A NOTE ON THE CONTINUITY OF BEST POLYNOMIAL APPROXIMATIONS

STANLEY POREDA
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S. J. POREDA

An example is given to show that while best uniform polynomial approximation in the complex plane is continuous, it is not in general uniformly continuous.

For a continuous complex valued function $f$ defined on $E$, a compact set in the plane and $n \in \{0, 1, 2, \ldots\}$, let $p_n(f, E)$ denote the polynomial of degree $n$ of best uniform approximation to $f$ on $E$ and let $\|f\|_E$ denote the uniform norm of $f$ on $E$. In [2] it was shown that for any such $f$ and $E$ there exists for each $n$ and each $\beta, 0 < \beta < 1/2$, a constant $M(n, \beta) > 0$ such that

$$\|p_n(f, E) - q_n\|_E \leq M(n, \beta)(\|f - q_n\|_E - \|f - p_n(f, E)\|_E)^\beta,$$

where $q_n$ is any polynomial of degree $n$. If we in fact let $M(n, \beta)$ denote the least such constant then we ask if the sequence $(M(n, \beta))_{n=0}^\infty$ is bounded. The purpose of this note is to show that in general it is not.

Let $f(z) = 1/z$ and $E = U = \{|z| = 1\}$. Then $p_n(f, U) \equiv 0$ for $n = 0, 1, 2, \ldots$. Now for each $k = 1, 2, \ldots$ and each $\beta, 0 < \beta < 1/2$, let $\Omega_{k, \beta}$ denote a simply connected Jordan region containing the origin such that

1. $\Omega_{k, \beta} \subset \{|z| < k^\beta\}$
2. $k^\beta \in \partial \Omega_{k, \beta}$
3. $\Omega_{k, \beta} \cap \{|z| > k(\beta - 1)/\beta\} \cap \{\text{Re } z \leq 0\} = \emptyset$.

The region $\Omega_{k, \beta}$ can in fact be chosen to be a displaced ellipse. Furthermore, there exists a conformal map $g_{k, \beta}$ of the unit disc $\{|z| < 1\}$ onto $\Omega_{k, \beta}$ such that $g_{k, \beta}(0) = 0$. Furthermore $g_{k, \beta}$ will be continuous in $\{|z| \leq 1\}$ and map $U$ onto the boundary of $\Omega_{k, \beta}$. As a consequence of these definitions we have that

$$\|1 - g_{k, \beta}(z)/k\|_U \leq 1 + (1/k)^{1/\beta}$$

and

$$\|g_{k, \beta}(z)/k\|_U = k^{\beta}/k.$$

Now there exists [1, p. 98] a polynomial $p$ such that

$$\|g_{k, \beta}(z)/z - p(z)\|_U < k^{\beta - 1/\beta}.$$

Thus

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\[ \|1/z - p(z)/k\|_V \leq 1 + 2(1/k)^{1/\beta} \]

and

\[ \|p(z)/k\|_V \geq \frac{k^\beta - k^{(\beta-1)/\beta}}{k} = \frac{k^\beta - k^{(\beta-1)/\beta}}{2^\beta} \left( \frac{2^\beta}{k} \right). \]

Consequently, we see that for this particular choice of \( f \) and \( \beta \), and for each \( 0 < \beta < 1/2 \),

\[ \lim_{n \to \infty} M(n, \beta) \geq \lim_{k \to \infty} \left( \frac{k^\beta - k^{(\beta-1)/\beta}}{2^\beta} \right) = \infty. \]

REFERENCES


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* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

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