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LAPLACE TRANSFORM METHODS IN MULTIVARIATE SPECTRAL THEORY

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The Laplace transform of the semigroup $\exp{(tA)}$ generated by an operator A gives the resolvent of A. An integral formula is obtained for the Laplace transform of $\exp{(tA+B)}$, where B is another operator which does not commute with A. The new transform has analytic continuation to the same domain as the resolvent, but the analytic continuation is not single-valued. The integral formula is then applied to the joint spectral theory of noncommutative operators. Explicit computations with matrices of degree two illustrate the results.

1. Introduction. Any bounded linear operator A on a Banach space generates a semigroup $\exp(tA)$, $0 \le t < \infty$, and the Laplace transform $\mathscr{L}(s,A)$ of this semigroup converges for Re s sufficiently large and equals the resolvent $(s-A)^{-1} \cdot \mathscr{L}(s,A)$ therefore has unique analytic continuation to the component containing ∞ of the resolvent set A.

Multivariate problems requiring integration of $\exp(\sum t_i A_i)$ one variable at a time, lead us to consider the Laplace transform $\mathcal{L}(s, A, B)$ of $\exp(tA + B)$, $0 \le t < \infty$, where B is a fixed bounded operator.

The main result is:

THEOREM 1. $\mathcal{L}(s, A, B)$ has the contour integral representation (1.1)

$$(s-A)^{-1} + \int_s^\infty (u-A)^{-1} B(u-A)^{-1} \exp\left[B(u-A)^{-1}(u-s)\right] du$$

valid for Res sufficiently large. Therefore, $\mathcal{L}(s, A, B)$ can be analytically continued along any arc not intersecting $\sigma(A)$.

Examples are given in §5 which show that the analytic continuation is not always unique.

In §§ 3 and 4 our result is applied to problems in spectral theory. According to the Weyl functional calculus [1], [2], [3], two self-adjoint operators have a joint spectral distribution in the plane, which if A and B commute is simply the tensor product of their spectral measures. Two operators A and B which are merely bounded have instead a two-dimensional Laplace transform $\mathcal{L}(s, \sigma, A, B)$ which if A and B commute is simply the product of their resolvents. $\mathcal{L}(s, \sigma, A, B)$ may be regarded as a functional on the space of entire

functions on C^2 . Its carrier may be regarded as a joint spectrum of A, B. Although there is no unique minimal carrier in general for functionals of this type, Theorem 1 can be exploited to obtain information about the carriers in terms of the spectral properties of A and B. It turns out that the actual spectrum of A can be used to construct a carrier, if accuracy with respect to B is sacrificed.

Suppose now that A is bounded and B is self-adjoint. The Weyl calculus for the three self-adjoint operators $\operatorname{Re} A$, $\operatorname{Im} A$, and B gives a spectrum projecting onto the whole numerical range of A. However, in §4 we construct a hybrid functional for A and B which is an analytic functional with respect to A. The motivating question is whether the carrier of this functional will still be the whole numerical range of A or whether the actual spectrum of A will reappear. Theorem 6 gives the transition between the two competing theories and offers no help in shrinking the carrier. But the examples of §5 show that in some cases the actual spectrum of A does suffice as the carrier.

2. Proof of Theorem 1.

Proof of Theorem 1. The Laplace transform of $\exp(tA + B)$ cannot be computed directly unless B commutes with A, in which case the trivial result is:

$$\mathcal{L}(s, A, B) = \exp(B)\mathcal{L}(s, A)$$
.

We therefore resort to the following contour integral method.

Let C be a simple closed curve containing $\sigma(A)$ (spectrum of A) in its interior. Then C also encloses $\sigma(A + t^{-1}B)$ for |t| greater than some constant k, and by the Riesz functional calculus [6] applied to the operator $A + t^{-1}B$, for |t| > k,

$$\exp(tA + B) = \exp(t(A + t^{-1}B)) = \frac{1}{2\pi i} \oint_C e^{tz} [z - (A + t^{-1}B)]^{-1} dz$$
.

