

Pacific Journal of Mathematics

FUNCTIONALS ON CONTINUOUS FUNCTIONS

JOHN ROBERT BAXTER AND RAFAEL VAN SEVEREN CHACON

FUNCTIONALS ON CONTINUOUS FUNCTIONS

J. R. BAXTER AND R. V. CHACON

Let $\mathcal{C}(M)$ be the space of continuous functions on a compact metric space M . In a previous paper a class of nonlinear functionals Φ on $\mathcal{C}([0, 1] \times [0, 1])$ was constructed, such that each Φ satisfied:

- (i) $\lim_{\|f\| \rightarrow 0} \Phi(f) = 0$,
- (ii) $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever $fg = 0$, and
- (iii) $\Phi(f + \alpha) = \Phi(f) + \Phi(\alpha)$ for any constant α .

In this paper we show that the dimensionality of $[0, 1] \times [0, 1]$ is what makes the construction work. More precisely, we show that if Φ is a functional on $\mathcal{C}(M)$ satisfying (i), (ii), and (iii), and if the dimension of M is less than two, then Φ must be linear.

1. Introduction. Let M be a compact metric space. Let $\mathcal{C}(M)$ be the space of continuous real-valued functions on M . In this paper, we will prove the following result:

THEOREM 1. *Let $\Phi: \mathcal{C}(M) \rightarrow \mathbf{R}$ (\mathbf{R} = the real numbers) be a functional such that:*

- (i) $\lim_{\|f\| \rightarrow 0} \Phi(f) = 0$, ($\|f\| = \sup_{x \in M} |f(x)|$)
- (ii) $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever $fg = 0$
- (iii) $\Phi(f + \alpha) = \Phi(f) + \Phi(\alpha)$ for all $f \in \mathcal{C}(M)$, $\alpha \in \mathbf{R}$.

Then if M has dimension no greater than one, Φ must be linear.

The additivity properties (ii) and (iii) may also be expressed by one condition:

- (ii)' $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever g is constant on $\{x \mid f(x) \neq 0\}$.

It is also easy to see that we must have $\Phi(\alpha) = \alpha\Phi(1)$ for all $\alpha \in \mathbf{R}$.

It has been shown in [2] that there exist nonlinear functionals Φ on $\mathcal{C}([0, 1] \times [0, 1])$ which are bounded, continuous, monotonic, and satisfy conditions (ii) and (iii). Thus Theorem 1 does not extend to spaces of dimension greater than one.

In [1], a proof of Theorem 1 is given for the special case $M = [0, 1]$. We will use this case of Theorem 1 to prove the general case. In § 2 it is shown that Theorem 1 is equivalent to the following result:

THEOREM 2. *For each $f \in \mathcal{C}(M)$, let $\mathcal{B}_f = \{f^{-1}(E) \mid E \subseteq \mathbf{R}, E \text{ Borel}\}$. Suppose a measure μ_f on \mathcal{B}_f is given, for each $f \in \mathcal{C}(M)$, such that:*

(i) the measures μ_f are uniformly bounded in total variation, and

(ii) the measures μ_f are consistent, in the sense that if $\mathcal{B}_f \subseteq \mathcal{B}_g$, then $\mu_f = \mu_g$ on \mathcal{B}_f .

Then if M has dimension no greater than one, a measure μ on the Borel sets of M can be found, which is the common extension of all the μ_f .

Theorem 2 is obvious if M is the unit interval, but not if M is the unit circle. Theorem 2 will be proved in § 3.

2. Construction of a set function. For each $f \in \mathcal{E}(M)$, let \mathcal{L}_f be the space of continuous functions $g \in \mathcal{E}(M)$ which are measurable with respect to \mathcal{B}_f . It is easy to see that $g \in \mathcal{L}_f$ if and only if $g(x) = g(y)$ whenever $f(x) = f(y)$, and that this means g is of the form $h \circ f$, where h is a continuous function on \mathbf{R} .

LEMMA 1. Φ satisfies conditions (i), (ii), and (iii) of Theorem 1 if and only if:

(i) Φ is bounded, that is, there exists k such that $|\Phi(f)| \leq k \|f\|$ for all $f \in \mathcal{E}(M)$,

(ii) Φ is linear on each space \mathcal{L}_f .

Proof. Assume Φ satisfies (i), (ii) and (iii) of Theorem 1. Fix $f \in \mathcal{E}(M)$. Let I be a compact interval containing $f(M)$.

