MAXIMAL PURE SUBGROUPS OF TORSION COMPLETE
ABELIAN $p$-GROUPS

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Let $N$ be the set of nonnegative integers, and let $B = \Sigma \oplus [b_i](i \in N)$ be the direct sum of cyclic groups with $0(b_i) = p^{i+1}$. Denote by $\bar{B}$ the torsion-completion of $B$. This paper is concerned with pure subgroups of the group $\bar{B}$. If $G$ is such a group, let

$$I(G) = \{i | i^{th} \text{ Ulm invariant of } G \text{ is nonzero}\}.$$

Beaumont and Pierce introduced a further invariant for $G$, namely,

$$U(G) = \{I(A) | A \text{ is a pure torsion-complete subgroup of } G\}.$$

$U(G)$ is a (boolean) ideal in $\mathcal{P}(N)$, the power set of $N$.

If $\mathcal{I}$ is an ideal in $\mathcal{P}(N)$, then the canonical example of a pure subgroup, $G$, of $\bar{B}$ with $U(G) = \mathcal{I}$ is constructed as follows:

$$G = \mathcal{G}(\mathcal{I}) = \Sigma A_I(i \in \mathcal{I})$$

where $A_I$ is the torsion-completion of $\Sigma \oplus [b_i](i \in I)$.

Beaumont and Pierce showed that if $\mathcal{P}(N)/\mathcal{I}$ has no atoms and $\mathcal{I}$ is free, then there exist maximal pure subgroups $G$ of $\bar{B}$ such that $G \supset \mathcal{G}(\mathcal{I})$ and $U(G) = \mathcal{I}$. The purpose of this paper is to give necessary and sufficient conditions for the existence of such a $G$ in the case that $\mathcal{P}(N)/\mathcal{I}$ is finite. In the process, some information is obtained about the number of nonisomorphic extensions of $\mathcal{G}(\mathcal{I})$.

I. Preliminaries. For the basic background on $p$-groups without elements of infinite height see [2] and [3]. The groups $G$ that we consider in this paper will all be pure subgroups of $\bar{B}$, where $B$ is a standard basic subgroup as above. The following definitions and facts may be found in [1].

(i) \textbf{Definition.} $I(G) = \{n | nth \text{ Ulm invariant of } G \text{ is not zero}\}$.

(ii) \textbf{Definition.} If $x \in \bar{B}$ and $x = \Sigma r_i b_i(i \in N)$, then $\delta(x) = \{i | r_i b_i \neq 0\}$.

(iii) \textbf{Proposition.} If $\mathcal{I}$ is an ideal in $\mathcal{P}(N)$, then $\mathcal{G}(\mathcal{I}) = \{x \in \bar{B} | \delta(x) \in \mathcal{I}\}$.

(iv) \textbf{Proposition.} $\mathcal{G}(\mathcal{I})$ is a pure subgroup of $\bar{B}$ and $U(\mathcal{G}(\mathcal{I})) = \mathcal{I}$.

(v) \textbf{Proposition.} If $\mathcal{I}$ contains all finite subsets of $N$ (such an ideal is called free) and is maximal in $\mathcal{P}(N)$, then $\mathcal{G}(\mathcal{I})$ is a maximal pure subgroup of $\bar{B}$.

In [1] Beaumont and Pierce give an example to show that it is not always possible to extend $\mathcal{G}(\mathcal{I})$ to a maximal pure subgroup $G$. 
with $U(G) = \mathcal{I}$, when $\mathcal{I}$ is the intersection of two maximal ideals. It turns out that this is the case which causes most of the difficulties, and the majority of the paper is devoted to showing that their example is typical of the situation where no such $G$ exists.

II. Throughout this section $\mathcal{I}$ will be the intersection of two maximal free ideals. Let $\mathcal{V}$ and $\mathcal{W}$ be distinct maximal free ideals of $\mathcal{F}(N)$ and let $\mathcal{I} = \mathcal{V} \cap \mathcal{W}$. Let $V \in \mathcal{V} - \mathcal{I}$ and let $W = \mathcal{N} - V$. Then $W \in \mathcal{W} - \mathcal{I}$ and by the maximality of $\mathcal{V}$ and $\mathcal{W}$ we have $\mathcal{V} = [V, \mathcal{I}]$ and $\mathcal{W} = [W, \mathcal{I}]$. Note that $\mathcal{F}(V) \cap \mathcal{I}$ and $\mathcal{F}(W) \cap \mathcal{I}$ are maximal ideals of $\mathcal{F}(V)$ and $\mathcal{F}(W)$ respectively, and that $\mathcal{F}(\mathcal{I}) = \mathcal{F}(\mathcal{V}) \cap \mathcal{F}(\mathcal{W})$.

Our purpose in this section is to give a necessary and sufficient condition for a group $G$ with $\mathcal{F}(\mathcal{I}) \subset G \subset \mathcal{B}$ and $G/\mathcal{F}(\mathcal{I}) \cong \mathbb{Z}_p(\infty)$ to be of the form $\mathcal{F}(\mathcal{V})$ or $\mathcal{F}(\mathcal{W})$.

II. A. Notation.

(i) Let $A_1$ be the closure in $\mathcal{B}$ of $\sum b_i(i \in V)$.

Let $A_2$ be the closure in $\mathcal{B}$ of $\sum b_i(i \in W)$.

