LOCAL IDEALS IN A TOPOLOGICAL ALGEBRA OF ENTIRE FUNCTIONS CHARACTERIZED BY A NON-RADIAL RATE OF GROWTH

JAMES JEROME METZGER
LOCAL IDEALS IN A TOPOLOGICAL ALGEBRA OF ENTIRE FUNCTIONS CHARACTERIZED BY A NON-RADIAL RATE OF GROWTH

JAMES J. METZGER

In this paper a class of locally convex algebras of entire functions is considered: For fixed $\rho > 0$, $\sigma > 0$, and $n$ a positive integer, let $E[\rho, \sigma; n]$ denote the space of all entire functions $f$ in $n$ variables which satisfy $|f(x + iy)| = O\{\exp [A(|x|^\rho + |y|^\sigma)]\}$ for some $A > 0$. Sufficient conditions are given in order that the local ideal generated by a family in $E[\rho, \sigma; n]$ coincides with the closed ideal generated by the family.

For $z = x + iy = (x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n) \in C^n$, write $||z||^2 = ||x||^2 + ||y||^2 = \sum_{k=1}^n (x_k^2 + y_k^2)$. For $f: C^n \to C$ and $A > 0$, let $||f||_A = \sup \{|f(z)| \exp [-A(|x|^\rho + |y|^\sigma)]: z \in C^n\}$. The space $E = E[\rho, \sigma; n]$ is a locally convex algebra over $C$, with the natural inductive limit topology from the Banach spaces $\{f$ entire: $||f||_A < \infty\}$, $A > 0$.

For $\mathcal{I}$ a family of functions in $E$, write $\mathcal{I}(\mathcal{I})$, $\mathcal{I}^{-}(\mathcal{I})$, and $\mathcal{I}_{\text{loc}}(\mathcal{I})$, respectively, for the ideal, closed ideal, and local ideal in $E$ generated by $\mathcal{I}$. The local ideal $\mathcal{I}_{\text{loc}}(\mathcal{I})$ consists of all $H \in E$ such that in a neighborhood of each $z \in C^n$, $H$ has the form $H = \sum_{j=1}^n h_j F_j$ for some $F_1, F_2, \ldots, F_r \in \mathcal{I}$ and $h_1, h_2, \ldots, h_r$ analytic in a neighborhood of $z$. The ideal $\mathcal{I}_{\text{loc}}(\mathcal{I})$ is closed in $E$ and contains $\mathcal{I}$; hence $\mathcal{I}(\mathcal{I}) \subseteq \mathcal{I}^{-}(\mathcal{I}) \subseteq \mathcal{I}_{\text{loc}}(\mathcal{I})$. The problem to be considered is: Under what conditions is $\mathcal{I}^{-}(\mathcal{I}) = \mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E$?

Problems of this type have been studied in various algebras $E$ by many authors, among them: L. Ehrenpreis [2, 3], L. Schwartz [14], H. Cartan [1], L. Hörmander [5, 6], B. A. Taylor [15], J. J. Kelleher and B. A. Taylor [7, 8, 9], J. Metzger [11], I. F. Krasičkov [10], P. K. Raševskiǐ [13], and K. V. Rajeswara Rao [12].