Define Φ^* on $\mathcal{E}(I)$ by the equation $\Phi^*(h) = \Phi(h \circ f)$. Clearly Φ^* satisfies conditions (i), (ii), and (iii) of Theorem 1. By the special case of Theorem 1 that is proved in [1], Φ^* must be linear. It follows at once that Φ is linear on \mathcal{L}_f .

Since Φ is continuous at 0, there exists $r > 0$ such that

$$\|f\| \leq r \text{ implies } |\Phi(f)| \leq 1.$$

Then for any $f \in \mathcal{E}(M)$, $f \neq 0$,

$$|\Phi(f)| = \left| \frac{\|f\|}{r} \Phi\left(\frac{rf}{\|f\|}\right) \right| \leq \frac{1}{r} \|f\|.$$

Thus Φ is bounded.

Now assume Φ satisfies conditions (i) and (ii) of Lemma 1. Then condition (i) of Theorem 1 clearly holds.

To prove that condition (ii) of Theorem 1 holds, let us first assume that f and g are in $\mathcal{E}(M)$, with $f \geq 0$, $g \leq 0$, and $fg = 0$.

Then $f = (f + g) \vee 0$ and $g = (f + g) \wedge 0$, so that f and g are both in \mathcal{L}_{f+g} . Hence $\Phi(f + g) = \Phi(f) + \Phi(g)$.

Now assume that $f \geq 0$, $g \geq 0$, and $fg = 0$. Then by the preceding argument f and g are both in \mathcal{L}_{f-g} , so again $\Phi(f+g) = \Phi(f) + \Phi(g)$.

Finally, for arbitrary f and g in $\mathcal{C}(M)$ with $fg = 0$, let $f_1 = f \vee 0$, $f_2 = f \wedge 0$, $g_1 = g \vee 0$, $g_2 = g \wedge 0$. Then

$$\begin{aligned} \Phi(f+g) &= \Phi(f_1 + f_2 + g_1 + g_2) \\ &= \Phi(f_1 + g_1) + \Phi(f_2 + g_2) \quad \text{by the first case,} \\ &= \Phi(f_1) + \Phi(g_1) + \Phi(f_2) + \Phi(g_2) \quad \text{by the second case,} \\ &= \Phi(f_1 + f_2) + \Phi(g_1 + g_2) \quad \text{by the first case,} \\ &= \Phi(f) + \Phi(g). \quad \text{Thus condition (ii) of Theorem 1 holds.} \end{aligned}$$

Condition (iii) of Theorem 1 clearly holds, so Lemma 1 is proved.

Using Lemma 1 and the Riesz representation theorem it is easy to see that for each functional Φ satisfying conditions (i), (ii), and (iii) of Theorem 1 we can find a system of measures μ_f satisfying conditions (i) and (ii) of Theorem 2, and such that $\Phi(f) = \int f d\mu_f$ for each $f \in \mathcal{C}(M)$. Conversely, if μ_f , $f \in \mathcal{C}(M)$, is a system of measures satisfying conditions (i) and (ii) of Theorem 2, then Lemma 1 implies that the functional Φ defined by $\Phi(f) = \int f d\mu_f$ must satisfy conditions (i), (ii), and (iii) of Theorem 1. It follows at once that Theorems 1 and 2 are equivalent.

In what follows we will use both Φ and the corresponding system of measures μ_f .

LEMMA 2. *Let f and g be in $\mathcal{C}(M)$. Let K be a closed set in $\mathcal{B}_f \cap \mathcal{B}_g$. Then $\mu_f(K) = \mu_g(K)$.*

Proof. $f(K)$ is a compact set in \mathbf{R} . It is easy to see that one can find a sequence of continuous functions h_n on \mathbf{R} such that $0 \leq h_n \leq 1$, $h_n = 1$ on a neighborhood of $f(K)$, $h_n = 0$ on the support of h_{n+1} , and the intersection of the supports of the h_n is $f(K)$.

Let $f_n = h_n \circ f$. Then clearly $0 \leq f_n \leq 1$, $f_n = 1$ on a neighborhood of K , $f_n = 0$ on the support of f_{n+1} , and the intersection of the supports of the f_n is K .

Let $g_n = p_n \circ g$ be a sequence having the same properties as the f_n . Fix f_n . Then $f_n = 1$ on a neighborhood, A , of K . Since the intersection of the supports of the g_n is K , it follows that for sufficiently large m the support of g_m will be contained in A . Hence, by choosing subsequences and relabelling, we may assume that, in addition to the properties mentioned above, f_n and g_n are also such that $f_n = 1$ on a neighborhood of the support of g_n , and $g_n = 1$ on a neighborhood of the support of f_{n+1} .

