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In this paper the structure theory of convolution measure algebras due to J. L. Taylor is used in studying the multiplier algebra $M(A)$ of a commutative semi-simple convolution measure algebra A . A criterion is given for the embeddability of $M(A)$ in the measure algebra $M(S)$ on the structure semigroup S of A , and the relationship between the structure semigroups of A and $M(A)$ is investigated in case $M(A)$ is also a convolution measure algebra and S has an identity.

1. Introduction. A convolution measure algebra A is a complex L -space with a multiplication which gives A the structure of a Banach algebra and satisfies certain additional requirements. For precise definitions and the basic theory of convolution measure algebras we refer to J. L. Taylor's paper [11]. A central role in Taylor's theory is played by the structure semigroup S of a commutative convolution measure algebra A . The maximal regular ideal space of A may be identified with the set of semicharacters of the compact commutative topological semigroup S , and some properties of A are reflected in those of S .

For any (complex) commutative Banach algebra A , let $\mathcal{A}(A)$ denote the spectrum of A , that is, the space of nonzero multiplicative linear functionals on A , equipped as usual with the relative weak* topology. If A is in addition semisimple, then we denote by A^m the space of all complex-valued functions on $\mathcal{A}(A)$ that keep the space \hat{A} of the Gelfand transforms \hat{x} of the elements x of A invariant by pointwise multiplication, i.e., $A^m = \{f: \mathcal{A}(A) \rightarrow \mathbb{C} \mid f\hat{x} \in \hat{A} \text{ for all } x \in A\}$. It can be easily shown that each $f \in A^m$ determines a unique bounded linear operator $T_f: A \rightarrow A$ satisfying $\widehat{T_f x} = f\hat{x}$, $x \in A$. Then $M(A) = \{T_f \mid f \in A^m\}$ is a Banach algebra under the uniform operator norm, called the *multiplier algebra* of A . For the general theory of multiplier algebras one may consult e.g. Larsen's book [5].

In this paper we study the multiplier algebra of a commutative semi-simple convolution measure algebra A . J. L. Taylor has shown in [11] that A may be naturally embedded in the convolution algebra $M(S)$ of finite regular Borel measures on the structure semigroup S . In §3 we show that $M(A)$ can be isometrically realized as a subalgebra of $M(S)$ containing the image of A if and only if S has an identity. As is to be expected, the measures then corresponding to isometric onto multipliers have one point support in S . Section 4 gives con-

ditions for $M(A)$ to be a convolution measure algebra, too, and §5 concentrates on describing the relationship that exists between S and the structure semigroup of $M(A)$ provided $M(A)$ is a convolution measure algebra and S has an identity. For related results in a somewhat different situation, see [13].

For any compact Hausdorff space S , $C(S)$ will denote the Banach space of continuous complex-valued functions on S with the supremum norm, and $M(S)$ is the conjugate space of $C(S)$. Of course, $M(S)$ may be interpreted as the space of finite regular Borel measures on S , and if S is also a topological semigroup, $M(S)$ is a Banach algebra under the convolution product

$$\mu * \nu(f) = \int_S \int_S f(xy) d\mu(x) d\nu(y).$$

2. **Taylor's structure semigroup of a commutative convolution measure algebra.** Preliminarily to our discussion of the multiplier algebra we give in this section the structure semigroup a description which differs slightly from Taylor's original construction. In special cases an essentially similar method has been used e.g. by Rennison in [8] and Ramirez in [7]. See also [6] and [13].

The conjugate space A' of any complex L -space A is a commutative C^* -algebra with identity. The corollary in [11, p. 157] says that if A is a commutative convolution measure algebra, then $\Delta(A) \cup \{0\}$ is a self-adjoint multiplicative subsemigroup of A' containing the identity, so that the norm closed linear span P of $\Delta(A)$ in A' is a C^* -algebra with identity. A *semicharacter* on a topological semigroup is a non-zero continuous homomorphism into the multiplicative semigroup of complex numbers z with $|z| \leq 1$.

