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FUNCTIONALS ON CONTINUOUS FUNCTIONS

JOHN ROBERT BAXTER AND RAFAEL VAN SEVEREN CHACON

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J. R. BAXTER AND R. V. CHACON

Let $\mathcal{C}(M)$ be the space of continuous functions on a compact metric space M . In a previous paper a class of nonlinear functionals Φ on $\mathcal{C}([0, 1] \times [0, 1])$ was constructed, such that each Φ satisfied:

- (i) $\lim_{\|f\| \rightarrow 0} \Phi(f) = 0$,
- (ii) $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever $fg = 0$, and
- (iii) $\Phi(f + \alpha) = \Phi(f) + \Phi(\alpha)$ for any constant α .

In this paper we show that the dimensionality of $[0, 1] \times [0, 1]$ is what makes the construction work. More precisely, we show that if Φ is a functional on $\mathcal{C}(M)$ satisfying (i), (ii), and (iii), and if the dimension of M is less than two, then Φ must be linear.

1. Introduction. Let M be a compact metric space. Let $\mathcal{C}(M)$ be the space of continuous real-valued functions on M . In this paper, we will prove the following result:

THEOREM 1. *Let $\Phi: \mathcal{C}(M) \rightarrow \mathbf{R}$ ($\mathbf{R} =$ the real numbers) be a functional such that:*

- (i) $\lim_{\|f\| \rightarrow 0} \Phi(f) = 0$, ($\|f\| = \sup_{x \in M} |f(x)|$)
- (ii) $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever $fg = 0$
- (iii) $\Phi(f + \alpha) = \Phi(f) + \Phi(\alpha)$ for all $f \in \mathcal{C}(M)$, $\alpha \in \mathbf{R}$.

Then if M has dimension no greater than one, Φ must be linear.

The additivity properties (ii) and (iii) may also be expressed by one condition:

- (ii)' $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever g is constant on $\{x \mid f(x) \neq 0\}$.

It is also easy to see that we must have $\Phi(\alpha) = \alpha\Phi(1)$ for all $\alpha \in \mathbf{R}$.

It has been shown in [2] that there exist nonlinear functionals Φ on $\mathcal{C}([0, 1] \times [0, 1])$ which are bounded, continuous, monotonic, and satisfy conditions (ii) and (iii). Thus Theorem 1 does not extend to spaces of dimension greater than one.

In [1], a proof of Theorem 1 is given for the special case $M = [0, 1]$. We will use this case of Theorem 1 to prove the general case. In §2 it is shown that Theorem 1 is equivalent to the following result:

THEOREM 2. *For each $f \in \mathcal{C}(M)$, let $\mathcal{B}_f = \{f^{-1}(E) \mid E \subseteq \mathbf{R}, E \text{ Borel}\}$. Suppose a measure μ_f on \mathcal{B}_f is given, for each $f \in \mathcal{C}(M)$, such that:*

(i) the measures μ_f are uniformly bounded in total variation, and

(ii) the measures μ_f are consistent, in the sense that if $\mathcal{B}_f \subseteq \mathcal{B}_g$, then $\mu_f = \mu_g$ on \mathcal{B}_f .

Then if M has dimension no greater than one, a measure μ on the Borel sets of M can be found, which is the common extension of all the μ_f .

Theorem 2 is obvious if M is the unit interval, but not if M is the unit circle. Theorem 2 will be proved in § 3.

2. Construction of a set function. For each $f \in \mathcal{E}(M)$, let \mathcal{L}_f be the space of continuous functions $g \in \mathcal{E}(M)$ which are measurable with respect to \mathcal{B}_f . It is easy to see that $g \in \mathcal{L}_f$ if and only if $g(x) = g(y)$ whenever $f(x) = f(y)$, and that this means g is of the form $h \circ f$, where h is a continuous function on R .

LEMMA 1. Φ satisfies conditions (i), (ii), and (iii) of Theorem 1 if and only if:

(i) Φ is bounded, that is, there exists k such that $|\Phi(f)| \leq k \|f\|$ for all $f \in \mathcal{E}(M)$,

(ii) Φ is linear on each space \mathcal{L}_f .

Proof. Assume Φ satisfies (i), (ii) and (iii) of Theorem 1. Fix $f \in \mathcal{E}(M)$. Let I be a compact interval containing $f(M)$.

Define Φ^* on $\mathcal{E}(I)$ by the equation $\Phi^*(h) = \Phi(h \circ f)$. Clearly Φ^* satisfies conditions (i), (ii), and (iii) of Theorem 1. By the special case of Theorem 1 that is proved in [1], Φ^* must be linear. It follows at once that Φ is linear on \mathcal{L}_f .

