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# THE GARABEDIAN FUNCTION OF AN ARBITRARY COMPACT SET

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If the outer boundary of the compact plane set E is the union of finitely many disjoint analytic Jordan curves, the Garabedian function of E is a familiar object. J. Garnett and S. Y. Havinson have each asked whether the Garabedian functions of a decreasing sequence of such sets must converge. The present paper shows that they do converge. This fact leads to a natural definition of the Garabedian function of an arbitrary compact plane set. As an intermediate step, an approximate formula is obtained for the analytic capacity of the union of a compact set E and a small disc not intersecting E.

1. Prerequisites and notation. Good introductions to Analytic Capacity are given in [2], pp. 1-26, and [1], Ch. 8; and so we shall give only a brief outline.

C denotes the complex plane.  $S^z$  denotes the extended complex plane with its usual topology. D(z; r) denotes the closed disc with centre z and radius r.

Let E be a compact subset of C.  $\Omega(E)$  denotes the component of  $S^2 \setminus E$  containing  $\infty$ . The *outer boundary* of E is the boundary  $\partial \Omega(E)$  of  $\Omega(E)$ . The *analytic capacity* of E is:

$$\gamma(E) = \sup\{|g'(\infty)|: g \text{ analytic on } \Omega(E), |g| < 1\}$$
.

This supremum is attained by a unique function, the Ahlfors function of E ([1], p. 197).

 $\mathscr S$  will denote the class of all compact plane sets whose outer boundary is the union of finitely many pairwise disjoint analytic Jordan curves. Let  $E\in\mathscr S$ , and write  $\Omega=\Omega(E)$ . The Hardy space  $H^p(\Omega)$   $(0< p<\infty)$  is the space of all analytic functions g on  $\Omega$  such that there exists a harmonic function u on  $\Omega$  with  $|g|^p < u$ . If  $g \in H^p(\Omega)$  then g has a finite nontangential limit g(z) at almost every point  $z \in \partial \Omega$ .  $H^p(\Omega)$  is a separable Hilbert space with the inner product:

$$(g,\,h)=\int_{z_0}g(z)h(z)^*ds\quad (g,\,h\in H^2(\mathcal{Q}))\;.$$

If  $\zeta \in \Omega$  there is a unique function  $K(z, \zeta)$  in  $H^2(\Omega)$ , the Szegö kernel function, such that:

$$g(\zeta) = \int_{\partial arrho} g(z) K(z, \, \zeta)^* ds \quad \left(g \in H^2(arrho)
ight) \, .$$

 $K(z, \zeta)$  is the inner product between the functionals on  $H^2(\Omega)$  given

by evaluation at z and  $\zeta$ , so that  $K(z, \zeta) = \sum u_n(z)u_n(\zeta)^*$ , whenever  $\{u_n\}$  is an orthonormal basis for  $H^2(\Omega)$ . The Garabedian function is most easily defined for our purpose as:

$$\psi(z) = rac{2\pi}{i} \gamma(E)^2 K(z, \infty)^2$$
 .

See [2], pp. 13-23.

Throughout, E will be a compact plane set,  $\Omega = \Omega(E)$ , and f will be the Ahlfors function of E. If  $E \in \mathcal{S}$ ,  $K(z, \zeta)$  will denote its Szegö kernel function, and  $\psi$  its Garabedian function.

We shall use the following results.

- 1.1. Let  $\{E_n\}$  be a decreasing sequence of compact sets with intersection E. Let  $f_n$  be the Ahlfors function of  $E_n$ . Then  $f_n \to f$  uniformly on compact subsets of  $\Omega$ , and  $\gamma(E_n) \to \gamma(E)$  ([1], p. 198).
  - 1.2. Let  $E \in \mathcal{S}$ . Then:
  - (1) f and  $\psi$  are analytic across  $\partial \Omega$ .
  - (2) |f|=1 on  $\partial \Omega$ .
  - (3)  $f(z)\psi(z)dz \ge 0$  on  $\partial \Omega$ .
  - (4)  $\psi(\infty) = 1/(2\pi i)$ .
  - (5)  $K(\infty, \infty) = 1/(2\pi\gamma(E))$ .
- ([2], pp. 18-23).
- 1.3. Let E, F be compact,  $\gamma(E) = 0$ . Then  $\gamma(E \cup F) = \gamma(F)$  (an immediate consequence of [2], Theorem 1.4, pp. 10-11).
- 1.4. Let E be compact,  $0 \notin E$ ,  $E \subset D(0; R)$ . Denote by  $E_*$  the inversion of E in the unit circle. Then:

$$\gamma(E_*) \ge \gamma(E)/8R^2$$

(proof similar to [1], Lemma 12.2, p. 229).

Finally we need the following result on Hilbert spaces.

PROPOSITION 1.5. Let h be a separable Hilbert space, and let  $\{u_n\}$  be a sequence of vectors in h whose closed linear span is h. Suppose that the infinite matrix T given by  $T_{ij} = (u_j, u_i)$  is bounded and invertible (as an operator on  $l_2$ ). Then for every bounded linear functional f on h the sequence  $\{f(u_i)\}$  is square-summable and:

$$||f||^2 = \sum_{i,j=1}^{\infty} (T^{-1})_{ij} f(u_i) f(u_j)^*$$
 .