THEOREM 2.1. *Let A be a commutative convolution measure algebra and P as above. For any $F, G \in P'$ there is a unique element, denoted FG , of P' such that $FG(\alpha) = F(\alpha)G(\alpha)$ for all $\alpha \in \Delta(A)$. The map $(F, G) \mapsto FG$ is a commutative Banach algebra product in P' . The spectrum $\Delta(P)$ of P is a multiplicative subsemigroup of P' . With the relative weak* topology $\Delta(P)$ is a compact topological semigroup, and the semicharacters of $\Delta(P)$ are precisely the Gelfand transforms of the elements of $\Delta(A)$. The structure semigroup S of A in the sense of Taylor [11] is topologically isomorphic to $\Delta(P)$.*

Proof. The product in P' that we are referring to is discussed in [1, p. 816] and [13, pp. 168–169]. In particular, since $FG(\alpha\beta) = F(\alpha\beta)G(\alpha\beta)$ for all $\alpha, \beta \in \Delta(A)$, $F, G \in P'$, even if $\alpha\beta = 0$, the proof of Theorem 2.3 in [13] is valid also in our present situation where,

in general, merely $\mathcal{A}(A) \cup \{0\}$ is a multiplicative subsemigroup of A' . Similarly, Theorem 2.4 in [13] is applicable, for the semi-simplicity of A is nowhere needed in its proof, and $\mathcal{A}(A)$ (rather than $\mathcal{A}(A) \cup \{0\}$) is assumed to be closed with respect to multiplication only to allow one to appeal to the above mentioned Theorem 2.3. From the proof of Theorem 2.2 in [11] it is clear that there is a homeomorphism φ from the structure semigroup S of A onto $\mathcal{A}(P)$ such that its natural dual map from $C(\mathcal{A}(P))$ onto $C(S)$ puts the sets of semicharacters on S and $\mathcal{A}(P)$ in a bijective correspondence. As in the proof of Theorem 6.5 in [7] it is seen that φ is also a semigroup isomorphism.

From now on we call $\mathcal{A}(P)$ with the product mentioned in the above theorem the *structure semigroup* of A and use the notation $S = \mathcal{A}(P)$.

THEOREM 2.2. *Let A and P be as in Theorem 2.1. If P' is given the product referred to in that theorem, then the isometric embedding $F \mapsto \mu_F$ from P' onto $M(S) = C(S)'$ defined by $\langle f, \mu_F \rangle = \langle f, F \rangle$ for $F \in P'$, $f \in P = C(S)$, is an algebra isomorphism.*

Proof. Suppose $F, G \in P'$. By the definition of the convolution $\mu_F * \mu_G$ we have for any $\alpha \in \mathcal{A}(A)$,

$$\begin{aligned} \langle \hat{\alpha}, \mu_F * \mu_G \rangle &= \int_S \int_S \hat{\alpha}(xy) d\mu_F(x) d\mu_G(y) = \int_S \hat{\alpha}(x) d\mu_F(x) \int_S \hat{\alpha}(y) d\mu_G(y) \\ &= \langle \alpha, F \rangle \langle \alpha, G \rangle = \langle \alpha, FG \rangle = \langle \hat{\alpha}, \mu_{FG} \rangle. \end{aligned}$$

Since the functions $\hat{\alpha}$, $\alpha \in \mathcal{A}(A)$, generate the Banach space $C(S)$, the equality $\langle h, \mu_F * \mu_G \rangle = \langle h, \mu_{FG} \rangle$ is valid for all $h \in C(S)$, i.e., $\mu_F * \mu_G = \mu_{FG}$.

3. Representing the multipliers as measures on the structure semigroup. Throughout the rest of the paper we assume that A is a commutative, *semi-simple* convolution measure algebra with structure semigroup $S = \mathcal{A}(P)$, where P is always the closed linear span of $\mathcal{A}(A)$ in A' . The set of semicharacters on S is denoted by \hat{S} . We give P' the Banach algebra product mentioned in Theorem 2.1.

LEMMA 3.1. *The natural embedding $\theta: A \rightarrow P'$ defined by $\langle f, \theta x \rangle = \langle x, f \rangle$, $x \in A$, $f \in P$, is an isometric and bipositive (i.e., $\theta x \geq 0$ if and only if $x \geq 0$) algebra homomorphism.*

Proof. From the definition of the product in P' it is clear that θ is a homomorphism. Theorem 2.3 in [11] along with the corollary in [11, p. 154] shows that it is isometric and bipositive. Alternatively, θ is isometric by virtue of the Kaplansky density theorem [10, p. 22], and bipositive by Propositions 1.5.1 and 1.5.2 in [10, p. 9], since P

contains the identity of A' .

