

# Pacific Journal of Mathematics

**HELLY AND RADON-TYPE THEOREMS IN INTERVAL  
CONVEXITY SPACES**

PHILLIP WAYNE BEAN

## HELLY AND RADON-TYPE THEOREMS IN INTERVAL CONVEXITY SPACES

PHILLIP W. BEAN

The notion of interval convexity  $T$  on a point set  $S$  is defined. If  $T$  is an interval convexity defined on  $S$ ,  $\mathcal{C}(T)$  will denote the collection of nonempty  $T$ -convex subsets of  $S$ . Properties  $k$ ,  $H(k)$  (a Helly property), and  $R(k, n)$  (a Radon property) are defined on  $\mathcal{C}(T)$ , and relationships between these properties are investigated.

A partial order convexity  $\leq$  on a point set  $S$  is a special type of interval convexity. Some sufficient conditions are imposed on  $\leq$  and  $\mathcal{C}(\leq)$  to insure the existence of certain Radon-type properties.

1. Introduction. Suppose  $S$  is a point set, and  $\mathcal{P}(S)$  is the collection of nonempty subsets of  $S$ . The statement that  $T$  is an interval convexity on  $S$  means that  $T$  is a transformation from  $S \times S$  into  $\mathcal{P}(S)$ . A subset  $M$  of  $S$  is said to be  $T$ -convex provided that  $T(x, y)$  is a subset of  $M$  for every  $x$  and  $y$  in  $M$ . Let  $\mathcal{C}(T)$  denote the collection of all nonempty  $T$ -convex subsets of  $S$ . For each  $M \in \mathcal{P}(S)$ , the convex hull of  $M$  relative to  $T$ , denoted by  $\text{Co}(M)$ , is the intersection of the elements of  $\mathcal{C}(T)$  which contain  $M$ . We assume that if each of  $x$  and  $y$  is in  $S$ , then  $T(x, y)$  is  $T$ -convex,  $T(x, y)$  contains  $x$  and  $y$ , and  $T(x, y) = T(y, x)$ .

Let  $m$ -set mean a set of  $m$  points of  $S$ . A subset  $M$  of  $S$  is said to be  $n$ -divisible provided it may be partitioned into  $n$  mutually exclusive subsets whose  $T$ -convex hulls have a common point of  $S$ . In this paper we consider the relationship of the following Helly and Radon-type properties on a set  $S$  with an interval convexity  $T$ .  $\mathcal{C}(T)$  has property  $R(k)$  if each  $(k+1)$ -set of  $S$  is 2-divisible, and more generally,  $\mathcal{C}(T)$  has property  $R(k, n)$  with respect to some integer valued function  $f$  if each  $[f(k, n)]$ -set is  $n$ -divisible. We say that  $\mathcal{C}(T)$  has property  $r(k)$  if  $k$  is the smallest integer for which  $\mathcal{C}(T)$  has property  $R(k)$ .  $\mathcal{C}(T)$  is said to have property  $H(k)$  provided that if  $\mathcal{C}$  is a finite subcollection of  $\mathcal{C}(T)$  containing at least  $k$  elements, then the following two statements are equivalent:

- (a) Each  $k$  elements of  $\mathcal{C}$  have a common point.
- (b) The elements of  $\mathcal{C}$  have a common point.

In (2) we give sufficient conditions for property  $R(k)$  to be equivalent to property  $H(k)$ . We also consider in (2) the existence of sets with property  $R(k)$  in partially ordered spaces and more generally, in (3) the existence of sets with property  $R(k, n)$ .