(ii) $G_1 = \mathcal{F}(\mathcal{F}(V) \cap \mathcal{I})$.

$G_2 = \mathcal{F}(\mathcal{F}(W) \cap \mathcal{I})$.

(iii) $v_n = \Sigma p^{i-n} i b_i(i \in V$ and $i \geq n - 1)$.

$w_n = \Sigma p^{i-n} i b_i(i \in W$ and $i \geq n - 1)$.

(iv) $\mathcal{H}_1 = \{G|G = \mathcal{F} \cap \mathcal{I} + [u_n| n \in N]\}$, where $u_n - pu_{n+1} \in \mathcal{F}(\mathcal{I})$ and $u_n = v_n + t_n w_n$ with $0 \leq t_n < p^n$.

$\mathcal{H}_2 = \{G|G = \mathcal{F} \cap \mathcal{I} + [u_n| n \in N]\}$ where $u_n - pu_{n+1} \in \mathcal{F}(\mathcal{I})$ and $u_n = w_n + s_n v_n$ with $0 \leq s_n < p^n$.

The following proposition records the obvious connections between these objects.

II. B. Proposition.

(i) $A_1 \oplus A_2 = \mathcal{B}; G_1 \oplus G_2 = \mathcal{F}(\mathcal{I})$.

(ii) $A_1 = [v_n| n \in N] + G_1$.

$A_2 = [w_n| n \in N] + G_2$.

(iii) $A_i/G_i \cong \mathbb{Z}_p(\infty)$ for $i = 1, 2$.

(iv) $\mathcal{F}(\mathcal{V}) = [v_n| n \in N] + \mathcal{F}(\mathcal{I})$.

$\mathcal{F}(\mathcal{W}) = [w_n| n \in N] + \mathcal{F}(\mathcal{I})$.

(v) $G$ is a pure subgroup of $\mathcal{B}$ with $G/\mathcal{F}(\mathcal{I}) = \mathbb{Z}_p(\infty)$ iff $G \in \mathcal{H}_1 \cup \mathcal{H}_2$.

At this point we have explicitly realized $\mathcal{B}/\mathcal{F}(\mathcal{I})$ as $A_1/G_1 \oplus A_2/G_2$ and have definite sets of representatives for $A_i/G_i \cong \mathbb{Z}_p(\infty)$ and $A_2/G_2 \cong \mathbb{Z}_p(\infty)$. By I.B.v. the groups $G$ that we are interested in are obtained by taking a rank 1 summand of $A_1/G_1 \oplus A_2/G_2$ and adding its representa-
tatives to \( \mathcal{S}(\mathcal{X}) \). Any such summand \( D \) will be complementary to either \( A_i/G_i \) or \( A_j/G_j \) (or both). Now if \( D \) is complementary to \( A_j/G_j \), for example, let \( \pi_1 \) and \( \pi_2 \) be the projections into \( A_i/G_i \) and \( A_j/G_j \), respectively, with respect to the decomposition \( A_i/G_i \oplus A_j/G_j \). Then, of course, for \( d \in D \) we have \( d = \pi_1(d) + \pi_2(d) \) and \( \phi \) defined by \( \phi(\pi_1(d)) = \pi_2(d) \) is an element of \( \text{Hom}(A_i/G_i, A_j/G_j) \). In fact, there is a one-to-one correspondence between \( H = \text{Hom}(A_i/G_i, A_j/G_j) \) and \( \mathcal{H}_1 \) and \( H_2 = \text{Hom}(A_j/G_j, A_i/G_i) \) and \( \mathcal{H}_2 \). The following definition and proposition set forth the precise situation.

II. C. The Correspondence.
(i) Let \( \phi \in H_i \), then \( \phi(v_n + G_i) = t_n w_n + G_j \). Let \( v_n = v_n + t_n w_n \) and define \( G[\phi] = \mathcal{S}(\mathcal{X}) + \{ [w_n | n \in N]\} \).
(ii) If \( G \in \mathcal{H}_i \) with \( G = \mathcal{S}(\mathcal{X}) + \{ [v_n + t_n w_n | n \in N]\} \), then define \( \phi[G] \in H_i \) by \( \phi[G](v_n + G_i) = t_n w_n + G_j \).

II. D. Proposition. For \( \phi \in H_i \) and \( G \in \mathcal{H}_i \)
(i) \( G[\phi] \) is a uniquely determined element of \( \mathcal{H}_i \).
(ii) \( \phi[G] \) is a uniquely determined element of \( H_i \).
(iii) \( G[\phi[G]] = G \).
(iv) \( \phi[G[\phi]] = \phi \).

By interchanging the roles of the \( w_n \) and \( v_n \) we get a similar one-to-one correspondence between \( H_2 \) and \( \mathcal{H}_2 \). In fact, \( H_i \) and \( H_2 \) coordinatize \( \mathcal{H}_i \) and \( \mathcal{H}_2 \) with some overlap as the following proposition makes clear.

II. E. Proposition. If \( \phi \) and \( \psi \) are distinct elements of \( H_i \cup H_2 \), then \( G[\phi] = G[\psi] \) iff \( \phi \) and \( \psi \) are isomorphisms with \( \phi = \psi^{-1} \).

Proof. This is the case where the summand defining \( G[\psi] \) is complementary to both \( A_i/G_i \) and \( A_j/G_j \).