We usually identify P' and $M(S)$ as ordered Banach spaces and Banach algebras in accordance with Theorem 2.2. Following Taylor [11], we use the notation $\theta(A) = A_s \subset M(S)$. It follows e.g. from [11, Theorem 2.3] and the corollary in [11, p. 154] that A_s is an L -subspace [11, p. 151] of the complex L -space $M(S)$.

LEMMA 3.2. *The convolution product in $M(S)$ is separately weak* continuous.*

Proof. Suppose $\nu \in M(S)$ and $f \in C(S)$. It is a simple matter to show that the function φ , $\varphi(y) = \int_S f(xy) d\nu(x)$, is continuous on S . (A much more general result may be proved using Grothendieck's weak compactness theorem, see [3, p. 205].) Since we have $|\langle f, \nu * \mu \rangle| = \left| \int_S \int_S f(xy) d\nu(x) d\mu(y) \right| = |\langle \varphi, \mu \rangle|$, the mapping $\mu \mapsto \nu * \mu$ is weak* continuous at zero, hence everywhere.

LEMMA 3.3. *Suppose that S has an identity and $\mu \in M(S)$. Then $\mu \geq 0$ if and only if $\mu * \nu \geq 0$ for all $\nu \geq 0$ in A_s .*

Proof. Clearly the latter condition is necessary. Suppose now that $\mu * \nu \geq 0$ for all $\nu \geq 0$ in A_s . Choose a basis \mathcal{U} of compact neighborhoods of the identity u of S , directed in the natural order opposite to inclusion. Each $\nu \in A_s$ is a linear combination of non-negative elements of A_s and if $\lambda \in M(S)$, $0 \leq \lambda \leq \nu \in A_s$, then $\lambda \in A_s$ since A_s is an L -subspace of $M(S)$. Furthermore, since A_s separates $P = C(S)$, A_s is a weak* dense subspace of $M(S)$. It follows easily that for each $U \in \mathcal{U}$ there exists a positive measure $\mu_U \in A_s$, $\|\mu_U\| = 1$, with support contained in U . The net $(\mu_U)_{U \in \mathcal{U}}$ converges to the Dirac measure δ_u in the weak* topology. By assumption, $\mu * \mu_U \geq 0$ for all $U \in \mathcal{U}$, and since the positive cone in $M(S)$ is weak* closed and the convolution is separately weak* continuous (Lemma 3.2), it follows that $\mu = \mu * \delta_u = \lim_U \mu * \mu_U \geq 0$.

We regard the multiplier algebra $M(A)$ as an ordered Banach space with positive cone $\{T \in M(A) \mid Tx \geq 0 \text{ for all } x \geq 0 \text{ in } A\}$.

THEOREM 3.1. *If S has an identity, then there exists a bipositive, isometric algebra isomorphism from $M(A)$ onto the subalgebra $B = \{\mu \in M(S) \mid \mu * \nu \in A_s \text{ for all } \nu \in A_s\}$ of $M(S)$. Conversely, if there exists an isometric algebra isomorphism ψ from $M(A)$ onto a subalgebra of $M(S)$ containing A_s , then S has an identity, and for any isometric and surjective $T \in M(A)$ we have $\psi(T) = c\delta_x$, where $c \in \mathbb{C}$, $|c| = 1$ and δ_x is the Dirac measure corresponding to some $x \in S$.*