Since Φ is continuous at 0, there exists $r > 0$ such that

$$\|f\| \leq r \text{ implies } |\Phi(f)| \leq 1.$$

Then for any $f \in \mathcal{E}(M)$, $f \neq 0$,

$$|\Phi(f)| = \left| \frac{\|f\|}{r} \Phi\left(\frac{rf}{\|f\|}\right) \right| \leq \frac{1}{r} \|f\|.$$

Thus Φ is bounded.

Now assume Φ satisfies conditions (i) and (ii) of Lemma 1. Then condition (i) of Theorem 1 clearly holds.

To prove that condition (ii) of Theorem 1 holds, let us first assume that f and g are in $\mathcal{E}(M)$, with $f \geq 0$, $g \leq 0$, and $fg = 0$.

Then $f = (f + g) \vee 0$ and $g = (f + g) \wedge 0$, so that f and g are both in \mathcal{L}_{f+g} . Hence $\Phi(f + g) = \Phi(f) + \Phi(g)$.

Now assume that $f \geq 0$, $g \geq 0$, and $fg = 0$. Then by the preceding argument f and g are both in \mathcal{L}_{f-g} , so again $\Phi(f+g) = \Phi(f) + \Phi(g)$.

Finally, for arbitrary f and g in $\mathcal{E}(M)$ with $fg = 0$, let $f_1 = f \vee 0$, $f_2 = f \wedge 0$, $g_1 = g \vee 0$, $g_2 = g \wedge 0$. Then

$$\begin{aligned} \Phi(f+g) &= \Phi(f_1 + f_2 + g_1 + g_2) \\ &= \Phi(f_1 + g_1) + \Phi(f_2 + g_2) \quad \text{by the first case,} \\ &= \Phi(f_1) + \Phi(g_1) + \Phi(f_2) + \Phi(g_2) \quad \text{by the second case,} \\ &= \Phi(f_1 + f_2) + \Phi(g_1 + g_2) \quad \text{by the first case,} \\ &= \Phi(f) + \Phi(g). \quad \text{Thus condition (ii) of Theorem 1 holds.} \end{aligned}$$

Condition (iii) of Theorem 1 clearly holds, so Lemma 1 is proved.

Using Lemma 1 and the Riesz representation theorem it is easy to see that for each functional Φ satisfying conditions (i), (ii), and (iii) of Theorem 1 we can find a system of measures μ_f satisfying conditions (i) and (ii) of Theorem 2, and such that $\Phi(f) = \int f d\mu_f$ for each $f \in \mathcal{E}(M)$. Conversely, if μ_f , $f \in \mathcal{E}(M)$, is a system of measures satisfying conditions (i) and (ii) of Theorem 2, then Lemma 1 implies that the functional Φ defined by $\Phi(f) = \int f d\mu_f$ must satisfy conditions (i), (ii), and (iii) of Theorem 1. It follows at once that Theorems 1 and 2 are equivalent.

In what follows we will use both Φ and the corresponding system of measures μ_f .

LEMMA 2. *Let f and g be in $\mathcal{E}(M)$. Let K be a closed set in $\mathcal{B}_f \cap \mathcal{B}_g$. Then $\mu_f(K) = \mu_g(K)$.*

Proof. $f(K)$ is a compact set in \mathbf{R} . It is easy to see that one can find a sequence of continuous functions h_n on \mathbf{R} such that $0 \leq h_n \leq 1$, $h_n = 1$ on a neighborhood of $f(K)$, $h_n = 1$ on the support of h_{n+1} , and the intersection of the supports of the h_n is $f(K)$.

Let $f_n = h_n \circ f$. Then clearly $0 \leq f_n \leq 1$, $f_n = 1$ on a neighborhood of K , $f_n = 1$ on the support of f_{n+1} , and the intersection of the supports of the f_n is K .

Let $g_n = p_n \circ g$ be a sequence having the same properties as the f_n . Fix f_n . Then $f_n = 1$ on a neighborhood, A , of K . Since the intersection of the supports of the g_n is K , it follows that for sufficiently large m the support of g_m will be contained in A . Hence, by choosing subsequences and relabelling, we may assume that, in addition to the properties mentioned above, f_n and g_n are also such that $f_n = 1$ on a neighborhood of the support of g_n , and $g_n = 1$ on a neighborhood of the support of f_{n+1} .