We are now ready for the fundamental definition and theorem of this section.

II. F. Definition. Let \( S_i, S_j, T_i, T_j \) be abelian groups and let \( \phi \in \text{Hom}(S_i/S_j, T_i/T_j) \). We say that \( \phi \) is liftable if there is a \( \Phi \in \text{Hom}(S_i, T_i) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
S_i & \xrightarrow{\phi} & T_i \\
\downarrow & & \downarrow \\
S_i/S_j & \xrightarrow{\phi} & T_i/T_j
\end{array}
\]

II. G. Theorem. Let \( G \in \mathcal{H}_i \) with \( G = G[\phi] \) for \( \phi \in H_i \). Then
Proof. Clearly \( G = \mathfrak{G}(\mathcal{Y}) \) if and only if \( \phi \) is liftable. A similar theorem holds for 
\( G \in \mathcal{H}_2 \).

Proof. Clearly \( G = \mathfrak{G}(\mathcal{Y}) \) if and only if there is a pure torsion-
complete subgroup \( A \) of \( G \) with \( I(A) = V \).

(i) If \( \phi \) is liftable, then \( \Phi \in \text{Hom}(A, A) \) by definition of liftable.
Now \( \Phi \) can be thought of as an endomorphism of \( \bar{B} \) by taking \( \Phi(A_2) = 0 \).
Let \( A = (1 + \Phi)(A_1) \). Then \( A \) is a pure torsion-complete with \( I(V) = V \).
If \( x \in A_2 \), then \( x = g + rv_n \), where \( g \in G_1 \) and \( 0 \leq r < p^n \) for some \( n \). Consequently, \( (1 + \Phi)(x) \in G \).

(ii) Assume that there exists a pure torsion-complete \( A \) with 
\( A \subseteq G \) and \( I(A) = V \). Since \( U(A) \cup U(G_2) = \mathfrak{S}(V) \cap (W \cap \mathcal{F}) \) is empty, we have \( A + G_2 = G \) and the sum is direct. We claim that, in fact, \( A \subseteq G \). To see this note that \( U(A \oplus G_2) = \mathfrak{S}(V) + (W \cap \mathcal{F}) \) so \( U(A \oplus G_2) \) is free. Hence any basic subgroup of \( A \oplus G_2 \) will be a basic subgroup of \( \bar{B} \).

One such basic subgroup is \( B' = B_1 \oplus B_2 \) where \( B_1 \) is a basic subgroup of \( A \) and \( B_2 \subseteq \bar{B} \cap G_2 \). One sees easily that with respect to \( B' \), \( A \oplus G_2 \) is of the form \( \mathfrak{S}(\mathcal{F}) \). Hence \( \bar{B}/(A \oplus G_2) = Z_\mathcal{F}(\infty) \) by 4.2 in [1]. And since \( A \oplus G_2 \in G \), \( \bar{B}/G = Z_\mathcal{F}(\infty) \), and both \( G \) and \( A \oplus G_2 \) are pure in \( \bar{B} \), we have \( A \oplus G_2 = G \). Now \( \bar{B} = A \oplus A_1 \) since \( I(A \oplus A_1) = N \). Let \( \Phi \) be \(- (\pi|A_1) \) where \( \pi \) is the projection of \( \bar{B} \) on \( A_2 \) associated with this decomposition of \( B \). To show that \( \Phi \) is a lifting of \( \phi \) we must show the following two things:

(a) \( \Phi(G) \subseteq G_2 \). This is clear because \( G = A \oplus G_2 \) implies \( (A \oplus G_2) \cap A_1 \subseteq \mathfrak{S}(\mathcal{F}) \subseteq A_i = G_i \), and \( \Phi \) maps into \( A_i \) which is the closure in \( \bar{B} \) of \( G_2 \).

(b) \( \Phi(v_n) \equiv t_n w_n \) mod \( G_2 \) where \( G = \mathfrak{G}(\mathcal{F}) + \{v_n + t_n w_n | n \in N\} \).
Because \( v_n + \Phi(v_n) \in A \subseteq G \), it follows that \( (v_n + \Phi(v_n)) - (v_n + t_n w_n) = \Phi(v_n) - t_n w_n \in G \).
Therefore, since \( \delta(\Phi(v_n)) \subseteq W \), \( \Phi(v_n) - t_n w_n \in G \cap A_2 = G_2 \); that is, \( \Phi(v_n) \equiv t_n w_n \) mod \( G_2 \).

II. G. Remark. Note that it is not possible for \( G = G[\phi] \in \mathcal{H}_1 \) to be isomorphic to \( \mathfrak{S}(\mathcal{F}) \) if \( \text{Ker} \phi > 0 \). This is the case because \( G \cong \mathfrak{S}(\mathcal{F}) \) implies that there is a pure \( A \subseteq \mathfrak{S}(\mathcal{F}) \) with \( A \subseteq A_2 \) and \( \text{Ker} \phi > 0 \) implies that \( A_1[2] \subseteq G \) which would give \( G[2] \subseteq A_1[2] + A[2] = \bar{B}[2] \). The purity of \( G \) would now give \( G = \bar{B} \), a contradiction. If \( \text{Ker} \phi = 0 \), then we might have \( G \cong \mathfrak{S}(\mathcal{F}) \), this will occur if \( \phi^{-1} \in H_2 \) is liftable. Therefore, this theorem together with the corre-
sponding theorem for \( G \in \mathcal{H}_2 \) give a necessary and sufficient condition
for \( G \) to be isomorphic to \( \mathfrak{S}(\mathcal{F}) \) or \( \mathfrak{S}(\mathcal{F}) \).
III. In this section we state the main theorems of the paper. The notation remains the same as that of II.