If in addition

$$|t| > k_1 = \sup_{z \in \text{int}C} ||B(z-A)^{-1}||$$

then

$$egin{aligned} [z-A-t^{-1}B]^{-1} &= [(I-t^{-1}B(z-A)^{-1})(z-A)]^{-1} \ &= (z-A)^{-1}\sum_{n=0}^{\infty} [B(z-A)^{-1}]^n t^{-n} \end{aligned}$$

and

$$egin{aligned} \exp{(tA+B)} &= rac{1}{2\pi i} \oint_{C} \sum\limits_{n=0}^{\infty} e^{zt} t^{-n} (z-A)^{-1} [B(z-A)^{-1}]^n dz \ &= rac{1}{2\pi i} \oint_{C} \sum\limits_{n,j=0}^{\infty} rac{z^j t^{j-n}}{j!} (z-A)^{-1} [B(z-A)^{-1}]^n dz \;. \end{aligned}$$

The double series in the integrand is absolutely and uniformly convergent on the domain $z \in C$, $\max(k, k_1) < |t| < k_2$ where k_2 is any constant. The contour integral can therefore be evaluated term-by-term. All terms having j < n are $O(|z|^{-2})$ for large z, and so by enlargement of the contour, they vanish. The remaining terms can therefore be rewritten as the sum

$$\sum_{q=0}^{\infty} t^q rac{1}{2\pi i} \int_{C} (z-A)^{-1} \sum_{p=0}^{\infty} rac{z^{p+q}}{(p+q)!} [B(z-A)^{-1}]^p dz$$
 .

This power series expansion of the entire function $\exp(tA + B)$ is valid in the annular region given above, and consequently holds for all t.

Since $\exp(tA+B)$ has exponential growth rate, the Laplace transform of its power series expansion may be taken term-by-term. This fact is discussed fully in Widder's book on the Laplace transform [8]. Therefore

$$\mathscr{L}(s, A, B) = \sum_{q=0}^{\infty} \frac{q!}{s^{q+1}} \frac{1}{2\pi i} \oint_{C} (z - A)^{-1} \sum_{p=0}^{\infty} \frac{z^{p+q}}{(p+q)!} [B(z - A)^{-1}]^{p} dz$$
.

Next we note that

$$\sum_{q,\,p=0}^{\infty} \frac{q!}{s^{q+1}} \frac{z^{p+q}}{(p+q)!} [B(z-A)^{-1}]^{p}$$

is absolutely and uniformly convergent when $z \in C$, |z/s| < l < 1, l being any constant less than one.

To reduce the double series to closed form, consider

$$F(a, b) = \sum_{q,p=0}^{\infty} a^q b^p \frac{q!}{(p+q)!}$$
.

If the series $\sum_{q=0}^{\infty} a^{q+p} q!/(p+q)!$ is differentiated p times, we obtain a geometric series which converges to $(1-a)^{-1}$. By elementary means, therefore,

$$\sum_{q=0}^{\infty} a^q rac{q!}{(p+q)!} = rac{1}{a^p} \int_0^a rac{(a-t)^{p-1}}{(p-1)!} rac{dt}{1-t} \,, \qquad p \geqq 1$$

and

$$egin{align} F(a,\,b) &= rac{1}{1-a} + \sum\limits_{p=1}^{\infty} \left(rac{b}{a}
ight)^p \!\!\int_0^a rac{(a-t)^{p-1}}{(p-1)!} rac{dt}{1-t} \ &= rac{1}{1-a} + b\sum\limits_{p=1}^{\infty} b^{p-1} \!\!\int_0^1 rac{(1-u)^{p-1}}{(p-1)!} rac{du}{1-au} \ &= rac{1}{1-a} + b \!\!\int_0^1 \!\!e^{b(1-u)} rac{du}{1-au} \,. \end{split}$$

