HELLY AND RADON-TYPE THEOREMS IN INTERVAL CONVEXITY SPACES

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The notion of interval convexity $T$ on a point set $S$ is defined. If $T$ is an interval convexity defined on $S$, $\mathcal{C}(T)$ will denote the collection of nonempty $T$-convex subsets of $S$. Properties $k, H(k)$ (a Helly property), and $R(k, n)$ (a Radon property) are defined on $\mathcal{C}(T)$, and relationships between these properties are investigated.

A partial order convexity $\leq$ on a point set $S$ is a special type of interval convexity. Some sufficient conditions are imposed on $\leq$ and $\mathcal{C}(\leq)$ to insure the existence of certain Radon-type properties.

1. Introduction. Suppose $S$ is a point set, and $\mathcal{P}(S)$ is the collection of nonempty subsets of $S$. The statement that $T$ is an interval convexity on $S$ means that $T$ is a transformation from $S \times S$ into $\mathcal{P}(S)$. A subset $M$ of $S$ is said to be $T$-convex provided that $T(x, y)$ is a subset of $M$ for every $x$ and $y$ in $M$. Let $\mathcal{C}(T)$ denote the collection of all nonempty $T$-convex subsets of $S$. For each $M \in \mathcal{P}(S)$, the convex hull of $M$ relative to $T$, denoted by $\text{Co}(M)$, is the intersection of the elements of $\mathcal{C}(T)$ which contain $M$. We assume that if each of $x$ and $y$ is in $S$, then $T(x, y)$ is $T$-convex, $T(x, y)$ contains $x$ and $y$, and $T(x, y) = T(y, x)$.

Let $m$-set mean a set of $m$ points of $S$. A subset $M$ of $S$ is said to be $n$-divisible provided it may be partitioned into $n$ mutually exclusive subsets whose $T$-convex hulls have a common point of $S$. In this paper we consider the relationship of the following Helly and Radon-type properties on a set $S$ with an interval convexity $T$. $\mathcal{C}(T)$ has property $R(k)$ if each $(k + 1)$-set of $S$ is 2-divisible, and more generally, $\mathcal{C}(T)$ has property $R(k, n)$ with respect to some integer valued function $f$ if each $[f(k, n)]$-set is $n$-divisible. We say that $\mathcal{C}(T)$ has property $r(k)$ if $k$ is the smallest integer for which $\mathcal{C}(T)$ has property $R(k)$. $\mathcal{C}(T)$ is said to have property $H(k)$ provided that if $\mathcal{C}$ is a finite subcollection of $\mathcal{C}(T)$ containing at least $k$ elements, then the following two statements are equivalent:

(a) Each $k$ elements of $\mathcal{C}$ have a common point.

(b) The elements of $\mathcal{C}$ have a common point.

In (2) we give sufficient conditions for property $R(k)$ to be equivalent to property $H(k)$. We also consider in (2) the existence of sets with property $R(k)$ in partially ordered spaces and more generally, in (3) the existence of sets with property $R(k, n)$. 

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2. Theorems concerning properties $k$, $R(k)$, and $H(k)$. From a theorem of Levi [7] we have that property $R(k)$ implies property $H(k)$. In [1] Calder introduces the following property: $\mathcal{E}(T)$ has property $k$ provided that if $M$ is a finite point set containing at least $k + 1$ points, then there exists a point $p$ such that $p \in \text{Co } [M \sim \{m\}]$ for each $m$ in $M$. He proves that property $k$ is equivalent to property $H(k)$ and then proves that property $R(k)$ is equivalent to property $H(k)$ in a partially ordered space. It should be noted that the partial order does not have to be antisymmetric. Calder also gives an example of an interval convexity $T$ such that property $H(k)$ is not equivalent to property $R(k)$ in $\mathcal{E}(T)$. In the first two theorems of this section we give sufficient conditions on $T$ for properties $H(k)$ and $R(k)$ to be equivalent.

If each of $A$ and $B$ is in $\mathcal{P}(S)$, then $A \ast B$ denotes the set

$$\bigcup_{a \in A, b \in B} T(a, b).$$

**Theorem 2.1.** Let $T$ be an interval convexity on $S$ such that for each $M$ in $\mathcal{P}(S)$, $\text{Co } (M) = M \ast M$; and if $a, b, c,$ and $d$ are four distinct points such that $d$ is in $T(a, b)$ and $T(a, c)$, then $b$ is in $T(a, c)$, or $c$ is in $T(a, d)$. Then property $H(k) \Rightarrow property R(k)$ in $\mathcal{E}(T)$.