Since the f_n are uniformly bounded, and $f_n \rightarrow \chi_K$ pointwise as

$n \rightarrow \infty$, we have $\Phi(f_n) = \int f_n d\mu_f \rightarrow \mu_f(K)$ as $n \rightarrow \infty$. Similarly $\Phi(g_n) \rightarrow \mu_g(K)$ as $n \rightarrow \infty$. Suppose $\mu_f(K) > \mu_g(K)$. Choose $\delta > 0$, $\delta < \mu_f(K) - \mu_g(K)$. For sufficiently large n we must have $\Phi(f_n) > \Phi(g_n) + \delta$. By relabelling we may assume that $\Phi(f_n) > \Phi(g_n) + \delta$ for all n .

Let u_n be a continuous function on M such that $0 \leq u_n \leq 1$, $u_n = 0$ on the support of g_n , and $u_n = 1$ on $\{x \mid f_n(x) < 1\}$. Let

$$v_n = f_n - u_n f_n - g_n.$$

It is easy to check that $0 \leq v_n \leq 1$, and the support of v_n is contained in

$$\{x \mid f_n(x) = 1\} - \{x \mid g_n(x) = 1\}.$$

Hence $\Phi(-v_n + f_n) = \Phi(-v_n) + \Phi(f_n)$, by the additivity property (ii)' of Φ . That is, $\Phi(u_n f_n + g_n) = \Phi(-v_n) + \Phi(f_n)$. Since $u_n f_n = 0$ on the support of g_n , we have $\Phi(u_n f_n + g_n) = \Phi(u_n f_n) + \Phi(g)$ by the additivity of Φ again. Thus $\Phi(u_n f_n) + \Phi(g_n) = \Phi(-v_n) + \Phi(f_n)$. Hence $\Phi(u_n f_n) > \Phi(-v_n) + \delta$, and so $\sum_{n=1}^m \Phi(u_n f_n) > \sum_{n=1}^m \Phi(-v_n) + m\delta$, for all m .

It is easy to check that the supports of the $u_n f_n$ are pairwise disjoint, as are the supports of the v_n . Hence

$$\Phi\left(\sum_{n=1}^m u_n f_n\right) > \Phi\left(\sum_{n=1}^m (-v_n)\right) + m\delta,$$

by additivity, for all m .

The functions $\sum_{n=1}^m u_n f_n$ and $\sum_{n=1}^m (-v_n)$ are uniformly bounded in m . Hence the last inequality contradicts the boundedness of Φ . Hence our original supposition, $\mu_f(K) > \mu_g(K)$, was false. This proves Lemma 2.

Since M is a metric space, it is easy to see that every closed set E and every open set E occurs in some \mathcal{B}_f .

DEFINITION 1. Let us write $\mu_f(E) = \mu(E)$ for E closed or E open, since the number has been shown to be independent of f .

LEMMA 3. *The set function μ is bounded and additive wherever defined.*

Proof. μ is bounded because the total variation of the μ_f 's is uniformly bounded.

Let E_1 and E_2 be sets, with $E_1 \cap E_2 = \phi$, such that $\mu(E_1)$, $\mu(E_2)$, and $\mu(E_1 \cup E_2)$ are defined. We may have E_1, E_2 open, E_1, E_2 closed, E_1 open, E_2 closed, and $E_1 \cup E_2$ open, or E_1 open, E_2 closed, and $E_1 \cup E_2$ closed. In each of the four possible cases it is easy to find a function $f \in \mathcal{C}(M)$ such that E_1 and E_2 are in \mathcal{B}_f . This proves Lemma 3.

LEMMA 4. Let G_n be a monotone increasing sequence of open sets, with union G . Let F_n be a sequence of closed sets such that $G_n \subseteq F_n \subseteq G$ for all n . Then $\mu(G_n) \rightarrow \mu(G)$ and $\mu(F_n) \rightarrow \mu(G)$ as $n \rightarrow \infty$.

Proof. Suppose $\mu(G_n) \not\rightarrow \mu(G)$ or $\mu(F_n) \not\rightarrow \mu(G)$. Then there exists a $\delta > 0$ and a subsequence n_j such that

$$|\mu(G_{n_j}) - \mu(G)| + |\mu(F_{n_j}) - \mu(G)| > \delta$$

for all j . Since the F_n are compact we can choose n_j so that $F_{n_j} \subseteq G_{n_{j+1}}$. It is then a straightforward matter to construct $f \in \mathcal{C}(M)$ such that $G_{n_j}, F_{n_j} \in \mathcal{B}_f$ for all j . This contradiction proves the lemma.