Proof. Suppose that S has an identity u . The net $(\mu_U)_{U \in \mathcal{U}}$ constructed in the proof of Lemma 3.3 converges to the Dirac measure δ_u in the weak* topology of $M(S)$. In particular, $\lim_U \langle \mu_U, \gamma \rangle = 1$ for each $\gamma \in \mathcal{A}(A) = \hat{S}$. If we denote by $T_f \in M(A)$ the operator corresponding to the function $f \in A^m$ (see the introduction), an argument given in [1, p. 817] shows that $|\sum_{k=1}^n a_k f(\gamma_k)| \leq \|T_f\| \|\sum_{k=1}^n a_k \gamma_k\|$ for all $\gamma_k \in \mathcal{A}(A)$, $a_k \in \mathbb{C}$, $k = 1, \dots, n$. It follows that f can be extended as a continuous linear functional f to the whole of P with norm $\|f\| \leq \|T_f\|$. Since the embedding $\theta: A \rightarrow P'$ is isometric (Lemma 3.1), we have, using the definition of the product in P' , $\|f\| \geq \sup \{\|f\theta(x)\| \mid x \in A, \|\theta(x)\| \leq 1\} = \sup \{\|f\theta(x)\| \mid x \in A, \|x\| \leq 1\} = \sup_{\substack{\|x\| \leq 1 \\ x \in A}} \|T_f x\| = \|T_f\|$. Thus $\|f\| = \|T_f\|$. From the definition of the product in P' it is obvious that the above embedding of $M(A)$ in P' is an algebra homomorphism, so that it may be interpreted as an isometric algebra homomorphism $\pi: M(A) \rightarrow M(S)$ (Theorem 2.2). Since the embedding of A in $M(S)$ is bipositive (Lemma 3.1), it is clear from Lemma 3.3 that π is bipositive. Denote $\pi(M(A)) = B \subset M(S)$. For functions in $A^m(\supset \hat{A})$, pointwise multiplication corresponds to the convolution of the respective measures on S (see the proof of Theorem 2.2). Therefore, A_S is an ideal in B . Also, if $\mu \in M(S)$, and $\mu * \nu \in A_S$ for all $\nu \in A_S$, then the function $f_\mu: \mathcal{A}(A) \rightarrow \mathbb{C}$ obtained by restricting μ to $\hat{S} = \mathcal{A}(A)$ belongs to A^m . The first part of the theorem is thus proved. To prove the converse assertion we note that $M(A)$ has always an identity. The hypothesis then implies that a weak* dense subalgebra of $M(S)$ has an identity η . It follows from Lemma 3.2 that η is also an identity for $M(S)$. But it is well known that the identity of any Banach algebra is an extreme point in its unit ball (see e.g. [10, p. 13]). Hence (see [2, p. 441]) we have $\eta = c\delta_u$ for some $u \in S$ and $c \in \mathbb{C}$, $|c| = 1$. In fact $c = 1$, since $c\delta_u = c\delta_u * c\delta_u = c^2\delta_{u^2}$, so that $u = u^2$ and $c = c^2$. Now, u is an identity for S , since $\delta_{ux} = \delta_u * \delta_x = \delta_x$ for all $x \in S$. For the last statement, it is enough to show that $\psi(T)$ is an extreme point of the unit ball of $M(S)$ [2, p. 441]. If $0 \leq \lambda \leq 1$ and $\mu_1, \mu_2 \in M(S)$ are such that $\psi(T) = \lambda\mu_1 + (1 - \lambda)\mu_2$ and $\|\mu_1\| \leq 1$, $\|\mu_2\| \leq 1$, we have for the identity η of $M(S)$, since also $T^{-1} \in M(A)$ [5, p. 15],

$$\eta = \psi(T^{-1}) * \psi(T) = \lambda \psi(T^{-1}) * \mu_1 + (1 - \lambda) \psi(T^{-1}) * \mu_2,$$

where $\|\psi(T^{-1}) * \mu_1\| \leq 1$ and $\|\psi(T^{-1}) * \mu_2\| \leq 1$. Since η is an extreme point of the unit ball of $M(S)$, we have $\lambda = 0$ or $\lambda = 1$. Therefore, $\psi(T)$ is also an extreme point of the unit ball of $M(S)$.

NOTE. From the proof of the above theorem it is clear that S has an identity if and only if A has a weak bounded approximate

identity [1, p. 817] of norm one. (Compare [11, Theorem 3.1].)

4. $M(A)$ as a convolution measure algebra. If S has an identity, then $M(A)$ may be embedded in $M(S)$ in accordance with Theorem 3.1. Unfortunately, the nature of the image $B \subset M(S)$ of $M(A)$ is not sufficiently clear on the basis of that result. For example, we should like to conclude that B is a so-called L -subalgebra of $M(S)$, which turns out to be equivalent to saying that $M(A)$ with its natural order is a convolution measure algebra. The next theorem gives two other necessary and sufficient conditions for this to be case.