*Proof.* T is positive, and so is the matrix of a positive operator

 $P \in B(l_2)$ . P has a positive square root  $P^{1/2}$ , which is invertible since P is invertible. For  $i=1,\,2,\,3,\,\cdots$ , write  $w_i=P^{1/2}e_i$ , where  $e_i$  is the vector with 1 in its ith place and 0 elsewhere. Since  $P^{1/2}$  is invertible,  $l_2$  is the closed linear span of the  $w_i$ .  $(w_j,\,w_i)=(P^{1/2}e_j,\,P^{1/2}e_i)=(Pe_j,\,e_i)=T_{i,j}=(u_j,\,u_i)$ ; so we can define a unitary  $J\colon l_2\to h$  by  $J(w_i)=u_i$  for all i, extended to the whole of  $l_2$  by linearity and continuity. The bounded linear functional  $J^*f$  on  $l_2$  is represented by some  $s\in l_2$ .  $(e_i,\,P^{1/2}s)=(P^{1/2}e_i,\,s)=(w_i,\,s)=(J^*f)(w_i)=f(Jw_i)=f(u_i)$ . Hence  $\{f(u_i)\}$  is square-summable. Also:

$$\|f\|^2 = \|s\|^2 = (P^{-1}(P^{1/2}s), (P^{1/2}s))$$

$$= \sum_{i,j=1}^{\infty} (T^{-1})_{ij} (e_i, P^{1/2}s) (e_j, P^{1/2}s)^* = \sum_{i,j=1}^{\infty} (T^{-1})_{ij} f(u_i) f(u_j)^*.$$

2. The slope function. The purpose of this section is to establish Theorem 2.2, which gives an expression, up to first order in  $\varepsilon$ , for the analytic capacity of a set of the form  $E \cup D(z; \varepsilon)$ , where  $E \in \mathscr{S}$  and  $z \in \Omega(E)$ . This will be extended to arbitrary compact sets E in §3. First we need a lemma which gives bounds on the Szegö kernel function.

LEMMA 2.1. Let  $E \in \mathcal{S}$ ,  $\zeta \in \Omega(E)$ ,  $\zeta \neq \infty$ . Let r, R be the least and greatest distances of points of E from  $\zeta$ . Then:

$$rac{r^2}{16\pi R^2 \gamma(E)} \le extit{K}(\zeta,\,\zeta) \le rac{8R^2}{2\pi r^2 \gamma(E)} \;.$$

*Proof.* We prove the upper bound: the lower one is similar. We may assume that  $\zeta=0$ . Let  $g\in H^2(\Omega)$ ,  $||g||\leq 1$ . Denote inversion in the unit circle by  $_*$ . Define  $g_*$  on  $\Omega_*$  by  $g_*(z)=g(z_*)^*$ . Clearly  $g_*\in H^2(\Omega_*)$  and  $||g_*||\leq 1/r$ . Hence:

$$|g(0)|^2 = |g_*(\infty)|^2 \le \frac{||g_*||^2}{2\pi\gamma(E_*)} \le \frac{1}{2\pi r^2 \gamma(E_*)} \le \frac{8R^2}{2\pi r^2 \gamma(E)}$$

by 1.4. So:

$$K(0, \, 0) = \sup \{ \mid g(0) \mid^2 : g \in H^2(\Omega), \mid \mid g \mid \mid \leq 1 \} \leq \frac{8R^2}{2\pi r^2 \gamma(E)} \; .$$

There is a simpler bound for the Garabedian function: for, in the above notation:

$$\left|\psi(\zeta)-rac{1}{2\pi i}
ight|=\left|rac{1}{2\pi i}\int_{sa}rac{\psi(z)dz}{z-\zeta}
ight| \leq rac{1}{2\pi r}\int_{sa}\left|\psi\left|ds
ight| = rac{\gamma(E)}{2\pi r} \ .$$

Theorem 2.2. Let  $E \in \mathcal{S}$ . There is a positive real-valued func-

tion  $a_E(\zeta)$ , the slope function of E, defined on  $\Omega$ , with the property that for all  $\zeta \in \Omega$ :

$$\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2)$$
.

 $a_E(\zeta)$  is given explicitly by:

$$a_E(\zeta) = 2\pi |\psi(\zeta)| \{1 - |f(\zeta)|^2\}.$$

The bound in the error term depends only on  $\gamma(E)$  and on the ratio of the greatest and least distances of points of E from  $\zeta$ .

*Proof.* We may suppose that  $\zeta = 0$ . Let r, R be the least and greatest distances of points of E from 0. Note that  $E \subset D(0; R)$ , so that  $\gamma(E) \leq R$ . We shall prove the theorem by showing that:

$$egin{aligned} arepsilon < 10^{-5} (r/R)^5 \gamma(E) &\Longrightarrow &| \gamma(E \cup D(0;arepsilon)) - \gamma(E) - arepsilon a_{\scriptscriptstyle E}(0) \,| \ &\le 10^9 (R/r)^{10} \gamma(E)^{-1} arepsilon^2 \;. \end{aligned}$$

Fix  $\varepsilon < 10^{-5}(r/R)^5\gamma(E)$ . Since r < R and  $\gamma(E) \le R$ , we have  $\varepsilon < 10^{-5}r$ ; so  $D(0;\varepsilon)$  does not meet E. Write  $E_1 = E \cup D(0;\varepsilon)$ ,  $\Omega_1 = \Omega(E_1)$ ,  $H^2 = H^2(\Omega)$ ,  $H^2 = H^2(\Omega_1)$ , Y = Y(E),  $Y_1 = Y(E_1)$ . Choose an orthonormal basis  $\{u_n\}$  for  $H^2$ . Now we can use the Cauchy integral to express any element of  $H^2$  as the sum of an element of  $H^2$  and an element of  $H^2(S^2\backslash D(0;\varepsilon))$ . The latter space is the closed linear span of  $\{z^{-n}: n \ge 0\}$ . It follows that if, for  $n \ge 1$ ,  $v_n$  is any function analytic on  $\Omega$  except for a pole of order n at 0, then  $H^2$  is the closed linear span of  $\{u_n\} \cup \{v_n\}$ . To be specific, we shall put:

$$v_n(z) = rac{arepsilon^{n-1/2}}{\sqrt{(2\pi)} K(0,0)} rac{K(z,0)}{z^n} .$$

1.2 (5) says that  $1/(2\pi\gamma_1)$  is the square of the norm of evaluation at  $\infty$  in  $H_1^2$ . Our proof consists of calculating this by applying Proposition 1.5 to  $\{u_n\} \cup \{v_n\}$ .