We now substitute a = z/s and $b = zB(z - A)^{-1}$.

$$egin{align} \mathscr{L}(s,\,A,\,B) &= rac{1}{2\pi i s} \oint_{c} (z-A)^{-1} \Big\{ \Big(1-rac{z}{s}\Big)^{-1} \\ &+ z B (z-A)^{-1} \int_{0}^{1} \exp\left[z B (z-A)^{-1} (1-u)\right] \Big(1-rac{z}{s} u\Big)^{-1} du \Big\} \, dz \ &= (s-A)^{-1} + \int_{0}^{1} rac{1}{2\pi i} \oint_{c} z (z-A)^{-1} B (z-A)^{-1} \\ &\cdot \exp\left[z B (z-A)^{-1} (1-u)\right] \Big(rac{s}{u}-z\Big)^{-1} dz rac{du}{u} \, . \end{split}$$

Since the integrand of the contour integral is holomorphic in the neighborhood of $z = \infty$ ontside C, the Cauchy integral formula yields

$$\mathscr{L}(s, A, B) = (s - A)^{-1} + \int_0^1 \frac{s}{u} \left(\frac{s}{u} - A\right)^{-1} B\left(\frac{s}{u} - A\right)^{-1} \exp\left[\frac{s}{u} B\left(\frac{s}{u} - A\right)^{-1} (1 - u)\right] \frac{du}{u}.$$

Replacing u by s/u, we get Theorem 1:

$$\mathscr{L}(s, A, B) = (s - A)^{-1} + \int_{s}^{\infty} (u - A)^{-1} B(u - A)^{-1} \exp \left[B(u - A)^{-1}(u - s)\right] du.$$

COROLLARY 2. $\mathcal{L}(s, A, B)$ has unique analytic continuation to R_{∞} , the component of the resolvent set of A containing ∞ , iff for every component σ_i of $\sigma(A)$ meeting \bar{R}_{∞} , any contour C_i enclosing component σ_i only, and for all $j \geq 1$,

(2.1)
$$\oint_{C_i} (u - A)^{-1} [B(u - A)^{-1}]^j \exp [B(u - A)^{-1}u] du = 0.$$

Proof. Suppose the contour integral of Theorem 1 is continued along two different arcs terminating at s. The difference between the values of s so obtained is, by homotopy arguments, an integral combination of the closed contour integrals

$$\oint_{G_s} (u-A)^{-1} B(u-A)^{-1} \exp \left[B(u-A)^{-1} (u-s) \right] du$$

 \mathbf{or}

$$\frac{d}{ds} \oint_{c_i} (u - A)^{-1} \exp \left[B(u - A)^{-1} (u - s) \right] du.$$

The result follows by power series expansion in s.

COROLLARY 3. The Laplace transform $\mathcal{L}(s, \sigma, A, B)$ of $\exp(tA + \xi B)$ is given by

(2.2)
$$\sigma^{-1}(s-A)^{-1} + \int_{s}^{\infty} (u-A)^{-1}B(u-A)^{-1}[\sigma-B(u-A)^{-1}(u-s)]^{-2}du$$
 for $\sigma>0$, $s>0$ sufficiently large.

Proof. Replace B by ξB in the formula for $\mathcal{L}(s, A, B)$. The integration with respect to ξ is elementary.

3. Analytic functionals in spectral theory. Suppose the bounded operators A_1, \dots, A_n are all self-adjoint, so that for $\xi \in R^n$, $\exp(i\xi \cdot A)$ is a unitary operator. Then by Fourier inversion a tempered distribution $\mathscr{F}^{-1} \exp(i\xi \cdot A)$ is determined. In previous papers by the present author, [1], [2], [3], this distribution was called the "joint spectral distribution" of A_1, \dots, A_n and denoted T(A).