3. Proof of the theorems. In this section we will prove:

THEOREM 3. Let μ be a real-valued set function defined for closed subsets and for open subsets of M , such that:

- (i) μ is bounded and additive wherever defined, and
- (ii) μ has the continuity property described in Lemma 4.

Then if M has dimension no greater than one, μ can be extended to a measure on the Borel sets of M .

We can apply Theorem 3 to the set function μ constructed in the previous section. The Borel measure $\hat{\mu}$ which is an extension of μ agrees with each measure μ_f on all closed sets in \mathcal{B}_f . Since each μ_f is obviously regular, $\hat{\mu}$ must be an extension of μ_f . Thus Theorem 2 is proved, and hence Theorem 1 also.

From now on let μ be any set function satisfying conditions (i) and (ii) of Theorem 3.

LEMMA 5. Let F_n be a monotone decreasing sequence of closed sets, having intersection F . Let G_n be a sequence of open sets such that $F_n \supseteq G_n \supseteq F$ for all n . Then $\mu(F_n) \rightarrow \mu(F)$ and $\mu(G_n) \rightarrow \mu(F)$ as $n \rightarrow \infty$.

Proof. Follows from condition (ii) by taking complements and using the additivity property.

DEFINITION 2. For any set $E \subseteq M$, define

$$\nu(E) = \sup \{ \mu(F) \mid F \subseteq E, F \text{ closed} \}.$$

Since μ is bounded, so is ν . Clearly ν is monotone.

LEMMA 6. Let E_1 and E_2 be disjoint subsets of M . Then $\nu(E_1 \cup E_2) \geq \nu(E_1) + \nu(E_2)$. If E_1 and E_2 are either both open or both closed,

then $\nu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2)$.

Proof. Follows from the additivity of μ .

LEMMA 7. *Let G be open. Then*

$$\nu(G) = \sup \{ \mu(H) \mid H \subseteq G, H \text{ open} \} .$$

Proof. Follows from the continuity of μ .

We pause now for a general topological lemma.

LEMMA 8. *Let X be a locally compact separable metric space of dimension 0. Then X is a countable union of monotone increasing sets that are both compact and open.*

Proof. From the definition of dimension 0, each point x has arbitrarily small neighborhoods G_x which are both closed and open.

By choosing G_x small enough, it can therefore be made both compact and open.

Since $X = \bigcup_{x \in X} G_x$, and X has a countable base, we can find x_1, x_2, \dots such that $X = \bigcup_{n=1}^{\infty} G_{x_n}$. Let $K_n = \bigcup_{j=1}^n G_{x_j}$. Then each K_n is both compact and open, and $K_n \uparrow X$.

Now we return to M, μ , and ν .

LEMMA 9. *Let G be open. Let E be open, $E \subseteq G$, such that $\partial E \cap G$ has dimension 0. Then $\mu(G) \leq \nu(E) + \nu(G - E)$.*

Proof. Let $D = \partial E \cap G$. Let $H = G - \bar{E}$. Then the sets E, D , and H are mutually disjoint, and $G = E \cup D \cup H$.

Since D is a closed subset of the locally compact separable metric space G , D is a locally compact separable metric space also.

By Lemma 8, we can find sets K_n which are both compact and open in D , such that $K_n \uparrow D$.

Let $K_n = A_n \cap D$, where A_n is open. Since K_n is compact we may choose A_n such that $\bar{A}_n \subseteq G$. By taking unions if necessary we may choose the A_n to be increasing.

Let E_n and H_n be open sets such that $\bar{E}_n \subseteq E$, $\bar{H}_n \subseteq H$ for all n , $E_n \uparrow E$ and $H_n \uparrow H$. Let $G_n = E_n \cup A_n \cup H_n$. Then G_n is open, $\bar{G}_n \subseteq G$, and $G_n \uparrow G$. Then $\mu(G_n) \rightarrow \mu(G)$ as $n \rightarrow \infty$, by continuity.

$$\begin{aligned} \text{But for all } n, G_n &= (G_n \cap E) \cup (G_n \cap D) \cup (G_n \cap H) \\ &= (G_n \cap E) \cup K_n \cup (G_n \cap H) . \end{aligned}$$

Thus $\mu(G_n) = \mu(G_n \cap E) + \mu(K_n) + \mu(G_n \cap H)$, by additivity,
 $\leq \nu(G_n \cap E) + \nu(K_n) + \nu(G_n \cap H)$
 $\leq \nu(E) + \nu(D) + \nu(H) \leq \nu(E) + \nu(G - E)$.