2. **Theorems concerning properties  $k$ ,  $R(k)$ , and  $H(k)$ .** From a theorem of Levi [7] we have that property  $R(k)$  implies property  $H(k)$ . In [1] Calder introduces the following property:  $\mathcal{C}(T)$  has property  $k$  provided that if  $M$  is a finite point set containing at least  $k + 1$  points, then there exists a point  $p$  such that  $p \in \text{Co}[M \sim \{m\}]$  for each  $m$  in  $M$ . He proves that property  $k$  is equivalent to property  $H(k)$  and then proves that property  $R(k)$  is equivalent to property  $H(k)$  in a partially ordered space. It should be noted that the partial order does not have to be antisymmetric. Calder also gives an example of an interval convexity  $T$  such that property  $H(k)$  is not equivalent to property  $R(k)$  in  $\mathcal{C}(T)$ . In the first two theorems of this section we give sufficient conditions on  $T$  for properties  $H(k)$  and  $R(k)$  to be equivalent.

If each of  $A$  and  $B$  is in  $\mathcal{S}(S)$ , then  $A * B$  denotes the set

$$\bigcup_{a \in A, b \in B} T(a, b).$$

**THEOREM 2.1.** *Let  $T$  be an interval convexity on  $S$  such that for each  $M$  in  $\mathcal{S}(S)$ ,  $\text{Co}(M) = M * M$ ; and if  $a, b, c$ , and  $d$  are four distinct points such that  $d$  is in  $T(a, b)$  and  $T(a, c)$ , then  $b$  is in  $T(a, c)$ , or  $c$  is in  $T(a, b)$ . Then property  $H(k) \Leftrightarrow$  property  $R(k)$  in  $\mathcal{C}(T)$ .*

**THEOREM 2.2.** *Let  $T$  be an interval convexity on  $S$  such that for each  $M$  in  $\mathcal{S}(S)$ ,  $\text{Co}(M) = \bigcup_{m \in M} T(m, m)$ . Then property  $H(k) \Leftrightarrow$  property  $R(k)$  in  $\mathcal{C}(T)$ .*

The proofs of Theorems 2.1 and 2.2 are easy modifications of the proof of Theorem 3.2 of Calder [1].

**EXAMPLE 2.1.** Let  $M$  be a subset of a linear space  $S$ . A subset  $K$  of  $M$  is said to be *extremal* provided that if  $k$  is an element of  $K$ , and there exist elements  $x$  and  $y$  in  $M$  such that  $k = tx + (1 - t)y$  for some  $t \in (0, 1)$ , then  $x$  and  $y$  are elements of  $K$ . Obviously, the union and intersection of any collection of extremal subsets of  $M$  are extremal.

We define an interval convexity  $T$  on  $M$  as follows: If each of  $x$  and  $y$  is an element of  $M$ ,  $T(x, y)$  is the intersection of the extremal subsets of  $M$  which contain  $\{x, y\}$ .

For each subset  $K$  of  $M$ ,  $K \subset \bigcup_{k \in K} T(k, k)$ . Since  $\bigcup_{k \in K} T(k, k)$  is convex,  $\text{Co}(K) \subset \bigcup_{k \in K} T(k, k)$ . However,  $\bigcup_{k \in K} T(k, k) \subset \text{Co}(K)$ . Thus  $\text{Co}(K) = \bigcup_{k \in K} T(k, k)$ , and hence property  $H(k) \Leftrightarrow$  property  $R(k)$  in  $\mathcal{C}(T)$ .

Let  $\leq$  be a partial order on the set  $S$ . If each of  $x$  and  $y$  is a point of  $S$ ,  $[x, y] = \{p \mid p = x, \text{ or } p = y, \text{ or } x < p < y, \text{ or } y < p < x\}$ .

A subset  $M$  of  $S$  is said to be  $\leq$ -convex if for all elements  $x$  and  $y$  of  $M$ ,  $[x, y]$  is a subset of  $M$ . The collection of all  $\leq$ -convex subsets of  $S$  is denoted by  $\mathcal{C}(\leq)$ . In [5], Franklin shows that  $\text{Co}(M) = M * M$  for any  $M$  in  $\mathcal{P}(S)$ .