In order to gain further insight into this type of spectral distribution, we consider the slightly different case when iA_1, \dots, iA_n are assumed only to be the generators of contraction semigroups. This is equivalent to the condition that A_1, \dots, A_n have numerical range in the upper half plane. In this case, so does $\xi \cdot A$ if $\xi_1, \dots, \xi_n \geq 0$ (abbreviation $\xi \geq 0$), so $||\exp(i\xi \cdot A)|| \leq 1$ when $\xi \geq 0$.

DEFINITION. When iA_1, \dots, iA_n generate contraction semigroups, S(A) denotes the tempered distribution defined for $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$(3.1) S(A)f = (2\pi)^{-n/2} \int_{\xi>0} (\mathscr{F}f)(\xi) \exp(i\xi \cdot A) d\xi.$$

In one dimension, simple computation shows that

(3.2)
$$S(A)f = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(x) \mathcal{L}(x, A) dx$$

where the Laplace transform $\mathcal{L}(x, A) = (x - A)^{-1}$, provided the spectrum of A does not intersect the real line.

However, $f \in L^2(\mathbb{R}^1)$ may be written as $f = f_+ + f_-$, where $\mathscr{F} f_+ = \mathscr{F} f$ for $x \geq 0$, $\mathscr{F} f_- = \mathscr{F} f$ for $x \leq 0$. f_+ is the boundary value of a function $f_+(z)$ holomorphic in the upper half plane, and $|f_+(z)| = f_+(z)$

 $0((\operatorname{Im} z)^{-1})$. If C is any contour in the upper half plane enclosing spectrum A, we obtain

(3.3)
$$S(A)f = \frac{1}{2\pi i} \oint_{c} f_{+}(z) \mathscr{L}(z, A) dz.$$

This is just the Riesz calculus (see Ch. XI of [6]), but in two dimensions we can similarly obtain the formula

$$(3.4) \hspace{1cm} S(A, B)f = \Big(\frac{1}{2\pi i}\Big)^2 \oint_{c_1} \oint_{c_2} f_+(s, \sigma) \mathscr{L}(s, \sigma, A, B) ds \, d\sigma$$

where $\mathscr{L}(s,\sigma)$ is holomorphic for s, σ outside C_1 , C_2 respectively, and $\mathscr{F}f_+ = \mathscr{F}f$ for $\xi \geq 0$, $\mathscr{F}f_+ = 0$ otherwise.

Formula (3.4) defines a continuous linear functional on the space of entire functions in two complex variables, and the numerical range of A, B need not be restricted. Such functionals are discussed, for example, in Hormanders' book [5]. Such functionals are in one-to-one correspondence with entire functions of exponential growth, in our case $\exp(i\xi \cdot A)$. In one dimension, there is a canonical representation of a functional similar to (3.2), but not in higher dimensions. If K_i denotes the compact set bounded by C_i , and $K = K_1 \times K_2$, then $||S(A, B)f|| \leq c \sup_{s,\sigma \in K} |f(s,\sigma)|$, so K is an example of a "carrier" of S(A, B). In general, there is no unique minimal carrier of a functional in dimension greater than 1.

LEMMA 4. If K_1 , K_2 contain neighborhoods of the numerical ranges of A, B resp., then $K = K_1 \times K_2$ is a carrier of S(A, B). If K_1 is simply connected and contains the spectrum of A in its interior, then there exists K_2 such that $K = K_1 \times K_2$ is a carrier of S(A, B).

Proof. $\mathcal{L}(s, \sigma, A, B)$ is holomorphic when $s, \sigma > 0$ if A and B have numerical range in the left half-plane. By translating and rotating A and B independently, the general result is obtained. The second result follows by inspection of formula (2.2) in Corollary 3.