This proves Lemma 9.

LEMMA 10. *Let G be an open set. Let E be open, $E \subseteq G$, such that $\partial E \cap G$ has dimension 0. Then $\nu(G) = \nu(E) + \nu(G - E)$.*

Proof. Let $\varepsilon > 0$ be given. Choose H open, $H \subseteq G$, such that $\mu(H) \geq \nu(G) - \varepsilon$. This is possible by Lemma 7.

Then $\partial(E \cap H) \cap H = \partial E \cap H \subseteq \partial E \cap G$. Hence $\partial(E \cap H) \cap H$ has dimension 0. By Lemma 7, $\mu(H) \leq \nu(E \cap H) + \nu(H - E \cap H) \leq \nu(E) + \nu(G - E)$. Hence $\nu(G) \leq \nu(E) + \nu(G - E)$.

The reverse inequality holds by Lemma 6, so Lemma 10 is proved.

From now on in this section, let M have dimension at most one.

LEMMA 11. *Let G_1 and G_2 be open, with union G . Then $\nu(G) \leq \nu(G_1) + \nu(G_2)$.*

Proof. $G_1 - G_2$ and $G_2 - G_1$ are disjoint and relatively closed in G . G is a separable metric space of dimension no larger than 1. Hence by Theorem 1 in [3], section 27II, page 290, we can find an open set $E \subseteq G$ such that $E \supseteq G_1 - G_2$, $\bar{E} \cap (G_2 - G_1) = \emptyset$, and $\partial E \cap G$ has dimension 0.

By Lemma 10,

$$\nu(G) = \nu(E) + \nu(G - E) \leq \nu(G_1) + \nu(G_2).$$

LEMMA 12. *Let G_n be a sequence of open sets. Let $G = \bigcup_{n=1}^{\infty} G_n$. Then $\nu(G) \leq \sum_{n=1}^{\infty} \nu(G_n)$.*

Proof. Let $\varepsilon > 0$ be given. Choose F closed, $F \subseteq G$ such that $\mu(F) \geq \nu(G) - \varepsilon$.

Then there exists n such that $F \subseteq \bigcup_{j=1}^n G_j$. Hence $\sum_{j=1}^{\infty} \nu(G_j) \geq \sum_{j=1}^n \nu(G_j) \geq \nu(\bigcup_{j=1}^n G_j)$, by Lemma 11, $\geq \mu(F)$ by definition.

This proves Lemma 12.

DEFINITION 3. For any set $E \subseteq M$, define $\nu^*(E) = \inf \{ \nu(G) \mid E \subseteq G, G \text{ open} \}$. Clearly $\nu^*(E) = \nu(E)$ when E is open.

LEMMA 13. ν^* is an outer measure.

Proof. Follows from Lemma 12.

LEMMA 14. *Every open set is measurable with respect to ν^* , in the sense of Caratheodory.*

Proof. Let G be open. Let E be any set. We know

$$\nu^*(E) \leq \nu^*(E \cap G) + \nu^*(E - G),$$

since ν^* is an outer measure. We must show that

$$\nu^*(E) \geq \nu^*(E \cap G) + \nu^*(E - G).$$

Choose any open set H such that $E \subseteq H$. Let $\varepsilon > 0$ be given. Choose F closed, $F \subseteq G \cap H$, such that $\nu(F) \geq \nu(G \cap H) - \varepsilon$. Then $\nu(H) \geq \nu(F) + \nu(H - F)$, by Lemma 6, $\geq \nu(G \cap H) - \varepsilon + \nu(H - F) \geq \nu^*(E \cap G) - \varepsilon + \nu^*(E - G)$ by definition.

Hence $\nu(H) \geq \nu^*(E \cap G) + \nu^*(E - G)$. By definition, then, $\nu^*(E) \geq \nu^*(E \cap G) + \nu^*(E - G)$, and Lemma 14 is proved.

Because of Lemma 14 we know that ν^* defines a measure on a σ -algebra of sets that includes the Borel sets of M .