**THEOREM 2.3.** *Suppose  $\leq$  is a partial order on  $S$ , and  $S$  is the union of  $n$  linearly ordered sets,  $S_1, S_2, \dots, S_n$ . Then  $\mathcal{C}(\leq)$  has property  $R(2n)$ .*

*Proof.* Suppose  $M = \{x_1, x_2, \dots, x_{2n+1}\}$  is a  $(2n + 1)$ -set. Then for some  $i, 1 \leq i \leq n$ ,  $S_i$  contains at least three points,  $z_1, z_2, z_3$ , of  $M$  such that  $z_1 < z_2 < z_3$ . Thus  $\text{Co}\{z_2\}$  and  $\text{Co}\{z_1, z_3\}$  have a common point, and therefore  $\mathcal{C}(\leq)$  has property  $R(2n)$ .

It is easy to show that  $\mathcal{C}(\leq)$  has property  $r(2)$  if and only if  $\leq$  linearly orders  $S$ . Suppose  $\leq$  is a partial order on  $S$  which does not linearly order  $S$ . Under these conditions on  $\leq$ , does  $\mathcal{C}(\leq)$  have property  $r(3)$  if and only if  $S$  is union of two mutually exclusive, linearly ordered subsets  $S_1$  and  $S_2$ ? The following example shows the answer to this question is no.

**EXAMPLE 2.2.** Let  $S = \{(x, y) \in R^2 \mid y = 0 \text{ or } y = 1\}$ . Define  $\leq$  on  $S$  as follows:  $(x_1, y_1) \leq (x_2, y_2)$  if  $y_1 = y_2$  and  $x_1 \leq x_2$ . Thus  $\leq$  is a partial order on  $S$  which does not linearly order  $S$ . However,  $\leq$  does linearly order  $S_1 = \{(x, 1) \in R^2\}$  and  $S_2 = \{(x, 0) \in R^2\}$ , and  $S = S_1 \cup S_2$ . To show that  $\mathcal{C}(\leq)$  does not have property  $r(3)$  we choose  $M = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ . Obviously  $M$  is not 2-divisible.

**3. Property  $R(k, n)$ .** Tverberg shows in [11] that the collection on convex sets in  $R^{k-1}$  has property  $R(k, n)$  with respect to  $f(k, n) = (n - 1)k + 1$  for  $n, k \geq 2$ . By putting suitable restrictions on  $T$ , we have the following:

**THEOREM 3.1.** *Suppose  $T$  is an interval convexity on  $S$  such that if  $M \in \mathcal{P}(S)$ , then  $\text{Co}(M) = \bigcup_{m \in M} T(m, m)$ . If  $\mathcal{C}(T)$  has property  $R(k)$ , then  $\mathcal{C}(T)$  has property  $R(k, n)$  with respect to  $f(k, n) = (n - 1)k + 1$  for  $n \geq 2$ .*

*Proof.* (We use induction on  $n$ .) Suppose  $\mathcal{C}(T)$  has property  $R(k)$ , i.e.,  $\mathcal{C}(T)$  has property  $R(k, 2)$ . Suppose further that  $\mathcal{C}(T)$  has property  $R(k, m)$  for some  $m \geq 2$ , and let  $M = \{x_1, x_2, \dots, x_{mk+1}\}$  be an  $[mk + 1]$ -set. Let  $M_1 = \{x_1, x_2, \dots, x_{(m-1)k+1}\}$  be the subset of  $M$  containing the first  $(m - 1)k + 1$  points of  $M$ . Then there exist  $m$  points,  $y_{11}, y_{12}, \dots, y_{1m}$ , of  $M_1$  and a point  $p_1$  such that

$$p_1 \in \bigcap_{i=1}^m \text{Co} \{y_{1i}\} .$$

Now choose  $M_2 = \{x_1, x_2, \dots, x_{(m-1)k+1}, x_{(m-1)k+2}\} \sim \{y_{11}\}$ . Thus  $M_2$  is an  $[(m-1)k+1]$ -set, and hence there exist  $m$  points,  $y_{21}, y_{22}, \dots, y_{2m}$ , of  $M_2$  and a point  $p_2 \in \bigcap_{i=1}^m \text{Co} \{y_{2i}\}$ .