4. A hybrid functional.

DEFINITION. Let iA generate a contraction semigroup and let B be self-adjoint. Then the tempered distribution ST (A, B) is defined for $f \in \mathcal{S}(R^2)$ by

$$(4.1) \qquad \operatorname{ST}(A, B) f = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{0}^{\infty} (\mathscr{F} f)(t, \, \xi) \exp\left(i(tA + \xi B)\right) dt \, d\xi \, .$$

NOTATION. Given $f \in \mathcal{S}(\mathbb{R}^2)$, let $f_+(z, x)$ denote the analytic con-

tinuation to Im $z \ge 0$ of the function $f_+ \in L^2(R^2)$ satisfying

$$(\mathscr{F}f_+)(t,\,\hat{\xi})=(\mathscr{F}f)(t,\,\hat{\xi})\;,\qquad t\geqq 0 \ t<0\;.$$

Note that for each fixed $z_0, f_1(z_0, x) \in \mathcal{S}(R^1)$.

LEMMA 5. For each z outside the closure of the numerical range of A, there is a tempered distribution $\Phi(z) \in \mathscr{S}'(R^1)$ acting on $\mathcal{P}(x) \in \mathscr{S}(R^1)$, such that $\Phi(z)$ is (weakly) holomorphic in z, and such that for a contour C enclosing the numerical range of A,

(4.2) ST
$$(A, B)f = \frac{1}{2\pi i} \oint_{C} \Phi(z) f_{+} dz$$
.

Proof. It is easily checked that

$$\|\exp(tA + i\xi B)\| \le \exp(\|t\|\|A\|)$$
.

Therefore, $\exp{(tA+i\xi B)}=\sum_{j=0}^\infty t^jG_j(\xi)$ where for all $j,\,G_j(\xi)\in C^\infty(R^i)$ and

$$||G_j(\hat{\xi})|| \le \left(\frac{e||A||}{j}\right)^j$$
 uniformly in $\hat{\xi}$.

Therefore, for $\varphi(x) \in \mathcal{S}(R^1)$,

$$(2\pi)^{-1/2}\!\int_{-\infty}^{\infty}(\mathscr{F}arphi)(\hat{arphi})\exp{(tA+i\hat{arphi}B)}d\hat{arphi}=\sum_{j=0}^{\infty}t^{j}arPhi_{j}(arphi)$$

where $\varPhi_j(arphi)=(2\pi)^{-1/2}\!\int_{-\infty}^\infty(\mathscr{F}arphi)(\xi)G_j(\xi)d\xi$ satisfies

$$|| \varPhi_j(arphi) || \leq (2\pi)^{-1/2} \left(\frac{e||A||}{j} \right)^j || (\mathscr{F} arphi) ||.$$

In particular, Φ_i is a tempered distribution on $\mathscr{S}(R^i)$. Define

$$\Phi(z) = \mathscr{L}\left(\sum_{j=0}^{\infty} t^j \Phi_j\right) = \sum_{j=0}^{\infty} \frac{j!}{z^{j+1}} \Phi_j$$

which converges when |z| > ||A||.

By trivial arguments, $\Phi(z)$ has analytic continuation to all z not in the closure of the numerical range of A. The lemma follows immediately for f of the form $\psi(z)\varphi(x)$, which suffices.

THEOREM 6. Suppose A, B act on Hilbert space, and let $A = \operatorname{Re} A + i \operatorname{Im} A$, where $\operatorname{Re} A$, $\operatorname{Im} A$ are self-adjoint. Then for $\varphi \in \mathscr{S}(R^1)$ and |z| large,

(4.3)
$$\Phi(z)\varphi = T(\operatorname{Re} A, \operatorname{Im} A, B) \frac{\varphi(x_3)}{z - (x_i + ix_2)}$$

where T is defined for $g(x_1, x_2, x_3) \in \mathcal{S}(R^3)$ as stated at the beginning of §3 and in [1].

Note. The support of the distribution T contains only (x_1, x_2, x_3) such that $x_1 + ix_2$ is in the closure of the numerical range of A. See [1]. Therefore, (4.3) extends at least to all z outside the closed numerical range of A.