We shall calculate various bounds now, so as not to break continuity later. Throughout, "|| ||" and "norm" will refer to the norm of an element of a Hilbert space, or the norm of an infinite matrix considered as a bounded operator on  $l_2$ ; and "||  $||_{\infty}$ " will denote the supremum of the absolute value of a function on the set  $D(0; \varepsilon)$ .

Let 
$$z_0 \in C$$
,  $|z_0| \leq \varepsilon$ . For  $n \geq 1$ :

$$u_n(z_0) = \frac{1}{2\pi i} \int_{|z|=r/2} \frac{u_n(z)dz}{z-z_0}.$$

Hence:

$$||u_n||_{\infty} \leq rac{1}{2\pi(r/2-arepsilon)} \int_{|z|=r/2} |u_n| ds$$
,

so that by Schwarz's inequality:

$$||u_n||_\infty^2 \leq rac{\pi r}{4\pi^2(r/2-arepsilon)^2} \int_{|z|=r/2} |u_n|^2 ds$$
 .

Summing over n and using Lemma 2.1 gives:

$$\begin{array}{l} \sum ||\,u_{_{n}}\,||_{_{\infty}}^{2} \leq \frac{r}{4\pi(r/2\,-\,\varepsilon)^{2}} \int_{|z|=r/2} K(z,\,z) ds \\ \\ \leq \frac{r \cdot \pi r}{4\pi(r/2\,-\,\varepsilon)^{2}} \frac{8(R\,+\,r/2)^{2}}{2\pi(r/2)^{2}\gamma} \leq 3R^{2} r^{-2} \gamma^{-1} \end{array}$$

since  $\varepsilon < 10^{-5}r$  and r < R. Analogous computation gives:

(2) 
$$\sum ||u_n'||_{\infty}^2 \leq 50R^2 r^{-4} \gamma^{-1}.$$

In particular:

$$\sum |u'_n(0)|^2 \leq 50R^2r^{-4}\gamma^{-1}.$$

Next we want a bound for  $||d^k/dz^k|K(z,0)||_{\infty}$ . Let  $z_0 \in C$ ,  $|z_0| \le \varepsilon$ . Then for  $k \ge 1$  and for all s < r:

$$igg|rac{d^k}{dz^k}K(z,\ 0)|_{z=z_0}igg| = igg|rac{k!}{2\pi i}\!\int_{|z|=s}\!rac{K(z,\ 0)dz}{(z-z_0)^{k+1}}igg| \ \le rac{k!}{2\pi}rac{2\pi s}{(s-arepsilon)^{k+1}}rac{8R(R+s)}{2\pi r(r-s)\gamma} \ .$$

(Here we have estimated |K(z, 0)| by  $K(z, z)^{1/2}K(0, 0)^{1/2}$  and then used Lemma 2.1.) In particular, putting s = kr/(k+1):

$$igg|rac{d^k}{dz^k}K(z,\,0)ig|_{z=z_0}igg| \leq rac{k!}{2\pi}rac{8R\cdot 2R}{r(r-(k+1)arepsilon/k)^{k+1}\gamma}rac{(k+1)^{k+1}}{k^k} \ \leq rac{(k+1)!}{2\pi}rac{16R^2}{r(r-2arepsilon)^{k+1}\gamma}e \leq rac{7R^2(k+1)!}{r(r-2arepsilon)^{k+1}\gamma}\,.$$

This holds also for k=0 by Lemma 2.1. Hence for  $k=0, 1, 2, \cdots$ :

$$\left\| \frac{d^k}{dz^k} K(z, 0) \right\|_{\infty} \leq \frac{7R^2(k+1)!}{r(r-2\varepsilon)^{k+1}\gamma}.$$

We need one more estimate. Since  $\varepsilon < 10^{-5}r$ , (4) gives:

$$\left\| rac{dK(z, 0)}{dz} 
ight\|_{\infty} \leq rac{15R^2}{r^3\gamma} \ .$$

Hence, using Lemma 2.1 and the fact that  $\varepsilon < 10^{-5} R^{-4} r^5$ , we have:

$$(5) || K(z, 0) ||_{\infty} \leq K(0, 0) + \varepsilon \left\| \frac{dK(z, 0)}{dz} \right\|_{\infty} \leq 1.01 K(0, 0).$$

We shall imagine the basis  $\{u_n\} \cup \{v_n\}$  to be partitioned into three sections. The first section consists of all the  $u_n$ , the second section consists of  $v_1$  alone, and the third section consists of  $v_2$ ,  $v_3$ ,  $v_4$ ,  $\cdots$ . The corresponding matrix T of inner products will be in block form:

$$T=I+M\,,\quad M=egin{bmatrix}A&B^H&C^H\B&D&E^H\C&E&F\end{bmatrix}.$$

Next we calculate the inner products. Denote inner products in  $H_1^2$  by ( , ). By a statement of the form "X=Y with error Z" we shall mean  $|X-Y| \leq Z$ , or  $||X-Y|| \leq Z$ , according to context.