We assume henceforth that S has an identity and let $\pi: M(A) \rightarrow M(S)$ be the bipositive, isometric homomorphism constructed in the proof of Theorem 3.1, and denote as before $B = \pi(M(A)) = \{\mu \in M(S) \mid \mu * \nu \in A_s \text{ for all } \nu \in A_s\}$. Since S has an identity, $\Delta(A)$ (and not merely $\Delta(A) \cup \{0\}$) is a multiplicative subsemigroup of A' , so that it makes sense to talk about translations of functions on $\Delta(A)$. A set \mathcal{F} of functions $f: \Delta(A) \rightarrow \mathbb{C}$ is *translation invariant*, if $f \in \mathcal{F}$ implies $f^\alpha \in \mathcal{F}$ for all $\alpha \in \Delta(A)$, where $f^\alpha(\beta) = f(\alpha\beta)$ for $\alpha, \beta \in \Delta(A)$.

A closed subalgebra N of the convolution measure algebra $M(S)$ is an L -subalgebra of $M(S)$, if for any $\mu \in N$ its total variation $|\mu|$ belongs to N , and if $\nu \in N$ whenever $\mu \in N$ and ν is absolutely continuous with respect to $|\mu|$ (denoted $\nu \ll |\mu|$) [12, p. 257]. This definition is easily seen to be equivalent to requiring that N be a subalgebra and an L -subspace of $M(S)$ in the sense of [11].

THEOREM 4.1. *The following conditions are equivalent:*

- (i) $M(A)$ is a convolution measure algebra (in the order defined before Theorem 3.1),
- (ii) B is an L -subalgebra of $M(S)$,
- (iii) A^m is a translation invariant set of functions on $\Delta(A)$,
- (iv) for any $\mu \in B$, $|\mu|$ also belongs to B .

Proof. We shall establish the following chain of implications: (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iv). If (ii) holds and $\mu \in B$, $f \in C(S)$, then the measure $f\mu$ (i.e., the functional $g \mapsto \mu(fg)$ on $C(S)$) belongs to B . But if $f \in A^m$ and $\mu_f = \pi(T_f)$, then we have $f^\alpha(\beta) = \mu_f(\alpha\beta) = \alpha\mu_f(\beta)$ for all $\alpha, \beta \in \Delta(A) = \hat{S}$, so that $f^\alpha \in A^m$, since $\alpha\mu_f \in B$ and A_s is an ideal in B . Thus (ii) implies (iii). Next, assume (iii) and choose any $\mu \in B$. Then the function $f_\mu: \Delta(A) \rightarrow \mathbb{C}$ defined by $f_\mu(\alpha) = \mu(\alpha)$ for $\alpha \in \Delta(A) = \hat{S}$ belongs to A^m . By assumption, $(f_\mu)^\alpha \in A^m$ for any $\alpha \in \Delta(A)$. But the measure in B corresponding to $(f_\mu)^\alpha$ when α is regarded as a semicharacter of S , is $\alpha\mu$. As \hat{S} generates the Banach space $C(S)$ and the mapping $f \mapsto f\mu$ is continuous from $C(S)$ to $M(S)$, which contains B as a closed subspace, we have $f\mu \in B$ for all $f \in C(S)$. Fur-

thermore, $C(S)$ is dense in $L^1(S, |\mu|)$ [4, p. 140] so that there is a sequence of functions in $C(S)$ converging to \bar{g} in $L^1(S, |\mu|)$, where g is a $|\mu|$ -measurable function with $|g(x)| = 1$ $|\mu|$ -a.e. and such that $\mu = g|\mu|$ [4, p. 171]. By virtue of the continuity of the mapping $f \mapsto f\mu$ from $L^1(S, |\mu|)$ to $M(S)$ and the fact that $f\mu \in B$ for all $f \in C(S)$, it follows that $|\mu| = \bar{g}\mu \in B$, i.e., (iv) holds. Assume now (iv) and that $\mu \in M(S)$ is absolutely continuous with respect to some $\lambda \geq 0$ in B . Then we have also $\mu_j \ll \lambda$, $j=1, \dots, 4$, in the Jordan decomposition $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where μ_1 and μ_2 (resp. μ_3 and μ_4) are mutually singular nonnegative measures. If $\nu \geq 0$ is in A_s , we have $\nu * \mu_j \ll \nu * \lambda$. This has been proved in a somewhat more general setting by Pym in [6, p. 630]. Since A_s is an L -subspace of $M(S)$, hence an L -subalgebra in the sense of [12], we have $\nu * \mu \in A_s$. It follows that in fact $\nu * \mu \in A_s$ for an arbitrary $\nu \in A_s$, so that $\mu \in B$. Thus (ii) holds. Since $M(A)$ is isometrically algebra and order isomorphic to B , and any L -subalgebra of $M(S)$ is a convolution measure algebra (see [11, p. 151 and Definition 2.1]), (ii) implies (i) at once. Finally, if $M(A)$ is a convolution measure algebra (hence a complex L -space), (iv) holds by virtue of Corollary 1.6 and Proposition 1.8 in [9].