III. A. DEFINITION. For \( \phi \in H_1 \cup H_2 \) let \( K(\phi) = n \) if \( |\text{Ker} \ \phi| = p^n \).

III. B. PROPOSITION. Let \( \phi, \psi \in H_1 \). If \( K(\phi) = K(\psi) \), then \( G[\phi] \) is isomorphic to \( G[\psi] \) and similarly for \( H_2 \).

Proof. \( G[\phi] = \mathcal{F}(\mathcal{F}) + \{[v_n + t_n w_n | n \in N] \} \) and \( G[\psi] = \mathcal{F}(\mathcal{F}) + \{[v_n + s_n w_n | n \in N] \} \). Let \( m = K(\phi) = K(\psi) \). Then \( t_n = p^n t_n' \mod p^m \) and \( s_n = p^n s_n' \mod p^m \) for \( (t_n', p) = 1 \) and \( (s_n', p) = 1 \) for \( n > m \). Let \( r_n \) be such that \( r_n t_n' = s_n' \mod p^m \). Define \( \alpha \) by:

\[
\begin{align*}
\alpha(b_k) &= b_k \quad \text{for } k \in V \cup \{ h \in W | h < m - 1 \} \\
\alpha(b_h) &= r_{h+1} b_h \quad \text{for } h \in W \text{ and } h \geq m - 1 .
\end{align*}
\]

III. C. DEFINITION.

(i) \( n_\mathcal{F}(\mathcal{F}, \mathcal{F}) = \min \{ K(\phi) | \phi \in H_1, U(G[\phi]) = \mathcal{F} \} \) if this exists, and \( \infty \) otherwise.

(ii) \( n_\mathcal{F}(\mathcal{F}, \mathcal{F}) = \min \{ K(\phi) | \phi \in H_2, U(G[\phi]) = \mathcal{F} \} \) if this exists, and \( \infty \) otherwise.

III. D. PROPOSITION. Let \( \phi \in H_1 \). Then \( \phi \) is liftable if \( \infty > K(\phi) \geq n_\mathcal{F}(\mathcal{F}, \mathcal{F}) \).

Proof. Let \( a = n_\mathcal{F}(\mathcal{F}, \mathcal{F}) \). By III. B. we have \( G[\phi] \cong \mathcal{F}(\mathcal{F}) \) for every \( \phi \in H_1 \) with \( K(\phi) = a \). By II. G. every such \( \phi \) is liftable. If \( K(\phi) > a \), then there is a \( \psi \in H_1 \) with \( p^{K(\phi) - a} \psi = \phi \) and \( K(\psi) = a \). Then \( p^{K(\phi) - a} \psi = \Phi \) is a lift of \( \phi \).

III. E. DEFINITION. If \( I \subset N \) and \( n \) any integer, then we write \( I - n \) for \( \{ i - n | i \in I \} \cap N \). If \( \mathcal{F} \) is an ideal of \( \mathcal{P}(N) \), then \( \mathcal{F}^n = \{ I - n | I \in \mathcal{F} \} \).

III. F. PROPOSITION. Let \( \mathcal{F} \) be a maximal free ideal of \( \mathcal{P}(N) \). Then

(i) \( \mathcal{F}^n \) is a maximal free ideal of \( \mathcal{P}(N) \).
(ii) If \( n \neq m \), then \( \mathcal{F}^n \neq \mathcal{F}^m \).
(iii) \( n_\mathcal{F}(\mathcal{F}, \mathcal{F}^m) = m; n_\mathcal{F}(\mathcal{F}, \mathcal{F}^n) = 0 \).

Proof. (i) Clearly \( \mathcal{F}^n \) is free. If \( V \subseteq N \) and \( V \in \mathcal{F}^n \), then \( V + n \in \mathcal{F} \). Since \( N - (V + n \cup (N - V) + n) \) is finite we have \( (N - V) + n \in \mathcal{F} \) by maximality of \( \mathcal{F} \). Therefore, \( N - V \in \mathcal{F}^n \) by definition and \( \mathcal{F}^n \) is maximal.
(ii) If \( I \in \mathcal{Y} \); then \( I - m \in \mathcal{Y}^m \) and \( I - n \in \mathcal{Y}^n \). Consequently, if \( \mathcal{Y}^m = \mathcal{Y}^n \), then \( I - m \cap I - n \in \mathcal{Y}^m \) where \( I - m \cap I - n = \{ k - m \mid \text{there is a } k' \in I \text{ with } k - m = k' - n \} = \{ k - m \mid \text{there exists } k' \in I \text{ with } k - k' = m - n \} \). However, since \( \mathcal{Y} \) is free, there exist \( I \)'s such that \( I \in \mathcal{Y} \) and having the property that if \( k, k' \in I \) with \( k \neq k' \), then \( |k - k'| > m - n \). For such an \( I \) we actually have \( I - m \) and \( I - n \) disjoint.