$$egin{aligned} (u_n,\,u_m) &= \int_{\partial arOmega_1} u_n u_m^* ds = \int_{\partial arOmega} u_n u_m^* ds + \int_{|arowining} u_n u_m^* ds = \delta_{mn} + 2\pi arepsilon u_m(0)^* u_n(0) \ &+ arepsilon \int_0^{2\pi} [u_m(0)^* (u_n(arepsilon e^{i heta}) - u_n(0)) + (u_m(arepsilon e^{i heta})^* - u_m(0)^*) u_n(arepsilon e^{i heta})] d heta \;. \end{aligned}$$

$$|(u_n, u_m) - \delta_{mn} - 2\pi \varepsilon u_m(0)^* u_n(0)| \leq 2\pi \varepsilon^2 (||u_m||_{\infty} ||u'_n||_{\infty} + ||u'_m||_{\infty} ||u_n||_{\infty}).$$

Now the matrix  $[2\pi\varepsilon u_m(0)^*u_n(0)]$  has norm  $2\pi\varepsilon(\sum |u_m(0)|^2|u_n(0)|^2)^{1/2} = 2\pi\varepsilon K(0,0) \le 8R^2r^{-2}\gamma^{-1}\varepsilon$  by Lemma 2.1. The norm of the matrix  $[2\pi\varepsilon^2(||u_m||_{\infty}||u_n'||_{\infty}+||u_m'||_{\infty}||u_n||_{\infty})]$  is at most  $4\pi\varepsilon^2(\sum ||u_m||_{\infty}||u_n'||_{\infty})^{1/2} \le 200R^2r^{-3}\gamma^{-1}\varepsilon^2$  by (1) and (2). So (see the format (6)):

(7) 
$$A = [2\pi\varepsilon u_m(0)^*u_n(0)] \text{ with error } 200R^2r^{-3}\gamma^{-1}\varepsilon^2.$$

Also,  $||A|| \le 8R^2r^{-2}\gamma^{-1}\varepsilon + 200R^2r^{-3}\gamma^{-1}\varepsilon^2 \le 9(R/r)^2\gamma^{-1}\varepsilon \le 9(R/r)^5\gamma^{-1}\varepsilon$  since  $\varepsilon < 10^{-5}r$  and r < R. In fact the cruder bound  $||A|| \le 2500(R/r)^5\gamma^{-1}\varepsilon$  will be sufficient. Observe that, since  $\varepsilon < 10^{-5}(r/R)^5\gamma$ , we have also  $||A|| \le 1/40$ .

The mth element of B is:

$$egin{aligned} (u_{\scriptscriptstyle m},\,v_{\scriptscriptstyle 1}) &= rac{arepsilon^{1/2}}{\sqrt{(2\pi)}}rac{K(0,\,0)}{K(0,\,0)}\int_{etaarrho}rac{K(z,\,0)^*u_{\scriptscriptstyle m}ds}{z^*} \ &+ rac{arepsilon^{1/2}}{\sqrt{(2\pi)}}rac{K(0,\,0)arepsilon i}{K(0,\,0)arepsilon i}\int_{|z|=arepsilon}K(z,\,0)^*u_{\scriptscriptstyle m}(z)dz \;. \end{aligned}$$

Now the second term on the right-hand side is:

$$\frac{\varepsilon^{-1/2}}{\sqrt{(2\pi)}} \frac{\int_{|z|=\varepsilon} (K(z, 0)^* - K(0, 0)^*) u_m(z) dz}{\int_{|z|=\varepsilon} (K(z, 0)^* - K(0, 0)^*) u_m(z) dz}$$

by Cauchy's theorem, and is therefore bounded in magnitude by  $(2\pi e^{3/2}/(\sqrt{(2\pi)}K(0,0)))$   $(15R^2/r^3\gamma) ||u_m||_{\infty}$  by (4). So:

(8) 
$$B = \left[\frac{\varepsilon^{1/2}}{\sqrt{(2\pi)}} \int_{\partial \mathcal{Q}} \frac{K(z, 0)^* u_m ds}{z^*}\right]$$

$$\text{with error } \frac{2\pi \varepsilon^{3/2}}{\sqrt{(2\pi)}} \frac{15R^2}{K(0, 0)} (\sum ||u_m||_{\infty}^2)^{1/2}$$

$$\leq 66K(0, 0)^{-1} R^3 r^{-4} \gamma^{-3/2} \varepsilon^{3/2}$$

by (1). The norm of the matrix in the square brackets is at most:

$$egin{aligned} & rac{arepsilon^{1/2}}{\sqrt{(2\pi)}} \frac{|K(z,0)|^2 ds}{|X(z,0)|^2} ^{1/2} \ & \leq rac{arepsilon^{1/2}}{\sqrt{(2\pi)}} \frac{|K(z,0)|^2 ds}{K(0,0)r} \Big( \int_{\partial arrho} |K(z,0)|^2 ds \Big)^{1/2} \leq rac{arepsilon^{1/2}}{\sqrt{(2\pi)}} rac{arepsilon^{1/2}}{K(0,0)^{1/2}r} \ . \end{aligned}$$

Hence, using Lemma 2.1 and the fact that  $\varepsilon < 10^{-5} (r/R)^4 \gamma$ , (8) gives  $||B|| \le 3R r^{-2} \gamma^{1/2} \varepsilon^{1/2}$ . The cruder bounds  $||B|| \le 1/40$  and  $||B||^2 \le 2500 (R/r)^5 \gamma^{-1} \varepsilon$  will suffice. Also, using (8) and the estimates calculated in the last few lines, we have:

(9) 
$$B^{_{H}}B = \left[\frac{\varepsilon}{2\pi K(0,\,0)^{_{2}}}\int_{^{_{2}\varOmega}}\frac{K(z,\,0)u_{_{m}}^{*}ds}{z}\int_{^{_{2}\varOmega}}\frac{K(z,\,0)^{*}u_{_{n}}ds}{z^{*}}\right]$$
 with error  $20000R^{6}r^{-8}\varepsilon^{2}$ .