Since the f_n are uniformly bounded, and $f_n \rightarrow \chi_K$ pointwise as

$n \rightarrow \infty$, we have $\Phi(f_n) = \int f_n d\mu_f \rightarrow \mu_f(K)$ as $n \rightarrow \infty$. Similarly $\Phi(g_n) \rightarrow \mu_g(K)$ as $n \rightarrow \infty$. Suppose $\mu_f(K) > \mu_g(K)$. Choose $\delta > 0$, $\delta < \mu_f(K) - \mu_g(K)$. For sufficiently large n we must have $\Phi(f_n) > \Phi(g_n) + \delta$. By relabelling we may assume that $\Phi(f_n) > \Phi(g_n) + \delta$ for all n .

Let u_n be a continuous function on M such that $0 \leq u_n \leq 1$, $u_n = 0$ on the support of g_n , and $u_n = 1$ on $\{x \mid f_n(x) < 1\}$. Let

$$v_n = f_n - u_n f_n - g_n .$$

It is easy to check that $0 \leq v_n \leq 1$, and the support of v_n is contained in

$$\{x \mid f_n(x) = 1\} - \{x \mid g_n(x) = 1\} .$$

Hence $\Phi(-v_n + f_n) = \Phi(-v_n) + \Phi(f_n)$, by the additivity property (ii)' of Φ . That is, $\Phi(u_n f_n + g_n) = \Phi(-v_n) + \Phi(f_n)$. Since $u_n f_n = 0$ on the support of g_n , we have $\Phi(u_n f_n + g_n) = \Phi(u_n f_n) + \Phi(g)$ by the additivity of Φ again. Thus $\Phi(u_n f_n) + \Phi(g_n) = \Phi(-v_n) + \Phi(f_n)$. Hence $\Phi(u_n f_n) > \Phi(-v_n) + \delta$, and so $\sum_{n=1}^m \Phi(u_n f_n) > \sum_{n=1}^m \Phi(-v_n) + m\delta$, for all m .

It is easy to check that the supports of the $u_n f_n$ are pairwise disjoint, as are the supports of the v_n . Hence

$$\Phi\left(\sum_{n=1}^m u_n f_n\right) > \Phi\left(\sum_{n=1}^m (-v_n)\right) + m\delta ,$$

by additivity, for all m .

The functions $\sum_{n=1}^m u_n f_n$ and $\sum_{n=1}^m (-v_n)$ are uniformly bounded in m . Hence the last inequality contradicts the boundedness of Φ . Hence our original supposition, $\mu_f(K) > \mu_g(K)$, was false. This proves Lemma 2.

Since M is a metric space, it is easy to see that every closed set E and every open set E occurs in some \mathcal{B}_f .

DEFINITION 1. Let us write $\mu_f(E) = \mu(E)$ for E closed or E open, since the number has been shown to be independent of f .

LEMMA 3. *The set function μ is bounded and additive wherever defined.*

Proof. μ is bounded because the total variation of the μ_f 's is uniformly bounded.

Let E_1 and E_2 be sets, with $E_1 \cap E_2 = \phi$, such that $\mu(E_1)$, $\mu(E_2)$, and $\mu(E_1 \cup E_2)$ are defined. We may have E_1, E_2 open, E_1, E_2 closed, E_1 open, E_2 closed, and $E_1 \cup E_2$ open, or E_1 open, E_2 closed, and $E_1 \cup E_2$ closed. In each of the four possible cases it is easy to find a function $f \in \mathcal{C}(M)$ such that E_1 and E_2 are in \mathcal{B}_f . This proves Lemma 3.

LEMMA 4. *Let G_n be a monotone increasing sequence of open sets, with union G . Let F_n be a sequence of closed sets such that $G_n \subseteq F_n \subseteq G$ for all n . Then $\mu(G_n) \rightarrow \mu(G)$ and $\mu(F_n) \rightarrow \mu(G)$ as $n \rightarrow \infty$.*

Proof. Suppose $\mu(G_n) \not\rightarrow \mu(G)$ or $\mu(F_n) \not\rightarrow \mu(G)$. Then there exists a $\delta > 0$ and a subsequence n_j such that

$$|\mu(G_{n_j}) - \mu(G)| + |\mu(F_{n_j}) - \mu(G)| > \delta$$

for all j . Since the F_n are compact we can choose n_j so that $F_{n_j} \subseteq G_{n_{j+1}}$. It is then a straightforward matter to construct $f \in \mathcal{C}(M)$ such that $G_{n_j}, E_{n_j} \in \mathcal{B}_f$ for all j . This contradiction proves the lemma.