(iii) Let \( V \equiv N \) be such that \( V \in \mathcal{Y} \) and \( V \cap (V - m) \) is empty. Let \( W = (V - m) \cup (N - V), \mathcal{W} = \mathcal{Y}^m \), and use the notation of § 2.

Let \( \phi \in H_2 \) with \( \phi(w_n + G_1) = v_n + G_1 \). A computation shows that the following \( \phi \) is a lift of \( \phi \):

\[ \Phi(b_i) = p^m b_{i+m} \text{ for } j \in V - m \text{ and } \Phi(b_j) = 0 \text{ for } j \in W - (V - m). \]

For this \( \Phi \) we have \( K(\phi) = 0 \), so \( n_2(\mathcal{Y}, \mathcal{Y}^m) = 0 \).

Let \( \phi \in H_1 \) with \( \phi(v_n + G_1) = p^m w_n + G_2 \). Define a lifting \( \Phi \) of \( \phi \) as follows.

\[ \Phi(b_i) = b_{i-m} \text{ for } i \in V \text{ and } i \geq m \]
\[ \Phi(b_i) = 0 \text{ for } i \in V \text{ and } i < m. \]

Once again, a straightforward check shows that \( \Phi \) is a lift of \( \phi \).

Hence, \( n_i(\mathcal{Y}, \mathcal{Y}^m) \leq m \). By III. G. (ii) below, if \( n = n_i(\mathcal{Y}, \mathcal{Y}^m) < m \), we have \( W = \mathcal{Y}^m = \mathcal{Y}^n \) in contradiction to III. F. (ii).

We now show that if \( \mathcal{S}(\mathcal{P}) \) possesses at least one extension of the form \( \mathcal{S}(\mathcal{P}) \) and at least one of the form \( \mathcal{S}(\mathcal{W}) \) (other than the ones corresponding to the zero homomorphisms in \( H_1 \) and \( H_2 \)), then \( \mathcal{W} = \mathcal{Y}^m \) for some \( n \), or vice-versa.

III. G. THEOREM. Let \( \mathcal{Y} \) and \( \mathcal{W} \) be maximal free ideals. If \( n_1(\mathcal{Y}, \mathcal{W}) < \infty \) and \( n_2(\mathcal{Y}, \mathcal{W}) < \infty \), then

(i) \( n_i(\mathcal{Y}, \mathcal{W}) = 0 \) for one of \( i = 1 \) or \( i = 2 \).

(ii) If \( n_i(\mathcal{Y}, \mathcal{W}) = 0 \), then \( \mathcal{W} = \mathcal{Y}^m \), where \( n = n_i(\mathcal{Y}, \mathcal{W}) \).

Proof. Deferred to §IV.

III. H. COROLLARY. If \( \mathcal{Y} \) and \( \mathcal{W} \) are maximal free ideals with \( \mathcal{S} = \mathcal{Y} \cap \mathcal{W} \), then there does not exist a \( G \) with \( \mathcal{S}(\mathcal{P}) \subseteq G, G|\mathcal{S}(\mathcal{P}) \cong Z_n(\infty) \), and \( U(G) = \mathcal{S} \) if and only if \( \mathcal{W} = \mathcal{Y}^1 \) or \( \mathcal{W} = \mathcal{W}^1 \).

Proof. (i) If \( \mathcal{W} = \mathcal{Y}^1 \), then \( n_i(\mathcal{Y}, \mathcal{W}) = 0 \) and \( n_i(\mathcal{Y}, \mathcal{W}) = 1 \) by III. F., so \( G \in \mathcal{H}_1 \), implies \( G \cong \mathcal{S}(\mathcal{Y}) \) and \( G \in \mathcal{H}_2 - \mathcal{H}_1 \) implies \( G \cong \mathcal{S}(\mathcal{W}) \).

(ii) If the \( G \) mentioned does not exist, then for every \( G \in \mathcal{H}_1 \cap \mathcal{H}_2 \) we have either \( G \cong \mathcal{S}(\mathcal{Y}) \) or \( G \cong \mathcal{S}(\mathcal{W}) \). Hence, one of the \( n_i(\mathcal{Y}, \mathcal{W}) \) is zero and the other is 1. By III. G. we have either \( \mathcal{Y} = \mathcal{W}^1 \) or \( \mathcal{W} = \mathcal{Y}^1 \).

III. I. THEOREM. If \( \mathcal{Y}, \cdots, \mathcal{Y}_n \) are distinct maximal free ideals
and \( \mathcal{F} = \bigcap \mathcal{V}_i(1 \leq i \leq n) \), then there is a \( G \), pure in \( \bar{B} \), with \( \mathcal{D}(\mathcal{F}) \subset G \), \( U(G) = \mathcal{F} \) and \( \bar{B}/G \cong Z_p(\infty) \) if and only if there is no pair \( i, j \) with \( \mathcal{V}_j = \mathcal{V}_i^* \).