On the basis of the above theorem sufficient conditions (assuming that S has an identity) can be given to guarantee that $M(A)$ is a convolution measure algebra. Since A_s is an L -subalgebra of $M(S)$, an argument used in the proof of the above theorem shows that \hat{A} is translation invariant on $\mathcal{A}(A)$. If we assume for example that A is regular and has a bounded approximate identity consisting of elements with Gelfand transforms of compact support, then the theorem in [1, p. 819] shows that A^m consists of those functions on $\mathcal{A}(A)$ which belong locally to \hat{A} and may be extended to continuous linear functionals on P . Then Theorem 4.2 in [13] shows that A^m is translation invariant. Another case where the translation invariance of A^m follows immediately from that of \hat{A} arises, when S is a multiplicative group, for then we have $(f^\alpha \hat{x})(\beta) = f(\alpha\beta) \hat{x}^{\alpha^{-1}}(\alpha\beta) = \hat{y}(\beta)$ for some $y \in A$, if $f \in A^m$ and $x \in A$. For a discussion of this kind of a situation, see [12].

5. The structure semigroups of A and $M(A)$. We retain the general hypotheses and notational conventions made in §§3 and 4. In particular S has an identity. Let us make the additional assumption that $M(A)$ is a convolution measure algebra (in the order defined before Theorem 3.1). When A and $M(A)$ are realized as subalgebras of $M(S)$ (§3), it is seen that the embedding $j: A \rightarrow M(A)$ defined by $j(x) = T_{\hat{x}}$ is isometric and bipositive. It is readily seen to be an L -homomorphism [11, p. 152], since A_s is an L -subspace of $M(S)$. We let Q denote the closed linear span of $\mathcal{A}(M(A))$ in $M(A)'$. Then $T =$

$\mathcal{A}(Q)$ with the usual topology and product is the structure semigroup of $M(A)$. Before stating Theorem 5.1, which relates S and T to each other, we prove an auxiliary result.

LEMMA 5.1. *The mapping $\Phi: P \rightarrow M(A)'$ defined by $\langle \Phi f, L \rangle = \langle \pi(L), f \rangle$ for $L \in M(A)$, $f \in P = C(S)$, is an isometric C^* -algebra homomorphism which maps the identity of P to that of $M(A)'$, and we have*

$$(1) \quad j^* \circ \Phi(f) = f, \quad f \in P,$$

for the transpose $j^*: M(A)' \rightarrow A'$ of j . Furthermore, $\Phi(P) \subset Q$.

Proof. Equation (1) is immediate. Since $j: A \rightarrow M(A)$ is an L -homomorphism, $j^*: M(A)' \rightarrow A'$ is a C^* -algebra homomorphism which preserves the identity [11, p. 153]. Therefore,

$$(2) \quad j^*(\Phi\alpha\Phi\beta) = \alpha\beta \quad \text{for } \alpha, \beta \in \mathcal{A}(A) = \hat{S}.$$

As S has an identity, $\alpha\beta \neq 0$. A simple calculation shows that since π is a homomorphism, $\Phi\alpha$ and $\Phi\beta$ are multiplicative, so that by (2) their product is a multiplicative extension of $\alpha\beta$ to $M(A)$ (when $\alpha\beta$ is regarded as a functional on $j(A)$). Now, $\Phi(\alpha\beta)$ is also a multiplicative extension of $\alpha\beta$ to $M(A)$, and since there are only one of them [5, p. 24], we have $\Phi(\alpha\beta) = \Phi\alpha\Phi\beta$. A similar argument shows that $\Phi|_{\mathcal{A}(A)}$ preserves involution. It follows that Φ is a C^* -algebra homomorphism. Since the identity e_1 of A' belongs to $\mathcal{A}(A)$ and the identity e_2 of $M(A)'$ to $\mathcal{A}(M(A))$, the uniqueness of the multiplicative extension again shows that $\Phi e_1 = e_2$. Since any C^* -algebra homomorphism is norm-decreasing [10, p. 5] it follows from (1) that Φ is isometric. The last statement is a consequence of the fact that $\Phi(\mathcal{A}(A)) \subset \mathcal{A}(M(A))$.