Continuing this process we get  $M_j = [M_{j-1} \cup \{x_{(m-1)k+j}\}] \sim \{y_{j-11}\}$  for  $3 \leq j \leq k+1$ , and each of the sets is an  $[(m-1)k+1]$ -set. Thus there exist  $m$  points,  $y_{j1}, y_{j2}, \dots, y_{jm}$ , of  $M_j$  and a point  $p_j \in \bigcap_{i=1}^m \text{Co} \{y_{ji}\}$ .

Let  $K = \{p_1, p_2, \dots, p_{k+1}\}$ . If  $p_i = p_j$  for some  $i \neq j$ , the theorem is proved. Suppose  $p_i \neq p_j$  if  $i \neq j$ . Since  $\mathcal{C}(T)$  has property  $R(k)$ , there exist points,  $p_i$  and  $p_j, i < j$ , in  $K$  and a point

$$p_0 \in \text{Co} \{p_i\} \cap \text{Co} \{p_j\} .$$

Since for each  $x \in S, T(x, x)$  is convex, we have  $p_0 \in \text{Co} \{y_{i1}\} \cap \text{Co} \{y_{j1}\} \cap \dots \cap \text{Co} \{y_{jm}\}$ . Thus  $M$  is  $(m+1)$ -divisible and  $\mathcal{C}(T)$  has property  $R(k, m+1)$ . Therefore,  $\mathcal{C}(T)$  has property  $R(k, n)$  with respect to  $f(k, n) = (n-1)k+1$  for all  $n \geq 2$ .

EXAMPLE 3.1. In  $R^2$  let  $l/P$  and  $\overline{l/P}$  denote, respectively, the open and the closed half planes determined by the line  $l$  and containing the point  $P$ .  $PQ$  denotes the line determined by the points  $P$  and  $Q$ , and  $P[m]$  denotes the line through  $P$  with slope  $m$ . Let  $P_0 = (0, 0), P_1 = (1, 0), P_2 = (-1/2, \sqrt{3}/2), P_3 = (-1/2, -\sqrt{3}/2), P_4 = (1, 1), P_5 = (-1, 0), P_6 = (1, -1)$ . Choose  $S = S_1 \cup S_2 \cup S_3$  where  $S_1 = P_0P_1/P_4 \cap P_0P_2/P_4, S_2 = P_0P_2/P_5 \cap P_0P_3/P_5$ , and  $S_3 = P_0P_1/P_6 \cap P_0P_3/P_6$ . We define an interval convexity  $T$  on  $S$  as follows:

$$\begin{aligned} \text{(a)} \quad T(P, P) &= \begin{cases} S_1 \cap \overline{P[-\sqrt{3}]/P_0} & \text{if } P \in S_1; \\ S_2 \cap \overline{P[\sqrt{3}]/P_0} & \text{if } P \in S_2; \\ S_3 \cap \overline{P[0]/P_0} & \text{if } P \in S_3. \end{cases} \\ \text{(b)} \quad T(P, Q) &= T(P, P) \cup T(Q, Q) . \end{aligned}$$

Thus if  $M \in \mathcal{P}(S), \text{Co}(M) = \bigcup_{m \in M} T(m, m)$ . It is easily seen that  $\mathcal{C}(T)$  has property  $r(3)$ . Thus if  $k \geq 3, \mathcal{C}(T)$  has property  $R(k, n)$  with respect to  $f(k, n) = (n-1)k+1$  for  $n \geq 2$ .

THEOREM 3.2. Suppose  $\leq$  is a partial order on  $S$  such that  $\mathcal{C}(\leq)$  has property  $R(k)$ . Then  $\mathcal{C}(\leq)$  has property  $R(k, n)$  with respect to  $f(k, n) = (2n-3)k+1$  for all  $n \geq 2$ .