**Proof.** As in the case of \( n = 2 \) we may choose \( I_i \) such that \( I_i \in \mathcal{Y}_i \), \( N = \bigcup I_i(1 \leq i \leq n) \) with \( I_i \) and \( I_j \) disjoint for \( i \neq j \). Define \( A_i \) as the torsion-completion of \( \sum \oplus [b_j](j \in I_i) \) and \( G_i \) as \( \mathcal{D}(\mathcal{P}(I_i) \cap \mathcal{F}) \). For each \( k \in N \) and each \( 1 \leq i \leq n \) let \( v_{i,k} = \sum p^{-j+k} b_j(j \in I_i) \). Then \( \{v_{i,k} \mid k \in N\} \) is a canonical set of generators for \( A_i/G_i \cong Z_p(\infty) \). Let \( H_{ij} = \text{Hom}(A_i/G_i, A_j/G_j) \) and \( \mathcal{H}_{ij} = \{ G \mid G \text{ is pure in } A_i \oplus A_j \text{ with } (A_i \oplus A_j)/G \cong Z_p(\infty) \text{ and } G \supseteq G_i \oplus G_j \} \).

Since our earlier theorems could have been stated and proved for a standard subbasic we know that \( H_{ij} \) coordinizes \( \mathcal{Y}_i \), that if \( G \unlhd B \) then \( G \cong A_i \oplus G_j \) if and only if the associated \( \phi \in H_{ij} \) is liftable, and that Theorem III. G. holds.

We can realize \( \bar{B}/\mathcal{D}(\mathcal{F}) = D \) as \( \Sigma A_i/G_i(1 \leq i \leq n) \). Observe that if \( G \) is a pure subgroup of \( B \) with \( B/G \cong Z_p(\infty) \) such that \( G \supseteq \mathcal{F}(\mathcal{S}) \), then \( G \) is obtained by taking a rank \( n-1 \) summand \( Z_p \) and adding a set of representatives for \( D \mod G \) to \( gr(\mathcal{S}) \). Since \( D \mod G \) is of rank \( n-1 \) there must be at least one summand of the form \( A_i/G_i \) if \( D \mod G \).

For \( j \neq i \) let \( \phi_{j,i} = -\pi_{j,i} \), where \( \pi_{j,i} \) is the projection of \( A_j/G_j \) into \( A_i/G_i \) associated with this decomposition. Then \( D = \Sigma \oplus Z_j(\mathcal{I} = 1, \cdots, n \text{ and } j \neq i) \) where a complete set of generators for \( Z_j \) is \( \{v_{j,k} \mid k \in N\} \). Alternatively if \( S_{j,i} \in \mathcal{H}_{j,i} \) is the group associated with \( \phi_{j,i} \in H_{j,i} \), then \( G \) is the group generated by \( \{S_{j,i}\} \).

In fact, \( G \) will contain other groups of the form \( S_{j,h} \in \mathcal{H}_{j,h} \). If \( i \neq j \neq h \neq i \) and \( K(\phi_{j,i}) \supseteq K(\phi_{h,i}) \), then let \( \phi_{j,h} \in H_{j,h} \) be the map defined by \( \phi_{j,h}(v_{i,k}) = r_{i,k}v_{j,k} \) if \( \phi_{j,i}(v_{j,k}) = r_{i,k}\phi_{h,i}(v_{h,k}) \). If \( S_{j,h} \in \mathcal{H}_{j,h} \) is the group associated with \( \phi_{j,h} \), then we have \( S_{j,h} \subset G \). It follows that for every pair \( j,h \) with \( 1 \leq j, h \leq n \), \( G \) contains an element of either \( \mathcal{H}_{j,h} \) or \( \mathcal{H}_{h,j} \). Consequently, in view of III. H., if \( U(G) = \mathcal{F} \), then we cannot have \( \mathcal{Y}_i = \mathcal{Y}_j^* \) for any pair \( i, j \).

Suppose now that we do not have \( \mathcal{Y}_i = \mathcal{Y}_j^* \) for any pair \( i, j \). Then for every pair \( i, j \) either \( n_i(\mathcal{Y}_i, \mathcal{Y}_j) \geq 2 \) or \( n_j(\mathcal{Y}_i, \mathcal{Y}_j) \geq 2 \). A simple combinatorial argument shows we may assume that \( n_i(\mathcal{Y}_i, \mathcal{Y}_j) \geq 2 \) if \( j < i \). For \( i > 1 \) choose \( \phi_{i,i} \in H_{i,i} \) with \( K(\phi_{i,i}) = i - 1 \) and let \( S_{i,i} \in \mathcal{H}_{i,i} \) be the group associated with \( \phi_{i,i} \).

If \( G \) is generated by \( \{S_{i,j} \mid i = 2, \cdots, n\} \), then \( G \) is pure, \( \bar{B}/G \cong Z_p(\infty) \) and \( G \supseteq \mathcal{D}(\mathcal{F}) \). We claim that \( U(G) = \mathcal{F} \). If not, then \( I_i \in U(G) \) for some \( i \). Note that \( I_i \neq I_i \) since this would mean that there existed a pure torsion-complete \( A \) with \( A \subset G \) and \( I(A) = I_i \). However, \( A \oplus A_i \oplus \cdots \oplus A_n = \bar{B} \) and \( (A_i \oplus \cdots \oplus A_n)[p] \subset G \) by construction, so \( A \subset G \) would imply \( G[p] = \bar{B}[p] \). Since \( G \) is pure this would give
G = \bar{B}, a contradiction.