The elements of C are, for  $m \ge 1$ ,  $n \ge 2$ :

$$egin{aligned} (u_{\scriptscriptstyle m}, \, v_{\scriptscriptstyle n}) &= rac{arepsilon^{n-1/2}}{\sqrt{(2\pi)}} rac{K(0, \, 0)}{K(0, \, 0)} \int_{\partial arrho} rac{K(z, \, 0)^* u_{\scriptscriptstyle m} ds}{(z^*)^n} \ &+ rac{arepsilon^{-n+1/2}}{\sqrt{(2\pi)}} rac{K(0, \, 0)i}{K(0, \, 0)i} \int_{|z|=arepsilon} K(z, \, 0)^* u_{\scriptscriptstyle m}(z) z^{n-1} dz \; . \end{aligned}$$

Call the first and second terms of the above expression  $P_{mn}$  and  $Q_{mn}$  respectively. Then:

$$egin{aligned} \|P\| & \leq rac{1}{\sqrt{(2\pi)} \ K(0,\,0)} \Big(\sum\limits_{n=2}^{\infty} \int_{\partial\varOmega} rac{|\ K(z,\,0)\ |^2 ds}{|\ z\ |^{2n}} \, arepsilon^{2n-1} \Big)^{1/2} \ & \leq rac{1}{\sqrt{(2\pi)} \ K(0,\,0)} \Big(\sum\limits_{n=2}^{\infty} rac{arepsilon^{2n-1}}{r^{2n}} \, K(0,\,0) \Big)^{1/2} \ & \leq 3R r^{-3} \gamma^{1/2} arepsilon^{3/2} \leq (R/r)^5 \gamma^{-1} arepsilon \ . \end{aligned}$$

We estimate the integral in the expression for  $Q_{mn}$  as follows. Replace K(z,0) by K(z,0) minus its Taylor expansion about 0 as far as the term in  $z^{n-1}$ . By Cauchy's theorem, these added terms do not affect the integral. By Taylor's theorem, K(z,0) minus its Taylor expansion is bounded on  $|z| = \varepsilon$  by  $(\varepsilon^n/n!) ||d^n/dz^n K(z,0)||_{\infty}$ , and (4) gives an estimate for that. This procedure gives:

$$\begin{split} || \, Q \, || & \leq \frac{14\pi R^2}{r \sqrt{(2\pi)} \ K(0, \, 0) \gamma} \Big( \sum_{n=2}^{\infty} \frac{\varepsilon^{2n+1} (n \, + \, 1)^2}{(r \, - \, 2\varepsilon)^{2n+2}} \Big)^{1/2} (\sum || \, u_m \, ||_{\infty}^2)^{1/2} \\ & \leq \frac{14\pi R^2}{r \sqrt{(2\pi)} \, \gamma} \, \frac{16\pi R^2 \gamma}{r^2} \, \frac{4\varepsilon^{5/2}}{r^3} \, \frac{\sqrt{3}}{\sqrt{\gamma}} \, R \\ & \leq 6000 \, R^5 r^{-7} \gamma^{-1/2} \varepsilon^{5/2} \leq (R/r)^5 \gamma^{-1} \varepsilon \; . \end{split}$$

Hence  $||C|| \le ||P|| + ||Q|| \le 2(R/r)^5 \gamma^{-1} \varepsilon$ . Once again we shall need only  $||C|| \le 2500(R/r)^5 \gamma^{-1} \varepsilon \le 1/40$ .

It is convenient to deal with  $\begin{bmatrix} D & E^H \\ E & F \end{bmatrix}$  as a single matrix. Its (m, n)th element (see (6)) is, for  $m \ge 1$ ,  $n \ge 1$ :

$$rac{arepsilon^{m+n-1}}{2\pi K(0,0)^2} \int_{\partial \mathcal{Q}} rac{\mid K(z,0)\mid^2\!\!ds}{(z^*)^m z^n} + \Big(rac{arepsilon^{m+n-1}}{2\pi K(0,0)^2} \!\int_{|z|=arepsilon} rac{\mid K(z,0)\mid^2\!\!ds}{(z^*)^m z^n} - \delta_{mn} \Big) \,.$$

Denote by  $G_{mn}$ ,  $H_{mn}$  respectively the first term and the bracketed term of the above expression. We have:

$$||G_{mn}|| \leq rac{arepsilon^{m+n-1}}{2\pi K(0,0)^2} rac{1}{r^{m+n}} \int_{\partial arrho} |K(z,0)|^2 ds = rac{arepsilon^{m+n-1}}{2\pi K(0,0) r^{m+n}} \,.$$

Hence  $||G|| \le (1/(2\pi\varepsilon K(0,0))) \sum_{n=1}^{\infty} \varepsilon^{2n}/r^{2n} \le 9R^2r^{-4}\gamma\varepsilon \le 9(R/r)^5\gamma^{-1}\varepsilon$ . H is trickier to deal with. We have:

$$egin{align} H_{nn} &= rac{1}{2\pi K(0,\,0)^2arepsilon} \int_{|z|=arepsilon} |\,K(z,\,0)\,|^2 ds - 1 \ &= rac{1}{2\pi i K(0,\,0)^2} \int_{|z|=arepsilon} K(z,\,0) (K(z,\,0)^* - K(0,\,0)^*) z^{-1} dz \;. \end{align}$$

Lemma 2.1, (4) with k=1, and (5) now give  $|H_{nn}| \leq 800 R^4 r^{-5} \varepsilon$ . If m>n, then:

$$H_{mn} = rac{arepsilon^{n-m}}{2\pi i K(0,0)^2} \int_{|z|=arepsilon} K(z,0) K(z,0)^* z^{m-n-1} dz \; .$$