*Proof.* (The proof is a slight modification of the proof of Theorem 3.1.) The statement is true for  $n = 2$  since property  $R(k)$  is the same as property  $R(k, 2)$ . Now suppose the statement is true for

$n = m$ , and let  $M = \{x_1, x_2, \dots, x_{(2m-3)k+1}, \dots, x_{(2m-1)k+1}\}$  be a  $[(2m-1)k + 1]$ -set. (Note that  $[(2m-1)k + 1] - [(2m-3)k + 1] = 2k$ .) Let  $K_0 = \{x_1, x_2, \dots, x_{(2m-3)k+1}\}$  be the subset of  $M$  containing the first  $(2m-3)k + 1$  points of  $M$ . Thus there exist  $m$  mutually exclusive subsets,  $K_{01}, K_{02}, \dots, K_{0m}$ , of  $K_0$  and a point  $y_0 \in \bigcap_{i=1}^m \text{Co}(K_{0i})$ . It follows then that there exist points  $s_0$  and  $t_0$  in  $K_0$  such that  $s_0 < y_0 < t_0$ . Now let  $K_1$  be the  $[(2m-3)k + 1]$ -set  $[K_0 \sim \{s_0, t_0\}] \cup \{x_{(2m-3)k+2}, x_{(2m-3)k+3}\}$ . Again there exist  $m$  mutually exclusive subsets,  $K_{11}, K_{12}, \dots, K_{1m}$ , of  $K_1$  such that  $\bigcap_{i=1}^m \text{Co}(K_{1i}) \neq \emptyset$ . If  $y_0 \in \bigcap_{i=1}^m \text{Co}(K_{1i})$ , the theorem is proved.

Suppose  $y_0 \notin \bigcap_{i=1}^m \text{Co}(K_{1i})$ . Let  $y_1 \in \bigcap_{i=1}^m \text{Co}(K_{1i})$ . Then there exist points  $s_1$  and  $t_1$  in  $K_1$  such that  $s_1 < y_1 < t_1$ .

Continuing this process for  $2 \leq i \leq k$ , we obtain  $K_i = [K_{i-1} \sim \{s_{i-1}, t_{i-1}\}] \cup \{x_{(2m-3)k+2i}, x_{(2m-3)k+(2i+1)}\}$  and correspondingly  $m$  mutually exclusive subsets,  $K_{i1}, K_{i2}, \dots, K_{im}$ , of  $K_i$  such that  $\bigcap_{p=1}^m \text{Co}(K_{ip})$  contains a point  $y_i$ . Now if for some  $j$  and  $i$ ,  $0 \leq j < i \leq k$ ,  $y_j \in \bigcap_{p=1}^m \text{Co}(K_{ip})$ , the theorem is proved.

Suppose that if  $0 \leq j < i \leq k$ ,  $y_j \notin \bigcap_{p=1}^m \text{Co}(K_{ip})$ . Then the  $(k + 1)$ -set  $C = \{y_0, y_1, \dots, y_k\}$  is 2-divisible. Let  $C_1$  and  $C_2$  be mutually exclusive subsets of  $C$  such that  $\text{Co}(C_1) \cap \text{Co}(C_2) \neq \emptyset$ . It can be shown that if  $w \in \text{Co}(C_1) \cap \text{Co}(C_2)$  then there are  $m + 1$  mutually exclusive subsets,  $M_1, M_2, \dots, M_{m+1}$ , of  $M$  such that  $w \in \bigcap_{i=1}^{m+1} \text{Co}(M_i)$ . Hence  $M$  is  $m + 1$  divisible. Therefore,  $\mathcal{C}(\leq)$  has property  $R(k, n)$  with respect to  $f(k, n) = (2n - 3)k + 1$  for all  $n \geq 2$ .

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