Let $\mathcal{I} \subseteq E = E[\rho, \sigma; n]$. It is known (see B. A. Taylor [15]) that for $n = 1$ and $\rho = \sigma \geq 1$, $\mathcal{I}^{-}(\mathcal{I}) = \mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E$ for any $\mathcal{I}$. If $\rho = \sigma$ and $\mathcal{I} = \{F\}$, but $n$ is arbitrary, then $\mathcal{I}(F) = \mathcal{I}^{-}(F) = \mathcal{I}_{\text{loc}}(F)$ (see L. Ehrenpreis [2]). In [11] this author proved that if $n = 1$, and $\rho \geq 1$ or $\sigma \geq 1$, then $\mathcal{I}^{-}(F) = \mathcal{I}_{\text{loc}}(F)$ for any $F \in E$; if in addition $\rho \neq \sigma$, there exists an $F \in E$ for which $\mathcal{I}(F) \neq \mathcal{I}^{-}(F)$. Concerning the more general case where $n$ is arbitrary, and $\rho$ and $\sigma$ do not necessarily agree: Ehrenpreis's Quotient Structure Theorem (see [3]) implies that if $\rho > 1$ and $\sigma > 1$, and if $\mathcal{I} = \{F_1, F_2, \ldots, F_r\}$ consists of polynomials, then $\mathcal{I}(\mathcal{I}) = \mathcal{I}^{-}(\mathcal{I}) = \mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E$. Also, a result of Hörmander [5] implies that when $\rho \geq 1$ and $\sigma \geq 1$,
a family $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ in $E$ satisfies $\mathcal{F}(\mathcal{F}) = E$ if and only if there exist $\varepsilon > 0$ and $A > 0$ such that

$$\sum_{j=1}^{r} |F_j(z)| \geq \varepsilon \exp \left[-A(||x||^\sigma + ||y||^\sigma)\right]$$

for all $z \in \mathbb{C}^n$.

In this paper the following result is proved:

**Theorem 1.** Let $n$ be a positive integer, $\rho > 0$, $\sigma > 0$, and $\tau = \max(\rho, \sigma) \geq 1$; and let $\mathcal{F} \subseteq E[\rho, \sigma; n]$. If $\mathcal{F}(\mathcal{F}) = \mathcal{F}_{\text{iso}}(\mathcal{F})$ in $E[\tau, \tau; n]$, then $\mathcal{F}^-(\mathcal{F}) = \mathcal{F}_{\text{iso}}(\mathcal{F})$ in $E[\rho, \sigma; n]$.

Since $\mathcal{F}(F) = \mathcal{F}_{\text{iso}}(F)$ in $E[\tau, \tau; n]$, a consequence of Theorem 1 is:

**Corollary.** Let $n$ be a positive integer, $\rho > 0$, $\sigma > 0$, with $\max(\rho, \sigma) \geq 1$. Then $\mathcal{F}^-(F) = \mathcal{F}_{\text{iso}}(F)$ in $E = E[\rho, \sigma; n]$ for any $F \in E$.

This corollary generalizes to several variables the result proved by this author in [11] for the case of one variable.

Theorem 1 follows immediately from an approximation theorem which is proved in the next section. In the third section Theorem 1 is applied to several examples.

2. The main theorem. The approximation theorem stated below, Theorem 2, yields Theorem 1 as an immediate corollary. The proof of Theorem 2 is based on a technique of L. Hörmander given in [6], which in turn involves the solution of the $\overline{\partial}$ equation (see [4, Chapter IV]).

**Theorem 2.** Let $\sigma \geq 1$, and $H, F_1, F_2, \ldots, F_r, G_1, G_2, \ldots, G_r$ be entire functions in $n$ variables, with $H = \sum_{j=1}^{r} G_j F_j$ and

$$|G_j(z)| \leq C \exp(A \cdot ||z||^\sigma)$$

for all $z \in \mathbb{C}^n$, $j = 1, 2, \ldots, r$, where $A, C$ denote positive constants. Then there exist positive constants $B, K, M,$ and entire functions $g_{j, t}$, $0 < t < 1$, $j = 1, 2, \ldots, r$, such that:

$$\left|H(z) - \sum_{j=1}^{r} g_{j, t}(z) F_j(z)\right| \leq tK(1 + ||z||^\sigma)^\nu \left\{ ||H(z)|| + \left[ \sum_{j=1}^{r} |F_j(z)| \exp(B \cdot ||y||^\sigma) \right] \right\}$$

for all $z \in \mathbb{C}^n$, $0 < t < 1$, and
for all \( z \in C^n \), \( 0 < t < 1 \), \( j = 1, 2, \ldots, r \), where \( L(t) > 0 \) may depend on \( t \) but not on \( z \).