3. Proof of the theorems. In this section we will prove:

THEOREM 3. *Let μ be a real-valued set function defined for closed subsets and for open subsets of M , such that:*

- (i) μ is bounded and additive wherever defined, and
- (ii) μ has the continuity property described in Lemma 4.

Then if M has dimension no greater than one, μ can be extended to a measure on the Borel sets of M .

We can apply Theorem 3 to the set function μ constructed in the previous section. The Borel measure $\hat{\mu}$ which is an extension of μ agrees with each measure μ_f on all closed sets in \mathcal{B}_f . Since each μ_f is obviously regular, $\hat{\mu}$ must be an extension of μ_f . Thus Theorem 2 is proved, and hence Theorem 1 also.

From now on let μ be any set function satisfying conditions (i) and (ii) of Theorem 3.

LEMMA 5. *Let F_n be a monotone decreasing sequence of closed sets, having intersection F . Let G_n be a sequence of open sets such that $F_n \supseteq G_n \supseteq F$ for all n . Then $\mu(F_n) \rightarrow \mu(F)$ and $\mu(G_n) \rightarrow \mu(F)$ as $n \rightarrow \infty$.*

Proof. Follows from condition (ii) by taking complements and using the additivity property.

DEFINITION 2. For any set $E \subseteq M$, define

$$\nu(E) = \sup \{ \mu(F) \mid F \subseteq E, F \text{ closed} \} .$$

Since μ is bounded, so is ν . Clearly ν is monotone.

LEMMA 6. *Let E_1 and E_2 be disjoint subsets of M . Then $\nu(E_1 \cup E_2) \geq \nu(E_1) + \nu(E_2)$. If E_1 and E_2 are either both open or both closed,*

then $\nu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2)$.

Proof. Follows from the additivity of μ .

LEMMA 7. *Let G be open. Then*

$$\nu(G) = \sup \{ \mu(H) \mid H \subseteq G, H \text{ open} \}.$$

Proof. Follows from the continuity of μ .

We pause now for a general topological lemma.

LEMMA 8. *Let X be a locally compact separable metric space of dimension 0. Then X is a countable union of monotone increasing sets that are both compact and open.*

Proof. From the definition of dimension 0, each point x has arbitrarily small neighborhoods G_x which are both closed and open.

By choosing G_x small enough, it can therefore be made both compact and open.

Since $X = \bigcup_{x \in X} G_x$, and X has a countable base, we can find x_1, x_2, \dots such that $X = \bigcup_{n=1}^{\infty} G_{x_n}$. Let $K_n = \bigcup_{j=1}^n G_{x_j}$. Then each K_n is both compact and open, and $K_n \uparrow X$.

Now we return to M, μ , and ν .

LEMMA 9. *Let G be open. Let E be open, $E \subseteq G$, such that $\partial E \cap G$ has dimension 0. Then $\mu(G) \leq \nu(E) + \nu(G - E)$.*

Proof. Let $D = \partial E \cap G$. Let $H = G - \bar{E}$. Then the sets E, D , and H are mutually disjoint, and $G = E \cup D \cup H$.

Since D is a closed subset of the locally compact separable metric space G , D is a locally compact separable metric space also.

By Lemma 8, we can find sets K_n which are both compact and open in D , such that $K_n \uparrow D$.

Let $K_n = A_n \cap D$, where A_n is open. Since K_n is compact we may choose A_n such that $\bar{A}_n \subseteq G$. By taking unions if necessary we may choose the A_n to be increasing.

Let E_n and H_n be open sets such that $\bar{E}_n \subseteq E, \bar{H}_n \subseteq H$ for all n , $E_n \uparrow E$ and $H_n \uparrow H$. Let $G_n = E_n \cup A_n \cup H_n$. Then G_n is open, $\bar{G}_n \subseteq G$, and $G_n \uparrow G$. Then $\mu(G_n) \rightarrow \mu(G)$ as $n \rightarrow \infty$, by continuity.

$$\begin{aligned} \text{But for all } n, G_n &= (G_n \cap E) \cup (G_n \cap D) \cup (G_n \cap H) \\ &= (G_n \cap E) \cup K_n \cup (G_n \cap H). \end{aligned}$$

Thus $\mu(G_n) = \mu(G_n \cap E) + \mu(K_n) + \mu(G_n \cap H)$, by additivity,

$$\leq \nu(G_n \cap E) + \nu(K_n) + \nu(G_n \cap H)$$

$$\leq \nu(E) + \nu(D) + \nu(H) \leq \nu(E) + \nu(G - E).$$

This proves Lemma 9.