As noted above, for every pair k, j, G contains a subgroup of \( \mathcal{H}_{k,j} \) or \( \mathcal{H}_{j,k} \). Because of our assumed ordering of \( \mathcal{H}_1, \ldots, \mathcal{H}_n \), we know that for \( j < k \), \( G \supset S_{k,j} \in \mathcal{H}_{k,j} \). Let T be the group generated by \( \{S_{k,j} \mid k \neq i \neq j\} \). Then clearly \( G = T \oplus S_{k,j} \). On the other hand, \( U(T), U(A), \) and \( \bar{U}(G) \) are pairwise disjoint, so \( T \oplus A \oplus G \) is pure and since
\[
\frac{A_1 \oplus \cdots \oplus A_{i-1} \oplus A_{i+1} \oplus \cdots \oplus A_n}{(T \oplus G)} \cong \mathbb{Z}_p(\infty)
\]
we have \( G = T \oplus A \oplus G \). Hence \( S_{i,j} \cong A \oplus G \). But this contradicts our choice of \( S_{i,j} \). Hence \( U(G) = \mathcal{I} \).

IV. In this section we provide the proof of Theorem III. G.

For simplicity, the following remarks and proposition will be stated for \( \phi \in H \). The case of \( \phi \in H_2 \) is exactly the same. The notation is that of II.

Let \( \phi \in H \) and assume that \( \Phi \) is a lift of \( \phi \), with \( \Phi(b_k) = \Sigma m_k b_k(h \in W) \) for each \( k \in V \). If \( r \) is any integer, then we can write \( \Phi = \Phi_1 + \Phi_2, \) where \( \Phi_1(b_k) = \Sigma m_k b_k(h \in W \text{ and } h < k + r) \) and \( \Phi_2(b_k) = \Sigma m_k b_k(h \in W \text{ and } h \geq k + r) \) with the understanding that either of these sums is zero if its index set is empty. Clearly \( \Phi_1, \Phi_2 \) are elements of Horn \( (A, A_2) \).

IV. A. LEMMA. \( \Phi_1, (G) \subseteq G_2 \) and \( \Phi_2, (G) \subseteq G_2. \)

Proof. Since \( \Phi(G_1) \subseteq G_2 \) we need only show that \( \Phi_1, (G) \subseteq G_2. \) Let \( x \in G_1 \) and assume \( \Phi_1, (x) \in G_2. \) Then for some \( J \in \mathcal{J}, x = \Sigma m_k b_k(k \in J) \) with \( m_k b_k \neq 0. \) If \( J_s = \{k \in J \mid 0(m_k b_k) = p^{s+1}\} \) and \( x_s = \Sigma m_k b_k(k \in J_s), \) then \( x = \Sigma x_s(0 \leq s \leq n - 1) \text{ where } 0(x) = p^n. \) Since \( \Phi_1, (x) \in G_2, \) there exists an \( s \) such that \( \Phi_1, (x_s) \in G_1. \) Hence we may assume that \( x = x_s \) and \( J = J_s. \)

Since \( 0(m_k b_k) = p^{s+1} \) for \( k \in J, \) we may write \( m_k b_k = p^{k-s} l_k b_k \) where \( (l_k, p) = 1. \) It follows that

\[
\Phi_1, (m_k b_k) = \Sigma p^{k-s} l_k b_k(k - s - 1 < h < k + r)
\]

because \( h \leq k - s - 1 \) implies \( p^{k-s} l_k b_k = 0. \)

For \( 0 \leq t < s + r + 1 \) let \( K_t = \{t + n(s + r + 1) \mid n \in N\} \cap J. \) Since the \( K_t \) are pairwise disjoint and \( J = \bigcup K_t(0 \leq t < s + r + 1) \) we have \( x = \Sigma x_t(0 \leq t < s + r + 1) \text{ where } x_t = \Sigma m_k b_k(k \in K_t). \) It follows that there exists a \( t \) with \( \Phi_1, (x_t) \in G_2. \) Hence, we may assume that \( x = x_t \) and \( J = K_t. \) As a consequence, if \( j \in J \) and \( k \in J \) with \( j \neq k, \) then \( j \) and \( k \) differ by at least \( s + r + 1 \) in absolute value and by (1) above \( \delta(\Phi_1, (m_k b_k)) \) and \( \delta(\Phi_1, (m_k b_k)) \) are disjoint.

We now know that the \( \delta(\Phi_1, (m_k b_k)) \) are pairwise disjoint for \( k \in J \) and that their cardinality is less than \( s + r + 1. \) Hence, \( \delta(\Phi_1, (x)) = \)
\[ \bigcup \delta(\Phi_{i,r}(m_kb_k))(k \in J) \text{ and, since } \Phi_{i,r}(x) \in G_z, \delta(\Phi_{i,r}(x)) \in \mathcal{P}(W) \cap \mathcal{F}. \]

Therefore, it is possible to choose one integer \( t_k \) from each \( \delta(\Phi_{i,r}(m_kb_k)) \) such that \( \{t_k | k \in J\} \in \mathcal{P}(W) \cap \mathcal{F} \). For \( k \in J \) we define \( n_k b_k \) inductively as follows:

(i) \( n_k b_k = m_k b_k \) if \( k \) is the least element of \( \mathcal{F} \).