The proof of Theorem 2 is facilitated by the following:

**Lemma.** Let \( n \) and \( N \) be positive integers, with \( N \) even. There exist \( \alpha > 0 \) and \( \varepsilon > 0 \) such that: If \( z \in C^n \) with \( \alpha \| x \| \geq \| y \| \), then \( \operatorname{Re}(z^n) \geq \varepsilon \| z \|^N \).

Here \( z^n = \sum_{k=1}^{n} (x_k + i y_k)^n \).

**Proof.** Write \( q = N/2 \); then

\[
\operatorname{Re}(x_k^n + iy_k^n) = a_k^{2q} + \sum_{m=1}^{q} a_m a_k^{2q-m} y_k^m
\]

for all \( x_k + iy_k \in C \), where \( a_1, a_2, \ldots, a_q \) are integers depending only on \( N \). Hence for \( z \in C^n \),

\[
\operatorname{Re}(z^n) \geq \sum_{k=1}^{n} a_k^{2q} - \sum_{m=1}^{q} |a_m| \left( \sum_{k=1}^{n} a_k^{2(q-m)} y_k^m \right)
\]

\[
\geq 2^{(-n+1)(q-1)} \| x \|^{2q} - \sum_{m=1}^{q} |a_m| \| x \|^{2(q-m)} \| y \|^{2m}.
\]

The required condition is then satisfied with \( \varepsilon = 2^{-(n-1)(q-1)-(q-2)} \), and \( 0 < \alpha < 1 \) sufficiently small that \( \sum_{m=1}^{q} |a_m| \alpha^{2m} < 2^{-(n-1)(q-1)-1} \).

**Proof of Theorem 2.** Let \( N \) be an even integer, \( N > \sigma \). By the lemma there exist \( \alpha = \alpha(n, N) > 0 \) and \( \varepsilon = \varepsilon(n, N) > 0 \) such that \( \alpha \| x \| \geq \| y \| \) implies \( \operatorname{Re}(z^n) \geq \varepsilon \| z \|^N \). Set \( S = \{ z \in C^n : \alpha \| x \| \geq \| y \| \) and \( \operatorname{Re}(z^n) \geq 1 \} \). The bounds (1) imply that for some \( B > 0 \) and \( K_i > 0 \),

\[
|G_j(z)| \leq K_i \exp(B \| y \|^r)
\]

for all \( z \in C^n \setminus S, j = 1, 2, \ldots, r \).

Let \( \varphi : R \to R \) be a \( C^\infty \) function such that

\[
\varphi(u) = 0 \quad \text{if} \quad u \leq 0,
\]

\[
= 1 \quad \text{if} \quad u \geq 1,
\]

and \( 0 \leq \varphi(u) \leq 1 \) if \( 0 \leq u \leq 1 \). For \( 0 < t < 1 \) and \( z \in C^n \), set

\[
\omega_t(z) = \frac{\varphi(\operatorname{Re}(z^n))}{\exp(-t(z^n))} + [1 - \varphi(\operatorname{Re}(z^n))].
\]

Each \( \omega_t \) is a \( C^\infty \) function on \( C^n \); and \( |\omega_t(z)| \leq 1 \) for all \( z \in C^n \), while
$|\omega_t(z)| \leq \exp (-\varepsilon \|z\|^\nu)$ for all $z \in S$. Together with (1) and (4), this implies that for some $K_2 > 0$,

$$\omega_t(z)G_j(z) \leq K_2 \exp (B \|y\|^\sigma)$$

for all $z \in C^s$, $0 < t < 1$, $j = 1, 2, \cdots, r$. Since $|1 - e^{i\zeta}| \leq |\zeta|$ if $\Re \zeta \leq 0$, and since $|z^n| \leq \|z\|^n$, it follows that $|1 - \omega_t(z)| \leq tn \|z\|^n$ for all $z \in C^s$, $0 < t < 1$. Consequently,

$$H(z) - \sum_{j=1}^{r} (\omega_t(z)G_j(z))F_j(z) \leq tn \|z\|^\nu \|H(z)\|$$

for all $z \in C^s$, $0 < t < 1$. Thus the functions $\omega_tG_j$ satisfy conditions of the form (2) and (3).