Proof of Theorem 3. First suppose that μ is nonnegative. Let G be open. By Lemma 7, $\mu(G) \leq \nu(G)$. On the other hand, for any closed subset F of G , $\mu(F) \leq \mu(F) + \mu(G - F) = \mu(G)$. Thus $\mu(G) = \nu(G)$. ν^* is a measure on the Borel sets of M which agrees with μ on open sets and hence on all sets in the domain of μ .

Now let μ be arbitrary. Consider the set function $\omega = \nu^* - \mu$, defined for closed subsets of M and for open subsets of M . ω is nonnegative by Lemma 7. By what has already been proved, ω can be extended to a Borel measure. But then $\mu = \nu^* - \omega$ can be extended also, so the theorem is proved.

REFERENCES

1. J. R. Baxter and R. V. Chacon, *Almost linear operators and functionals on $\mathcal{C}([0, 1])$* , to appear in Proc. Amer. Math. Soc.
2. ———, *Nonlinear functionals on $\mathcal{C}([0, 1] \times [0, 1])$* , Pacific. J. Math., **48** (1973), 347-353.
3. K. Kuratowski, *Topology*, Vol. 1, Academic Press, New York, 1968.

Received January 19, 1973.

UNIVERSITY OF BRITISH COLUMBIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Robert F. V. Anderson, <i>Laplace transform methods in multivariate spectral theory</i>	339
William George Bade, <i>Two properties of the Sorgenfrey plane</i>	349
John Robert Baxter and Rafael Van Severen Chacon, <i>Functionals on continuous functions</i>	355
Phillip Wayne Bean, <i>Helly and Radon-type theorems in interval convexity spaces</i>	363
James Robert Boone, <i>On k-quotient mappings</i>	369
Ronald P. Brown, <i>Extended prime spots and quadratic forms</i>	379
William Hugh Cornish, <i>Crawley's completion of a conditionally upper continuous lattice</i>	397
Robert S. Cunningham, <i>On finite left localizations</i>	407
Robert Jay Daverman, <i>Approximating polyhedra in codimension one spheres embedded in S^n by tame polyhedra</i>	417
Burton I. Fein, <i>Minimal splitting fields for group representations</i>	427
Peter Fletcher and Robert Allen McCoy, <i>Conditions under which a connected representable space is locally connected</i>	433
Jonathan Samuel Golan, <i>Topologies on the torsion-theoretic spectrum of a noncommutative ring</i>	439
Manfred Gordon and Edward Martin Wilkinson, <i>Determinants of Petrie matrices</i>	451
Alfred Peter Hallstrom, <i>A counterexample to a conjecture on an integral condition for determining peak points (counterexample concerning peak points)</i>	455
E. R. Heal and Michael Windham, <i>Finitely generated F-algebras with applications to Stein manifolds</i>	459
Denton Elwood Hewgill, <i>On the eigenvalues of a second order elliptic operator in an unbounded domain</i>	467
Charles Royal Johnson, <i>The Hadamard product of A and A^*</i>	477
Darrell Conley Kent and Gary Douglas Richardson, <i>Regular completions of Cauchy spaces</i>	483
Alan Greenwell Law and Ann L. McKerracher, <i>Sharpened polynomial approximation</i>	491
Bruce Stephen Lund, <i>Subalgebras of finite codimension in the algebra of analytic functions on a Riemann surface</i>	495
Robert Wilmer Miller, <i>TTF classes and quasi-generators</i>	499
Roberta Mura and Akbar H. Rhemtulla, <i>Solvable groups in which every maximal partial order is isolated</i>	509
Isaac Namioka, <i>Separate continuity and joint continuity</i>	515
Edgar Andrews Rutter, <i>A characterization of QF - 3 rings</i>	533
Alan Saleski, <i>Entropy of self-homeomorphisms of statistical pseudo-metric spaces</i>	537
Ryōtarō Satō, <i>An Abel-maximal ergodic theorem for semi-groups</i>	543
H. A. Seid, <i>Cyclic multiplication operators on L_p-spaces</i>	549
H. B. Skerry, <i>On matrix maps of entire sequences</i>	563
John Brendan Sullivan, <i>A proof of the finite generation of invariants of a normal subgroup</i>	571
John Griggs Thompson, <i>Nonsolvable finite groups all of whose local subgroups are solvable, VI</i>	573
Ronson Joseph Warne, <i>Generalized ω - \mathcal{L}-unipotent bisimple semigroups</i>	631
Toshihiko Yamada, <i>On a splitting field of representations of a finite group</i>	649