In the following theorem \mathfrak{J} denotes the natural embedding of $M(A)$ in $M(T)$ [11, p. 158]. The identity map of a set D is denoted by id_D .

THEOREM 5.1. *There exist unique continuous semigroup homomorphisms $\psi: S \rightarrow T$ and $\varphi: T \rightarrow S$ such that*

$$(1) \quad \Phi f(t) = f \circ \varphi(t) \text{ and } \Psi g(s) = g \circ \psi(s)$$

for all $t \in T$, $f \in C(S) = P$, $s \in S$, $g \in C(T) = Q$, where $\Psi = j^*|_Q$ and Φ is the map defined in Lemma 5.1. Furthermore,

$$(a) \quad \varphi \circ \psi = id_S \text{ and } \Psi \circ \Phi = id_P.$$

$$(b) \quad \psi(S) \text{ is a closed ideal in } T.$$

$$(c) \quad \text{For the identity } u \text{ of } S \text{ we have } \psi \circ \varphi(t) = t\psi(u), t \in T.$$

(d) *If we denote $M_s(T) = \{\mu \in M(T) \mid |\mu|(T \setminus \psi(S)) = 0\}$, then $M_s(T)$ is an ideal in $M(T)$ and $\Psi^*(M(S)) = M_s(T)$ for the transpose $\Psi^*: M(S) \rightarrow M(T)$ of Ψ .*

(e) The diagram below commutes, and all maps appearing in it are algebra homomorphisms.

$$\begin{array}{ccc}
 A & \xrightarrow{j} & M(A) \\
 \downarrow \theta & & \downarrow \mathfrak{S} \\
 M(S) & \xrightarrow{\Psi^*} & M(T)
 \end{array}
 \begin{array}{c}
 \searrow \pi \\
 \searrow \Phi^* \\
 \searrow id \\
 \searrow \\
 M(S)
 \end{array}$$

Proof. It is clear that (1) holds if and only if the maps $\psi: \mathcal{A}(P) \rightarrow \mathcal{A}(Q)$ and $\varphi: \mathcal{A}(Q) \rightarrow \mathcal{A}(P)$ are defined by setting $\langle \psi(s), g \rangle = \Psi g(s)$ and $\langle \varphi(t), f \rangle = \Phi f(t)$ for $s \in S$, $t \in T$, $g \in Q = C(T)$, and $f \in P = C(S)$. From the definition of the product in S and T it follows that ψ and φ , obviously continuous, are semigroup homomorphisms. For example, if $\gamma \in \mathcal{A}(M(A))$ and $x, y \in S$, then $\Psi\gamma \in \mathcal{A}(A)$ or $\Psi\gamma = 0$, and in both cases $\langle \psi(xy), \gamma \rangle = \langle xy, \Psi\gamma \rangle = \langle x, \Psi\gamma \rangle \langle y, \Psi\gamma \rangle = \langle \psi(x), \gamma \rangle \langle \psi(y), \gamma \rangle$, i.e., $\psi(xy) = \psi(x)\psi(y)$. The second formula in (a) is a consequence of Lemma 5.1, and the first formula follows from the second by a simple calculation. The commutativity of the square in (e) is seen as follows: $\langle g, \Psi^* \circ \theta(x) \rangle = \langle \Psi g, \theta x \rangle = \langle x, \Psi g \rangle = \langle j(x), g \rangle = \langle g, \mathfrak{S} \circ j(x) \rangle$ for $x \in A$, $g \in C(T) = Q$, so that $\Psi^* \circ \theta = \mathfrak{S} \circ j$. The lower triangle commutes because of (a). As to the upper triangle, note that if γ belongs to $\mathcal{A}(A)$, then $\Phi\gamma$ is its unique multiplicative extension to $M(A)$ (see the proof of Lemma 5.1). Therefore, $\langle \gamma, \Phi^* \circ \mathfrak{S}(L) \rangle = \langle \Phi\gamma, \mathfrak{S}(L) \rangle = f_L(\gamma) = \langle \gamma, \pi(L) \rangle$, where f_L is the function in A^m corresponding to $L \in M(A)$. Since $\mathcal{A}(A) = \hat{S}$ generates $C(S)$, the equation $\Phi^* \circ \mathfrak{S} = \pi$ follows. The second statement in (e) is also easily proved. Next we show that $\Psi^*(M(S)) = M_s(T)$. Since Ψ and Φ are norm-decreasing, Ψ^* and Φ^* are so, and (e) implies that Ψ^* is isometric. On the other hand, Ψ^* is continuous from $\sigma(M(S), C(S))$ to $\sigma(M(T), C(T))$. Therefore, using the weak* compactness of $B_r = \{\mu \in M(S) \mid \|\mu\| \leq r\}$, we see that $\{\mu \in \Psi^*(M(S)) \mid \|\mu\| \leq r\}$ is weak* compact, hence closed for each $r \geq 0$. The Krein-Smulian theorem [2, p. 429] then shows that $\Psi^*(M(S))$ is weak* closed in $M(T)$. If S (resp. T) is considered naturally embedded in $M(S)$ (resp. $M(T)$), then $\Psi^*|_S = \psi$, so that $\Psi^*(S) = \psi(S)$. The linear combinations of the Dirac measures are weak* dense in $M(S)$. Similarly, since $\sigma(M_s(T), C(T))$ and $\sigma(M_s(T), C(\psi(S)))$ coincide on $M_s(T)$, the linear span of $\psi(S) = \Psi^*(S)$ is $\sigma(M_s(T), C(T))$ -dense in $M_s(T)$, which in turn is weak* closed in $M(T)$, as $\psi(S) \subset T$ is compact. From these remarks the equation $\Psi^*(M(S)) = M_s(T)$ follows by the weak* continuity of Ψ^* . From (e) and the fact that $j(A)$ is