As is done by Hörmander, the functions $\omega_tG_j$ will now be altered to obtain the desired analytic functions $\nu_{j,t}$. First of all, $\bar{\omega}_t = 0$ if $\Re (x^n) \leq 0$, and $\|\bar{d}(\varphi(\Re (x^n)))\| \leq K_3 \|x\|^\nu$ everywhere on $C^s$; therefore, $\|\bar{\omega}_t(z)\| \leq tnK_3 \|z\|^\nu$ for all $z \in C^s$, $0 < t < 1$. Also, $\bar{d}(\omega_tG_j) = (\bar{\omega}_t)G_j$; and $\bar{\omega}_t = 0$ on $S$. By (4) then, for $0 < t < 1$ and $j = 1, 2, \cdots, r$,

$$\|\bar{d}(\omega_t(z)G_j(z))\| \leq tK_4 \|z\|^{2\nu-1} \exp (B \|y\|^\sigma)$$

for all $z \in C^s$, and thus

$$\int_{C^s} \|\bar{d}(\omega_t(z)G_j(z))\|^2 \exp (-\psi(z) - 2 \log (1 + \|z\|^\nu))d\lambda(z) \leq t^2K_5$$

where $\psi(z) = 2B \|y\|^\sigma + (2N + n) \log (1 + \|z\|^\nu)$, and $\lambda$ denotes Lebesgue measure.

By applying Theorem 4.4.2 of Hörmander [4], functions $\nu_{j,t}$ of class $C^m$ on $C^s$ may be chosen such that $\bar{d}\nu_{j,t} = \bar{d}(\omega_tG_j)$ and

$$\int_{C^s} |\nu_{j,t}(z)|^2 \exp [-\psi(z) - 2 \log (1 + \|z\|^\nu)]d\lambda(z) \leq t^2K_5$$

for $0 < t < 1$, $j = 1, 2, \cdots, r$. Together with (7), this implies (see Hörmander [6, p. 314]) that

$$|\nu_{j,t}(z)| \leq tK_6 (1 + \|z\|^\nu)^M \exp (B \|y\|^\sigma)$$

for all $z \in C^s$, $0 < t < 1$, $j = 1, 2, \cdots, r$, where $M = N + 1 + (1/2)n$.

Each of the functions $\nu_{j,t} = \omega_tG_j - \nu_{j,t}$ is then entire. Further, (3) is satisfied because of (5) and (8). Lastly $H - \sum_{j=1}^{r} \nu_{j,t}F_j = [H - \sum_{j=1}^{r} (\omega_tG_j)F_j] + \sum_{j=1}^{r} \nu_{j,t}F_j$, and thus (2) follows from (6) and (8).

3. Examples and applications. In this section several examples are given where $\mathcal{F}^{-}(\mathcal{I}) = \mathcal{K}_{10}(\mathcal{I})$ in spaces of the form $E[\rho, \sigma; n]$.

**Example 1.** Let $E = E[\rho, \sigma; n]$, with $\tau = \max (\rho, \sigma) \geq 1$, and let $F \in E$. The corollary to Theorem 1 implies that $\mathcal{F}^{-}(F) = \mathcal{K}_{10}(F)$.
in $E$. Also, $\mathcal{I}(F) = \mathcal{I}^{-}(F) = \mathcal{I}_{\text{loc}}(F)$ in $E[\tau, \tau; n]$. However, it need not be the case that $\mathcal{I}(F) = \mathcal{I}^{-}(F)$ in $E$; indeed, if $\rho \neq \sigma$ then (see [11]) there exists an $F \in E$ for which $\mathcal{I}(F) \neq \mathcal{I}^{-}(F)$.