As before, we may replace the second occurrence of K(z, 0) in the integral by K(z, 0) minus its Taylor expansion, this time as far as the term in  $z^{m-n-1}$ . Then by (5), Lemma 2.1, and (4) with k=m-n:

$$egin{align} |H_{mn}| & \leq arepsilon^{m-n} 1.01 rac{16\pi R^2 \gamma}{r^2} rac{7R^2(m-n+1)}{r(r-2arepsilon)^{m-n+1} \gamma} \ & \leq 400 R^4 r^{-5} arepsilon (|m-n|+1) (1/99998)^{|m-n|-1} \end{aligned}$$

since  $\varepsilon < 10^{-5}r$ . This holds similarly for m < n. Combining the cases m = n, m > n, and m < n, we see that:

$$||H|| \le rac{arepsilon R^4}{r^5} \Big( 800 + 2 imes 400 \Big( 2 + rac{3}{99998} + rac{4}{(99998)^2} + \cdots \Big) \Big)$$
  
 $\le 2401 R^4 r^{-5} arepsilon \le 2401 (R/r)^5 \gamma^{-1} arepsilon$ .

So  $\begin{bmatrix} D & E^H \\ E & F \end{bmatrix}$  has norm at most  $||G|| + ||H|| \le 2500 (R/r)^5 \gamma^{-1} \varepsilon$ . Hence each of ||D||, ||E||,  $||F|| \le 2500 (R/r)^5 \gamma^{-1} \varepsilon \le 1/40$ .

To summarise: we have shown that:

(10) 
$$\begin{aligned} \|A\|, \|B\|^2, \|C\|, \|D\|, \|E\|, \|F\| &\leq 2500 (R/r)^5 \gamma^{-1} \varepsilon ; \\ \|A\|, \|B\|, \|C\|, \|D\|, \|E\|, \|F\| &\leq 1/40 . \end{aligned}$$

In particular we have verified that M is a bounded matrix: indeed that  $||M|| \le 3/40 < 1$ . Thus T = I + M is invertible, and Proposition 1.5 applies.

Our next step is to calculate the top left-hand block of the inverse of T. Since  $T^{-1} = I - M + M^2 - M^3 + \cdots$ , this top left-hand block is:

$$S = I$$
 $-A$ 
 $+ A^{2} + B^{H}B + C^{H}C$ 
 $- A^{3} - AB^{H}B - AC^{H}C - B^{H}BA - B^{H}DB - B^{H}E^{H}C - C^{H}CA$ 
 $- C^{H}EB - C^{H}FC$ 
 $+ \cdots$ 

The row of this expression containing products of degree  $n \ (n \ge 4)$  consists of  $3^{n-1}$  terms. Each of these terms has norm at most  $(2500)^2 (R/r)^{10} \gamma^{-2} \varepsilon^2 (1/40)^{n-4}$  by (10). Hence  $S = I - A + B^H B$  with error:

$$egin{aligned} & rac{(2500)^2 arepsilon^2 R^{10}}{\gamma^2 r^{10}} \Big(1 + 1 + rac{1}{40} + 1 + rac{1}{40} + 1 + 1 + rac{1}{40} + rac{1}{40}$$

Using (7) and (9), we have:

$$S = \left[ \delta_{mn} - 2\pi \varepsilon u_m(0)^* u_n(0) + \frac{\varepsilon}{2\pi K(0, 0)^2} \int \frac{K(z, 0) u_m^* ds}{z} \int \frac{K(z, 0)^* u_n ds}{z^*} \right]$$

with error  $200R^2r^{-3}\gamma^{-1}\varepsilon^2 + 20000R^6r^{-8}\varepsilon^2 + 3.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 \le 4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2$ . Here and subsequently all integrals are taken round  $\partial\Omega$ .

Finally we apply Proposition 1.5, which says that  $1/(2\pi\gamma_1) = \sum S_{mn}u_m(\infty)u_n(\infty)^*$  (since  $v_n(\infty) = 0$  for all n). Hence:

$$\begin{aligned} \frac{1}{2\pi\gamma_1} &= \sum |u_n(\infty)|^2 - 2\pi\varepsilon |\sum u_n(0)^* u_n(\infty)|^2 \\ &+ \frac{\varepsilon}{2\pi K(0,0)^2} \left|\sum \left(u_n(\infty) \int \frac{K(z,0) u_n^* ds}{z}\right)\right|^2 \end{aligned}$$

with error  $4.10^8 (R/r)^{10} \gamma^{-2} \varepsilon^2 \sum |u_m(\infty)|^2 = 4.10^8 (R/r)^{10} \gamma^{-2} \varepsilon^2/(2\pi\gamma)$ . Multiplying by  $2\pi\gamma$  and using the fact that  $\sum u_n(z) u_n(\zeta)^* = K(z,\zeta)$ , we have:

(11) 
$$\frac{\gamma}{\gamma_1} = 1 - 4\pi^2 \gamma \varepsilon |K(0, \infty)|^2 + \frac{\gamma \varepsilon}{K(0, 0)^2} \left| \int \frac{K(z, 0)K(z, \infty)^* ds}{z} \right|^2$$
 with error  $4.10^8 (R/r)^{10} \gamma^{-2} \varepsilon^2$ .