LEMMA 10. *Let G be an open set. Let E be open, $E \subseteq G$, such that $\partial E \cap G$ has dimension 0. Then $\nu(G) = \nu(E) + \nu(G - E)$.*

Proof. Let $\varepsilon > 0$ be given. Choose H open, $H \subseteq G$, such that $\mu(H) \geq \nu(G) - \varepsilon$. This is possible by Lemma 7.

Then $\partial(E \cap H) \cap H = \partial E \cap H \subseteq \partial E \cap G$. Hence $\partial(E \cap H) \cap H$ has dimension 0. By Lemma 7, $\mu(H) \leq \nu(E \cap H) + \nu(H - E \cap H) \leq \nu(E) + \nu(G - E)$. Hence $\nu(G) \leq \nu(E) + \nu(G - E)$.

The reverse inequality holds by Lemma 6, so Lemma 10 is proved.

From now on in this section, let M have dimension at most one.

LEMMA 11. *Let G_1 and G_2 be open, with union G . Then $\nu(G) \leq \nu(G_1) + \nu(G_2)$.*

Proof. $G_1 - G_2$ and $G_2 - G_1$ are disjoint and relatively closed in G . G is a separable metric space of dimension no larger than 1. Hence by Theorem 1 in [3], section 27II, page 290, we can find an open set $E \subseteq G$ such that $E \supseteq G_1 - G_2$, $\bar{E} \cap (G_2 - G_1) = \emptyset$, and $\partial E \cap G$ has dimension 0.

By Lemma 10,

$$\nu(G) = \nu(E) + \nu(G - E) \leq \nu(G_1) + \nu(G_2).$$

LEMMA 12. *Let G_n be a sequence of open sets. Let $G = \bigcup_{n=1}^{\infty} G_n$. Then $\nu(G) \leq \sum_{n=1}^{\infty} \nu(G_n)$.*

Proof. Let $\varepsilon > 0$ be given. Choose F closed, $F \subseteq G$ such that $\mu(F) \geq \nu(G) - \varepsilon$.

Then there exists n such that $F \subseteq \bigcup_{j=1}^n G_j$. Hence $\sum_{j=1}^{\infty} \nu(G_j) \geq \sum_{j=1}^n \nu(G_j) \geq \nu(\bigcup_{j=1}^n G_j)$, by Lemma 11, $\geq \mu(F)$ by definition.

This proves Lemma 12.

DEFINITION 3. For any set $E \subseteq M$, define $\nu^*(E) = \inf \{ \nu(G) \mid E \subseteq G, G \text{ open} \}$. Clearly $\nu^*(E) = \nu(E)$ when E is open.

LEMMA 13. ν^* is an outer measure.

Proof. Follows from Lemma 12.

LEMMA 14. *Every open set is measurable with respect to ν^* , in the sense of Caratheodory.*

Proof. Let G be open. Let E be any set. We know

$$\nu^*(E) \leq \nu^*(E \cap G) + \nu^*(E - G),$$

since ν^* is an outer measure. We must show that

$$\nu^*(E) \geq \nu^*(E \cap G) + \nu^*(E - G).$$

Choose any open set H such that $E \subseteq H$. Let $\varepsilon > 0$ be given. Choose F closed, $F \subseteq G \cap H$, such that $\nu(F) \geq \nu(G \cap H) - \varepsilon$. Then $\nu(H) \geq \nu(F) + \nu(H - F)$, by Lemma 6, $\geq \nu(G \cap H) - \varepsilon + \nu(H - F) \geq \nu^*(E \cap G) - \varepsilon + \nu^*(E - G)$ by definition.

Hence $\nu(H) \geq \nu^*(E \cap G) + \nu^*(E - G)$. By definition, then, $\nu^*(E) \geq \nu^*(E \cap G) + \nu^*(E - G)$, and Lemma 14 is proved.

Because of Lemma 14 we know that ν^* defines a measure on a σ -algebra of sets that includes the Borel sets of M .

Proof of Theorem 3. First suppose that μ is nonnegative. Let G be open. By Lemma 7, $\mu(G) \leq \nu(G)$. On the other hand, for any closed subset F of G , $\mu(F) \leq \mu(F) + \mu(G - F) = \mu(G)$. Thus $\mu(G) = \nu(G)$. ν^* is a measure on the Borel sets of M which agrees with μ on open sets and hence on all sets in the domain of μ .

Now let μ be arbitrary. Consider the set function $\omega = \nu^* - \mu$, defined for closed subsets of M and for open subsets of M . ω is nonnegative by Lemma 7. By what has already been proved, ω can be extended to a Borel measure. But then $\mu = \nu^* - \omega$ can be extended also, so the theorem is proved.

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