Proof. Both sides expand in Laurent series in z, with coefficient of z^{-j-1} on the left

$$=(j!)arPhi_j(arphi)=(j!)(2\pi)^{-1/2}\int_{-\infty}^{\infty}(\mathscr{F}arphi)(\xi)G_j(\xi)d\xi$$

and on the right

$$egin{aligned} T(\operatorname{Re} A, \operatorname{Im} A, B)[arphi(x_3)(x_i + ix_2)^j] \ &= (2\pi)^{-1/2} \! \int_{-\infty}^{+\infty} (\mathscr{F} arphi)(\xi) T(\operatorname{Re} A, \operatorname{Im} A, B)[e^{ix_3 \xi} (x_i + ix_2)^j] d\xi \ . \end{aligned}$$

Now $G_i(\xi)$ is the coefficient of t^i in $\exp(tA + i\xi B)$ or $\sum_{n=0}^{\infty} 1/n! (tA + i\xi B)^n$ or, by the monomial substitution rule for T in [1],

$$\sum_{n=0}^{\infty} \frac{1}{n!} T(\text{Re } A, \text{Im } A, B) [e^{t(x_1+ix_2)} e^{i\xi x_3}].$$

That is,

$$G_j(\xi) = rac{1}{j!} T(\operatorname{Re} A, \operatorname{Im} A, B) [e^{ix_3\xi}(x_i + ix_2)^j] .$$

Therefore, the two Laurent expansions coincide.

5. Examples. We first examine the hybrid functional in the case when A and B act on the two-dimensional complex Hilbert space.

Let M_1 , M_2 , M_3 be the three hermitian matrices with eigenvalues ± 1 , satisfying $M_iM_j+M_jM_i=0$, $i\neq j$. (E.g. the Pauli matrices.) Then every 2×2 complex matrix is a unique linear combination of M_1 , M_2 , M_3 , I. Since I commutes with everything, we may as well assume that A, B are linear combinations of M_1 , M_2 , M_3 , and up to unitary equivalence and scale changes we can assume $B=M_1$, $A=i\omega\cdot M$ where ω is a triple of complex numbers.

By simple calculations,

$$egin{aligned} \exp\left(A+i\xi B
ight) &= I\cos\sqrt{(\xi+\omega_{\scriptscriptstyle 1})^2+r^2} \ &+ (\xi M_{\scriptscriptstyle 1}+\omega\cdot M)rac{i\sin\sqrt{(\xi+\omega_{\scriptscriptstyle 1})^2+r^2}}{\sqrt{(\xi+\omega_{\scriptscriptstyle 1})^2+r^2}} \end{aligned}$$

where $r^2 = \omega_2^2 + \omega_3^2$. Let $\chi(x)$ denote the characteristic function of the interval $-1 \le x \le 1$, and let δ_1 , δ_{-1} denote the unit measures concentrated at the points 1, -1 respectively.

Essentially, the Fourier transforms we need are

$$egin{aligned} \mathscr{F}^{_{-1}} rac{\sin \sqrt{ar{\xi}^2 + r^2}}{\sqrt{ar{\xi}^2 + r^2}} &= \sqrt{rac{\pi}{2}} J_{\scriptscriptstyle 0}(r\sqrt{1-x^2}) \chi(x) \;. \ \\ \mathscr{F}^{_{-1}} &(\cos \sqrt{ar{\xi}^2 + r^2}) &= \sqrt{rac{\pi}{2}} igg[\delta_{\scriptscriptstyle 1} + \delta_{\scriptscriptstyle -1} + rac{r J_{\scriptscriptstyle 1}(r\sqrt{1-x^2})}{\sqrt{1-x^2}} \chi(x) igg] \;. \end{aligned}$$

Since these formulae are rather hard exercises in contour integration, it is worth mentioning that the first is a corollary of the standard formula (see [4]) in dimension 3:

$$\mathscr{F}^{-1} rac{\sin |\xi|}{|\xi|} = (2\pi)^{3/2} \mu$$
 ,

where μ is the uniformly distributed measure on the unit sphere. Our formula follows from this equation by taking the partial Fourier transform with respect to two variables.