Now the last term simplifies. On  $\partial \Omega$ ,  $f(z)\psi(z)dz \ge 0$ , so that  $ds = (\psi(z)/|\psi(z)|) f(z)dz = (K(z, \infty)/K(z, \infty)^*) (f(z)/i) dz$ . Therefore:

$$\int \frac{K(z, 0)K(z, \infty)^*ds}{z} = \frac{1}{i} \int \frac{K(z, 0)K(z, \infty)f(z)dz}{z}$$
$$= -2\pi K(0, 0)K(0, \infty)f(0)$$

since  $K(z, 0)K(z, \infty)f(z)$  is analytic on  $\overline{\Omega}$  and vanishes at  $\infty$ . Substituting in (11), we have:

$$rac{\gamma}{\gamma_1}=1-4\pi^2\gammaarepsilon|\,K(0,\,\infty)\,|^2\{1-|\,f(0)\,|^2\}\,\,\, ext{with error}\,\,\,4.10^8(R/r)^{10}\gamma^{-2}arepsilon^2\,\,.$$

Now  $4\pi^2\gamma\varepsilon |K(0, \infty)|^2\{1-|f(0)|^2\} \le 4\pi^2\gamma\varepsilon K(0, 0)K(\infty, \infty) \le 8(R/r)^2\gamma^{-1}\varepsilon < 10^{-4}$ . Also  $4.10^8(R/r)^{10}\gamma^{-2}\varepsilon^2 \le 1/25$ . So we can invert to obtain:

$$egin{aligned} rac{\gamma_1}{\gamma} &= 1 + 4\pi^2 \gamma arepsilon |K(0,\,\infty)|^2 \{1 - |f(0)|^2\} \; ext{with error} \; 10^9 (R/r)^{10} \gamma^{-2} arepsilon^2 \; ; \ \gamma_1 &= \gamma + 4\pi^2 \gamma^2 arepsilon |K(0,\,\infty)|^2 \{1 - |f(0)|^2\} \ &= \gamma + 2\pi arepsilon |\psi(0)| \{1 - |f(0)|^2\} \; ext{with error} \; 10^9 (R/r)^{10} \gamma^{-1} arepsilon^2 \; . \end{aligned}$$

It is as well to explain the curious choice of the functions  $v_n$  in the above proof. The only essential property of  $v_n$  we used is that it vanishes at  $\infty$  and is analytic on  $\bar{\Omega}$  except for a pole at 0 near which  $v_n(z) = (2\pi)^{-1/2} \varepsilon^{n-1/2} z^{-n} + \cdots$ . The simpler choice  $v_n(z) = (2\pi)^{-1/2} \varepsilon^{n-1/2} z^{-n}$  shortens the proof but yields an error bound dependent on the length of  $\partial \Omega$ , which would have been unsuitable for the next section.

3. Extension to arbitrary compact sets. We shall now show how the above results extend to arbitrary compact sets E. In particular, we show how to define the Garabedian function of E, thus solving a problem considered in [2] and [3].

Let E be compact. We shall suppose meantime that  $\gamma(E) > 0$ . E can be expressed as the intersection of a decreasing sequence  $\{E_n\}$  in  $\mathscr{S}$ . Hence  $\psi_{E_n}$  and  $a_{E_n}$  are defined. Fix  $\zeta \in \Omega(E)$ , and choose  $n_0$  so that  $\zeta \in \Omega(E_n)$  whenever  $n > n_0$ . By Theorem 2.2 there exist  $\varepsilon_0 > 0$ , k > 0, such that  $\forall n > n_0$ ,  $\forall \varepsilon < \varepsilon_0$ :

(12) 
$$|\gamma(E_n \cup D(\zeta; \varepsilon)) - \gamma(E_n) - \varepsilon a_{E_n}(\zeta)| \leq k\varepsilon^2.$$

That is, for all  $\varepsilon < \varepsilon_0$ , the sequence  $\{\varepsilon a_{E_n}(\zeta)\}_{n>n_0}$ , considered as an element of the Banach space of bounded sequences with the supremum norm, is within a distance  $k\varepsilon^2$  of the sequence  $\{\gamma(E_n \cup D(\zeta;\varepsilon)) - \gamma(E_n)\}_{n>n_0}$ , which converges to  $\gamma(E \cup D(\zeta;\varepsilon)) - \gamma(E)$  by 1.1. Thus  $\{a_{E_n}(\zeta)\}$  is within a distance  $k\varepsilon$  of the closed subspace c of convergent sequences, for all  $\varepsilon$ , and is therefore itself in c. Call its limit  $a_E(\zeta)$ .  $a_E$  is the slope function of E. Letting  $n \to \infty$  in (12) now gives, for all  $\varepsilon < \varepsilon_0$ :

$$|\gamma(E \cup D(\zeta; \varepsilon)) - \gamma(E) - \varepsilon a_E(\zeta)| \leq k\varepsilon^2$$
.

This shows also that the limit  $a_E(\zeta)$  is independent of the choice of the sequence  $\{E_n\}$ .

Now, for each n,  $|\psi_{E_n}(\zeta)| = a_{E_n}(\zeta)/(2\pi\{1-|f_{E_n}(\zeta)|^2\})$ , and this converges pointwise in  $\Omega(E)$ . Moreover,  $\{\psi_{E_n}\}$  is a normal sequence, since, if F is a compact subset of  $\Omega(E)$ ,  $\{\psi_{E_n}\}$  is uniformly bounded on F by the remark following Lemma 2.1. It follows that for some sequence  $\lambda_n$  of points on the unit circle,  $\{\lambda_n\psi_{E_n}\}$  converges uniformly on compact subsets of  $\Omega(E)$ . In fact we may take  $\lambda_n=1$ , since  $\psi_{E_n}(\infty)=1/(2\pi i)$ . So  $\{\psi_{E_n}\}$  converges uniformly on compact sets. Call its limit,  $\{\psi_E\}$ , the  $Garabedian\ function\ of\ E$ . Hence also  $a_{E_n}(\zeta)=2\pi|\psi_{E_n}(\zeta)|\{1-|f_{E_n}(\zeta)|^2\}$  converges uniformly on compact sets (and not merely pointwise, as ascertained already).

