the above argument shows that in Theorem 1 the separability of  $E$  can be replaced by the (weaker) condition: There exists a sequence  $\{u_n\}$  in the positive cone of  $E$  such that  $\text{sp } \bigcup_{n=1}^{\infty} [0, u_n]$  is dense in  $E$ .

THEOREM 2. *If  $X$  is a solid convex set with void interior in  $E$  ( $E$  being specified above), then every point of  $\partial X$  supports  $X$  if and only if  $\bar{X}$  does not contain a weak order unit of  $E$ .*

*Proof.* (Necessity) Let every point of  $\partial X$  support  $X$  and let us assume that  $\bar{X}$  contains a weak order unit  $x$  of  $E$ . Then from [3, Proposition 11] it follows that the span of  $[0, x]$  is dense in  $E$ . Since  $[0, x] \subset \bar{X}$  this contradicts Theorem 1. Hence  $\bar{X}$  does not contain a weak order unit.

(Sufficiency) If  $\bar{X}$  does not contain a weak order unit of  $E$ , suppose that  $\sup f_n(X) > 0$ ,  $n \in N$ . Then for every  $n$  there is a  $y_n \in X$  such that  $\sup f_n(X) \leq 2f_n(y_n)$ . Since  $X$  is solid this yields for all  $n$  that  $\sup f_n(X)x_n \leq 2|y_n|$ ; whence  $1/2 \sup f_n(X)x_n \in X$ . In the same way as in the proof of the necessity part of the preceding theorem we can now construct an element  $x \in \bar{X} \cap K$  such that  $x = \lim_n \sum_{i=1}^n a_i x_i$ , where  $a_i > 0$ ,  $i \in N$ . If  $y \in K \setminus \{0\}$  then there must be a positive integer  $n$  such that  $f_n(y) > 0$ . If  $z \in K$  is given by  $z = \inf \{a_n, f_n(y)\}x_n$  it follows that  $z \neq 0$  and  $z = x \wedge y$ , i.e.,  $x$  is a weak order unit of  $E$  in  $\bar{X}$ . By this contradiction to our assumption there is an  $n \in N$  such that  $\sup f_n(X) = 0$ . Therefore,  $\text{sp } X$  cannot be dense in  $E$  and an application of Theorem 1 finally completes the proof.

Concerning the real Banach lattice  $C(S)$ ,  $S$  compact Hausdorff, it is clear that there can exist solid subsets  $X$  of  $C(S)$  with void interior containing a weak order unit of  $C(S)$  and such that every boundary point of  $X$  is a support point of  $X$ . For instance, take  $X = \{y \in C[0, 1] : |y| \leq x\}$ , where  $x$ , given by  $x(s) = s$ ,  $s \in [0, 1]$ , is a weak order unit of  $C[0, 1]$ . Therefore, that  $\bar{X}$  contains no weak order unit of  $C(S)$  is not a necessary condition for  $X$  to be supported at each boundary point, as is also seen by the following theorem:

THEOREM 3. *If  $X$  is a convex solid set with void interior in  $C(S)$  then every boundary point of  $X$  supports  $X$ .*

*Proof.* We assume that  $\text{sp } X$  is dense in  $C(S)$ . If  $f_s$  is the point evaluation functional of a general point  $s$  of  $S$ , this implies that  $\sup f_s(X) > 0$ ,  $s \in S$ . In this case, since  $X$  is solid, there is for every  $s \in S$  an  $x_s \geq 0$  in  $X$  such that  $x_s(s) > 0$ . Hence for every  $s \in S$

there is an open neighborhood  $V_s$  of  $s$  in  $S$  such that  $\inf x_s(V_s) > 0$ . Since  $S$  is compact and  $\{V_s: s \in S\}$  is an open covering of  $S$  there is a finite subcovering  $\{V_{s(1)}, \dots, V_{s(m)}\}$  for  $S$ . Taking  $x = m^{-1} \sum_{n=1}^m x_{s(n)}$  it is clear that  $x$  is in  $X$  since  $X$  is convex, and that

$$\inf x(S) \geq m^{-1} \inf_{n \leq m} \inf x_{s(n)}(V_{s(n)}) > 0.$$

If  $U$  is the unit ball of  $C(S)$  we obtain  $(\inf x(S)) |y| \leq x, y \in U$ , which, since  $\overline{X}$  is solid, implies that  $(\inf x(S))U \subset X$ . This contradiction shows that  $\overline{\text{sp } X} \neq C(S)$  and the result follows in the same way as in the sufficiency part of the proof of Theorem 1.

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