**EXAMPLE 2.** Let $n = 1$, and $E = E[\rho, \sigma; 1]$, with $\tau = \max(\rho, \sigma) \geq 1$. Let $\mathcal{I} \subseteq E$ and suppose some $F_0 \in E$ has only finitely many zeros. Then $\mathcal{I}^{-}(\mathcal{I}) = \mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E$. To prove this, write $F_0 = PH$ where $P$ is a polynomial and $H \in E$ has no zeros. There exists a polynomial $Q$ such that $\mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E$ is $(G \in E; G/Q$ is analytic). Set $P = P_0Q$, so that $F_0 = P_0QH \in \mathcal{I} \subseteq \mathcal{I}^{-}(\mathcal{I})$. The factors of $P_0$ can be divided out (see Taylor [15]) to yield $QH \in \mathcal{I}^{-}(\mathcal{I})$ in $E$. Since $1/H \in E[\tau, \tau; 1]$, it follows that $Q \in \mathcal{I}^{-}(\mathcal{I})$ in $E[\tau, \tau; 1]$, which implies that $\mathcal{I}(\mathcal{I}) = \mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E[\tau, \tau; 1]$. Then by Theorem 1, $\mathcal{I}^{-}(\mathcal{I}) = \mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E = E[\rho, \sigma; 1]$. Note that if $1/H \in E$--e.g., if $\rho \geq 1$, $\sigma \geq 1$, and $F_0$ is an exponential polynomial $F_0(z) \equiv P(z)e^{\alpha z}$--then $Q \in \mathcal{I}_{\text{loc}}(\mathcal{I})$ in $E$ and thus $\mathcal{I}(\mathcal{I}) = \mathcal{I}^{-}(\mathcal{I}) = \mathcal{I}_{\text{loc}}(\mathcal{I})$ trivially. On the other hand, if $1/H \not\in E$ then $\mathcal{I}(\mathcal{I})$ need not coincide with $\mathcal{I}^{-}(\mathcal{I})$ in $E$--for instance, if $\rho = 1$, $\sigma = 2$, and $\mathcal{I} = \{e^{-iz}, e^{iz} - 1\}$.

**EXAMPLE 3.** Let $1 \leq \rho < \sigma$ and $E = E[\rho, \sigma; 1]$. Choose $\gamma, \rho < \gamma < \sigma$; let $\varepsilon_m = \exp(-(2^m\gamma))$, $a_m = 2^m$, $b_m = 2^m + \varepsilon_m$, $m = 1, 2, \cdots$; and let

$$F_1(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{a_m}\right)$$

$$F_2(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{b_m}\right)$$

for all $z \in C$. Each $F_j$ is an entire function of order 0 and thus is in $E$. Clearly $\mathcal{I}_{\text{loc}}(F_1, F_2) = E$. It is easily argued that for $\rho < \rho' < \gamma$, $|F_j(2^m)| = O[\exp(-(2^m\gamma))]$ as $m \to \infty$. Consequently $1 \in \mathcal{I}(F_j, F_j)$ in $E$. On the other hand, letting $\gamma < \sigma' < \sigma$ and using standard estimates on infinite products yields:

$$|F_1(z)| \geq \delta \exp(-|z|^{\sigma'}) \quad \text{if} \quad z \in \bigcup_m \left\{z: \left|z - a_m\right| < \frac{1}{2}\varepsilon_m\right\},$$

$$|F_2(z)| \geq \delta \exp(-|z|^{\sigma'}) \quad \text{if} \quad z \in \bigcup_m \left\{z: \left|z - b_m\right| < \frac{1}{2}\varepsilon_m\right\},$$

where $\delta > 0$. Thus $|F_1(z)| + |F_2(z)| \geq \delta \exp(-|z|^{\sigma'})$ for all $z \in C$. It then follows (Hörmander [5]) that $1 \in \mathcal{I}(F_1, F_2)$ in $E[\sigma, \sigma; 1]$. Hence $\mathcal{I}(F_1, F_2) = \mathcal{I}^{-}(F_1, F_2) = \mathcal{I}_{\text{loc}}(F_1, F_2)$ in $E[\sigma, \sigma; 1]$, while $\mathcal{I}(F_1, F_1) \not\subseteq \mathcal{I}^{-}(F_1, F_1) = \mathcal{I}_{\text{loc}}(F_1, F_1)$ in $E = F[\rho, \sigma; 1]$. 
REFERENCES


Received November 14, 1972 and in revised form July 20, 1973.