Now suppose that  $\gamma(E)=0$ . We define  $\psi_E(\zeta)=1/(2\pi i)$ ,  $a_E(\zeta)=1$  for  $\zeta\in \Omega(E)$ . This is consistent with the relation  $a_E(\zeta)=2\pi |\psi_E(\zeta)|$   $\{1-|f_E(\zeta)|^2\}$  since  $f_E(\zeta)=0$ .  $\gamma(E\cup D(\zeta;\varepsilon))=\varepsilon$  for all  $\varepsilon>0$  by 1.3, and so the relation  $\gamma(E\cup D(\zeta;\varepsilon))=\gamma(E)+\varepsilon a_E(\zeta)+O(\varepsilon^2)$  holds trivially. If  $\{E_n\}$  is a sequence in  $\mathscr S$  decreasing to E, then  $\psi_{E_n}(\zeta)\to 1/(2\pi i)=\psi_E(\zeta)$  uniformly on compact sets by the remark following Lemma 2.1.

Finally, if E is compact, and  $\{E_n\}$  is any sequence of compact sets decreasing to E, the same working as above shows that  $\psi_{E_n} \rightarrow \psi_E$  and  $a_{E_n} \rightarrow a_E$  uniformly on compact sets.

We have therefore proved:

THEOREM 3.1. The Garabedian function  $\psi_E(\zeta)$  and the slope function  $a_E(\zeta)$  can be defined for all compact sets E, in such a way that:

- (1) The definitions coincide with the existing meanings if  $E \in \mathscr{S}$ ;
- (2) If  $\{E_n\}$  is a sequence of compact sets decreasing to E, then  $\psi_{E_n} \to \psi_E$  and  $a_{E_n} \to a_E$  uniformly on compact subsets of  $\Omega(E)$ ;
- (3)  $\gamma(E \cup D(\zeta; \varepsilon)) = \gamma(E) + \varepsilon a_E(\zeta) + O(\varepsilon^2)$  for all  $\zeta \in \Omega(E)$ , and the bound in the error term depends only on  $\gamma(E)$  and on the ratio of the greatest and least distances of points of E from  $\zeta$ ; and
  - $(4) \quad a_{\scriptscriptstyle E}(\zeta) = 2\pi |\psi(\zeta)| \{1 |f(\zeta)|^2\} \text{ for all } \zeta \in \Omega(E).$

The slope function is related to the problem of subadditivity of  $\gamma$ . If E is connected, then  $a_E(\zeta) \leq 1$ : this is a re-statement of Bieberbach's distortion theorem. Subadditivity of  $\gamma$  would obviously imply  $a_E(\zeta) \leq 1$  for all compact E.

I should like to thank Dr. A. M. Davie for his invaluable supervision.

Added in proof. N. Suita recently has independently proved the uniqueness of the Garabedian function much more simply ("On a metric induced by Analytic Capacity," Kōdai Math. Sem. Rep. 25 (1973), 215-218).

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# **Pacific Journal of Mathematics**

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Zvi Arad, $\pi$ -homogeneity and $\pi$ -closure of finite groups	1
Ivan Baggs, A connected Hausdorff space which is not contained in a maximal connected space	11
Eric Bedford, The Dirichlet problem for some overdetermined systems on the unit ball in $\mathbb{C}^n$	19
R. H. Bing, Woodrow Wilson Bledsoe and R. Daniel Mauldin, Sets generated by rectangles	27
Carlo Cecchini and Alessandro Figà-Talamanca, $Projections\ of\ uniqueness\ for\ L^p(G)$	37
Gokulananda Das and Ram N. Mohapatra, <i>The non absolute Nörlund summability of Fourier series</i>	49
Frank Rimi DeMeyer, On separable polynomials over a commutative ring	57
Richard Detmer, Sets which are tame in arcs in $E^3$	67
William Erb Dietrich, Ideals in convolution algebras on Abelian groups	75
Bryce L. Elkins, A Galois theory for linear topological rings	89
William Alan Feldman, A characterization of the topology of compact convergence on $C(X)$	109
Hillel Halkin Gershenson, A problem in compact Lie groups and framed cobordism	121
Samuel R. Gordon, Associators in simple algebras	131
Marvin J. Greenberg, Strictly local solutions of Diophantine equations	143
Jon Craig Helton, Product integrals and inverses in normed rings	155
Domingo Antonio Herrero, Inner functions under uniform topology	167
Jerry Alan Johnson, Lipschitz spaces	177
Marvin Stanford Keener, Oscillatory solutions and multi-point boundary value	
functions for certain nth-order linear ordinary differential equations	187
John Cronan Kieffer, A simple proof of the Moy-Perez generalization of the Shannon-McMillan theorem	203
Joong Ho Kim, Power invariant rings	207
Gangaram S. Ladde and V. Lakshmikantham, On flow-invariant sets	215
Roger T. Lewis, Oscillation and nonoscillation criteria for some self-adjoint even order linear differential operators	221
Jürg Thomas Marti, On the existence of support points of solid convex sets	235
John Rowlay Martin, Determining knot types from diagrams of knots	241
James Jerome Metzger, Local ideals in a topological algebra of entire functions characterized by a non-radial rate of growth	251
K. C. O'Meara, Intrinsic extensions of prime rings	257
Stanley Poreda, A note on the continuity of best polynomial approximations	271
Robert John Sacker, Asymptotic approach to periodic orbits and local prolongations of maps	273
Eric Peter Smith, The Garabedian function of an arbitrary compact set	289
Arne Stray, Pointwise bounded approximation by functions satisfying a side condition	301
John St. Clair Werth, Jr., Maximal pure subgroups of torsion complete abelian p-groups	307
Robert S. Wilson, <i>On the structure of finite rings. II</i>	317
Kari Ylinen, The multiplier algebra of a convolution measure algebra	327
, and a supplier of the suppli	