(ii) \( n_k b_k = 0 \) if \( w_k \in \delta(\Phi(\Sigma n_j b_j)) \) for \( j < k \) and \( j \in J \).

\[ n_k b_k = m_k b_k \text{ if } w_k \in \delta(\Phi(\Sigma n_j b_j)) \text{ for } j < k \text{ and } j \in J. \]

Let

\[ y = \Sigma n_k b_k (k \in J). \]

Then \( \delta(y) \subset J \) so \( y \in G_z \). On the other hand, \( \delta(\Phi(y)) = \{w_k | k \in J\} \) since \( i \in \delta(\Phi(m_kb_k)) \) implies \( i > k - s - 1 \) and \( w_i \leq k - s - 1 \) for \( j < k \) and \( j \in J \). Hence \( \Phi(y) \in G_z \) and this is a contradiction to the assumption that \( \Phi \) is a lift of \( \phi \).

**IV. B. Proof of III. G.** Let \( n = n_z(\mathcal{V}, \mathcal{V} \cap \mathcal{F}) \) and \( m = n_z(\mathcal{V}, \mathcal{V} \cap \mathcal{F}) \). Choose \( \phi \in H_1 \) with \( (v_k + G_z) = p^s \psi_k + G_z \) and \( \psi \in H_2 \) with \( \psi(v_k + G_z) = p^s \psi_k + G_z \). Let \( \phi \) and \( \psi \) be the lifts of \( \phi \) and \( \psi \). In the notation of Lemma IV. A., let \( \phi = \Phi_{i_1} + \Phi_{i_2} \) and \( \psi = \Phi_{i_1} + \Phi_{i_2} \).

By IV. A., \( \Phi_{i_1} \) and \( \Phi_{i_2} \) induce maps \( \phi_2 \in H_2 \) and \( \psi_2 \in H_2 \). If \( (\psi_2 \phi_2)(v_k + G_z) = r_k \psi_k + G_z \), then \( \phi_2(b_k) = r_{k+1} b_k \) for \( k \in V \) is a lift of \( (\psi_2 \phi_2) \) since \( b_k = v_{k+1} - p^n \psi_{k+2} \) for \( k \in V \) by definition.

We claim that \( \psi \phi_2 = 0 \) or \( \phi_2 = 0 \) or, equivalently, that \( \mathcal{V}_{i_1}(A_i) \subset G_z \) or \( \Phi_{i_2}(A_i) \subset G_z \). To show this we need only prove that \( \psi_2 \phi_2 = 0 \) since the images of \( \psi_2 \) and \( \phi_2 \) must be either \( Z \) or \( 0 \). If \( \psi_2 \phi_2 \neq 0 \), then \( \{k | r_{k+1} b_k \neq 0\} \in \mathcal{P}(V) \cap \mathcal{F} \) since \( \theta \) is a lift of \( \psi_2 \phi_2 \). If \( j \in \delta(\psi_2 \phi_2), \( \theta \) implies \( h > k \) and \( j \in \delta(\psi_2 \phi_2) \) implies \( j > h \) by definition of \( \psi_2 \phi_2 \) and \( \psi_2 \phi_2 \). Hence \( (\theta - \psi_2 \phi_2)(b_k) = r_{k+1} b_k + y_k \) where \( j \in \delta(y_k) \) implies \( j > k \). Since \( \{k | r_{k+1} b_k \neq 0\} \in \mathcal{P}(V) \cap \mathcal{F} \), \( \psi_2 \phi_2 = 0 \).

Assume that \( \phi_2 = 0 \). It follows that \( \Phi_{i_1} \) is a lift of \( \phi \). Let \( \theta_i = \Phi_{i_1} - t_i \Phi_{i_1} \) for \( 0 \leq t \leq n - 1 \) and \( \theta_n = \Phi_{i_1} - 1 \). Then \( \theta_i(G_z) \subset G_z \) since \( \Phi_{i_1} \) has this property for every \( r \). Therefore, \( \theta_i \) induces \( \theta_i \in H_t \) and, since \( \Phi_{i_1} = \theta_i(0 \leq t \leq n) \) we have \( \phi = \theta_n(0 \leq t \leq n) \).

If \( \theta_i \neq 0 \), then there is a \( k \) with \( \theta_i(v_k) \equiv s \psi_k \psi_k \equiv 0 \) mod \( G_z \). Since \( \theta_i(v_k) = \theta_i(p^{i+1} m_{j-1} b_{j-1} (j \in V) \), we have

\[ V' = \{j \in V | p^{i+1} m_{j-1} b_{j-1} \neq 0 \} \in \mathcal{P}(V) \cap \mathcal{F} \]

and \( W' = V - t \in \mathcal{P}(W) \cap \mathcal{F} \). However, since \( \theta_i(G_z) \subset G_z \), it follows that for any subset \( K \) of \( V' \), \( \Sigma p^{i+1} b_j (j \in K) \) is an element of \( G_z \) if and only if \( \Sigma p^{i+1} m_{j-1} b_{j-1} (j \in K) \) is an element of \( G_z \). Equivalently,
$K \in \mathcal{P}(V') \cap \mathcal{I}$ if and only if $K - t \in \mathcal{P}(W') \cap \mathcal{I}$. Since $V' \subset V$, $W' \subset W$, $V' \in \mathcal{I}$, and $W' \in \mathcal{I}$ we clearly have $W = \mathcal{V}$. By III. F. (iii) we have $n = n_i(\mathcal{V}, \mathcal{W}) \leq t \leq n$. It follows that $\theta_t = 0$ for $t < n$ and, since $\phi \neq 0$, $\theta_n = \phi$ which implies $W = \mathcal{V}^n$.

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