UNIVERSITY OF GEORGIA
Zvi Arad, \( \pi \)-homogeneity and \( \pi' \)-closure of finite groups ................................................. 1
Ivan Baggs, A connected Hausdorff space which is not contained in a maximal connected space .......................................................... 11
Eric Bedford, The Dirichlet problem for some overdetermined systems on the unit ball in \( C^n \) .......................................................... 19
R. H. Bing, Woodrow Wilson Bledsoe and R. Daniel Mauldin, Sets generated by rectangles .......................................................... 27
Carlo Cecchini and Alessandro Figà-Talamanca, Projections of uniqueness for \( L^p(G) \) .......................................................... 37
Gokulananda Das and Ram N. Mohapatra, The non absolute Nörlund summability of Fourier series .......................................................... 49
Frank Rimi DeMeyer, On separable polynomials over a commutative ring .......................................................... 57
Richard Detmer, Sets which are tame in arcs in \( E^3 \) .......................................................... 67
William Erb Dietrich, Ideals in convolution algebras on Abelian groups .......................................................... 75
Bryce L. Elkins, A Galois theory for linear topological rings .......................................................... 89
William Alan Feldman, A characterization of the topology of compact convergence on \( C(X) \) .......................................................... 109
Hillel Halkin Gershenson, A problem in compact Lie groups and framed cobordism .......................................................... 121
Samuel R. Gordon, Associators in simple algebras .......................................................... 131
Marvin J. Greenberg, Strictly local solutions of Diophantine equations .......................................................... 143
Jon Craig Helton, Product integrals and inverses in normed rings .......................................................... 155
Domingo Antonio Herrero, Inner functions under uniform topology .......................................................... 167
Jerry Alan Johnson, Lipschitz spaces .......................................................... 177
Marvin Stanford Keener, Oscillatory solutions and multi-point boundary value functions for certain \( n \)th-order linear ordinary differential equations .......................................................... 187
John Cronan Kieffer, A simple proof of the Moy-Perez generalization of the Shannon-McMillan theorem .......................................................... 203
Joong Ho Kim, Power invariant rings .......................................................... 207
Gangaram S. Ladde and V. Lakshmikantham, On flow-invariant sets .......................................................... 215
Roger T. Lewis, Oscillation and nonoscillation criteria for some self-adjoint even order linear differential operators .......................................................... 221
Jürg Thomas Marti, On the existence of support points of solid convex sets .......................................................... 235
John Rowlay Martin, Determining knot types from diagrams of knots .......................................................... 241
James Jerome Metzger, Local ideals in a topological algebra of entire functions characterized by a non-radial rate of growth .......................................................... 251
K. C. O’Meara, Intrinsic extensions of prime rings .......................................................... 257
Stanley Poreda, A note on the continuity of best polynomial approximations .......................................................... 271
Robert John Sacker, Asymptotic approach to periodic orbits and local prolongations of maps .......................................................... 273
Eric Peter Smith, The Garabedian function of an arbitrary compact set .......................................................... 289
Arne Stray, Pointwise bounded approximation by functions satisfying a side condition .......................................................... 301
John St. Clair Werth, Jr., Maximal pure subgroups of torsion complete abelian \( p \)-groups .......................................................... 307
Robert S. Wilson, On the structure of finite rings. II .......................................................... 317
Kari Ylinen, The multiplier algebra of a convolution measure algebra .......................................................... 327