an ideal in $M(A)$ it follows that $\Psi^*(A_S)$ is an ideal in $\mathfrak{S}(M(A))$. Since A_S is weak* dense in $M(S)$ [11, p. 158], it follows from what was said above that the weak* closure of $\Psi^*(A_S)$ is $M_S(T) = \Psi^*(M(S))$. By virtue of the separate weak* continuity of the convolution in $M(T)$ (Lemma 3.2), M_S is therefore an ideal in $M(T)$, which contains $\mathfrak{S}(M(A))$ as a weak* dense subspace. Thus (d) is proved. Since multiplication in T corresponds to the convolution of Dirac measures, (b) is an immediate consequence of (d). Finally, (c) follows from the equation $\varphi(\psi \circ \varphi(t)) = \varphi(t\psi(u))$, i.e., $\varphi(t) = \varphi(t)u$, since φ is injective on $\psi(S)$ and $t\psi(u) \in \psi(S)$.

EXAMPLES. The above theorem is applicable e.g. in two classical situations, where the algebra A is defined in terms of a locally compact abelian topological group G . If A is $L^1(G)$, the convolution algebra of Haar integrable functions on G , then as is well known [5, p. 3] the multiplier algebra $M(A)$ may be identified with the convolution algebra $M(G)$ of bounded regular Borel measures on G . In this case, S is the Bohr compactification of G and $\psi(S)$ is the kernel (i.e., minimal ideal) of T [11, p. 164].

Similarly, the theorem yields a connection between the structure semigroups of the convolution measure algebras

$$L^1(G_+) = \left\{ f \in L^1(G) \mid \int_{G \setminus G_+} |f(x)| dx = 0 \right\}$$

and

$$M(G_+) = \{ \mu \in M(G) \mid \mu|(G \setminus G_+) = 0 \},$$

where G_+ is a closed subsemigroup of G containing the neutral element of G and such that the interior of G_+ generates G and is dense in G_+ . For $A = L^1(G_+)$ satisfies the hypotheses of the theorem (for example, S has an identity, since G_+ may be realized as a dense subsemigroup of S [11, p. 163]), and Birtel has shown in [1, p. 821] that $M(A) = M(G_+)$. In the case of $A = L^1(G_+)$ (and hence if $A = L^1(G)$) the usual order in $M(A)$ as a space of measures agrees with the order defined before Theorem 3.1, for it follows from Birtel's proof that there is a net $\{\mu_\sigma\}$ of positive $\mu_\sigma \in L^1(G_+)$ satisfying $\lim_\sigma \mu_\sigma * \mu(f) = \mu(f)$ for all $f \in C_0(G)$, $\mu \in M(G_+)$.

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