In order to obtain $\Phi(z)$, we replace A by tA (i.e., replace ω by $t\omega$) and compute the Laplace transform. For the Laplace transforms of the Bessel functions see Watson [7], p. 386. The result is:

THEOREM 7. If $B = M_1$, $A = i\omega \cdot M$, then

$$egin{aligned} arPhi(z) &= I\sqrt{rac{\pi}{2}} \Big[rac{\delta_{_1}}{z-i\omega_{_1}} + rac{\delta_{_{-1}}}{z+i\omega_{_1}} + rac{\chi(x)}{((z-i\omega_{_1}x)^2+r^2(1-x^2))^{3/2}} \Big] \ &+ \Big(-M_1rac{\partial}{\partial x} + i\omega\cdot Mrac{\partial}{\partial z}\Big)\sqrt{rac{\pi}{2}}rac{\chi(x)}{((z-i\omega_{_1}x)^2+r^2(1-x^2))^{1/2}}\,. \end{aligned}$$

In particular, $\Phi(z)$ can be analytically continued to the complement of the set $\{z \mid z = i\omega_1 x \pm ir \sqrt{1-x^2}, -1 \le x \le 1\}$ which is the ellipsoid parameterized by $0 \le \theta \le 2\pi$,

$$z = i\omega_1 \cos \theta + ir \sin \theta$$
.

In the case when ω_2/ω_3 is real, this ellipsoid is the boundary of the numerical range of iA, and Lemma 5 gives the actual domain of $\Phi(z)$. The other extreme is the case when $\omega_1=0$ and ω_2/ω_3 is imaginary. Then $\omega_1=r=0$ and $\Phi(z)$ is singular only at z=0, although the numerical range of iA is the nontrivial ellipsoid $\{z\,|\,z=i\omega_2\cos\theta+i\omega_3\sin\theta,\,0\le\theta\le2\pi\}$. In the latter case, the singularities of $\Phi(z)$ coincide with the spectrum of iA (i.e., z=0), but the former case shows that the analytic continuation established for $\mathscr{L}(s,A,B)$ does not carry over to the hybrid functional ST (A,B).

To obtain examples of the Laplace transform, we utilize the elementary fact that when $g \in \mathcal{S}(R^1)$, the Laplace transform is obtained from the Fourier transform by the formula

$$(\mathscr{L}g)(is) = \sqrt{2\pi} \int_{-\infty}^{\infty} \frac{(\mathscr{F}^{-1}g)(x)}{s-x} dx$$
.

One of the coefficients in $\mathcal{L}(s, iB, A)$, with A, B the 2×2 complex matrices described above, is therefore

$$\pi \int_{-1}^{1} \frac{J_0(r\sqrt{1-x^2})}{s-x} dx$$
.

This coefficient, like the others, has nonunique analytic continuation to all $s \neq \pm 1$. The difference between two values is an integer times $\pi J_0(r\sqrt{1-s^2})$.

This result is easily extended by analytic continuation to any 2×2 matrices A, B.

REFERENCES

- 1. R. F. V. Anderson, The Weyl functional calculus, J. Functional Analysis, 4 No. 2, (1969), 240-267.
- 2. ——, On the Weyl functional calculus, J. Functional Analysis, 6 No. 1, (1970), 110-115.
- 3. ———, The multiplicative Weyl functional calculus, J. Functional Analysis, 9 No. 4, (1972), 423-440.
- 4. J. Arsac, Fourier Transforms and the Theory of Distributions, Englewood Cliffs, N. J., 1966.
- 5. L. V. Hormander, An Introduction to Complex Analysis in Several Variables, New York, 1966.
- 6. F. Riesz and B. Sz-Nagy, Functional Analysis, New York, 1955.
- 7. G. N. Watson, A Treaties on the Theory of Bessel Functions, 2nd ed., Cambridge University Press, 1944.
- 8. D. V. Widder, The Laplace Transform, Princeton University Press, 1941.

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