**Theorem 2.2.** Let $T$ be an interval convexity on $S$ such that for each $M$ in $\mathcal{P}(S)$, $\text{Co } (M) = \bigcup_{m \in M} T(m, m)$. Then property $H(k) \Leftrightarrow property R(k)$ in $\mathcal{E}(T)$.

The proofs of Theorems 2.1 and 2.2 are easy modifications of the proof of Theorem 3.2 of Calder [1].

**Example 2.1.** Let $M$ be a subset of a linear space $S$. A subset $K$ of $M$ is said to be extremal provided that if $k$ is an element of $K$, and there exist elements $x$ and $y$ in $M$ such that $k = tx + (1 - t)y$ for some $t \in (0, 1)$, then $x$ and $y$ are elements of $K$. Obviously, the union and intersection of any collection of extremal subsets of $M$ are extremal.

We define an interval convexity $T$ on $M$ as follows: If each of $x$ and $y$ is an element of $M$, $T(x, y)$ is the intersection of the extremal subsets of $M$ which contain $\{x, y\}$.

For each subset $K$ of $M$, $K \subset \bigcup_{k \in K} T(k, k)$. Since $\bigcup_{k \in K} T(k, k)$ is convex, $\text{Co } (K) \subset \bigcup_{k \in K} T(k, k)$. However, $\bigcup_{k \in K} T(k, k) \subset \text{Co } (K)$. Thus $\text{Co } (K) = \bigcup_{k \in K} T(k, k)$, and hence property $H(k) \Rightarrow property R(k)$ in $\mathcal{E}(T)$.

Let $\leq$ be a partial order on the set $S$. If each of $x$ and $y$ is a point of $S$, $[x, y] = \{p | p = s, or p = y, or x < p < y, or y < p < x\}$. 
A subset $M$ of $S$ is said to be $\leq$-convex if for all elements $x$ and $y$ of $M$, $[x, y]$ is a subset of $M$. The collection of all $\leq$-convex subsets of $S$ is denoted by $\mathcal{C}(\leq)$. In [5], Franklin shows that $\text{Co}(M) = M \star M$ for any $M$ in $\mathcal{P}(S)$.

**Theorem 2.3.** Suppose $\leq$ is a partial order on $S$, and $S$ is the union of $n$ linearly ordered sets, $S_1, S_2, \ldots, S_n$. Then $\mathcal{C}(\leq)$ has property $R(2n)$.

**Proof.** Suppose $M = \{x_1, x_2, \ldots, x_{2n+1}\}$ is a $(2n+1)$-set. Then for some $i, 1 \leq i \leq n$, $S_i$ contains at least three points, $z_i, z_2, z_3$, of $M$ such that $z_i < z_2 < z_3$. Thus $\text{Co}\{z_i\}$ and $\text{Co}\{z_2, z_3\}$ have a common point, and therefore $\mathcal{C}(\leq)$ has property $R(2n)$.

It is easy to show that $\mathcal{C}(\leq)$ has property $r(2)$ if and only if $\leq$ linearly orders $S$. Suppose $\leq$ is a partial order on $S$ which does not linearly order $S$. Under these conditions on $\leq$, does $\mathcal{C}(\leq)$ have property $r(3)$ if and only if $S$ is union of two mutually exclusive, linearly ordered subsets $S_1$ and $S_2$? The following example shows the answer to this question is no.

**Example 2.2.** Let $S = \{(x, y) \in \mathbb{R}^2 | y = 0 \text{ or } y = 1\}$. Define $\leq$ on $S$ as follows: $(x_1, y_1) \leq (x_2, y_2)$ if $y_1 = y_2$ and $x_1 \leq x_2$. Thus $\leq$ is a partial order on $S$ which does not linearly order $S$. However, $\leq$ does linearly order $S_1 = \{(x, 1) \in \mathbb{R}^2\}$ and $S_2 = \{(x, 0) \in \mathbb{R}^2\}$, and $S = S_1 \cup S_2$. To show that $\mathcal{C}(\leq)$ does not have property $r(3)$ we choose $M = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Obviously $M$ is not 2-divisible.

3. **Property $R(k, n)$.** Tverberg shows in [11] that the collection on convex sets in $\mathbb{R}^{k-1}$ has property $R(k, n)$ with respect to $f(k, n) = (n - 1)k + 1$ for $n, k \geq 2$. By putting suitable restrictions on $T$, we have the following:

**Theorem 3.1.** Suppose $T$ is an interval convexity on $S$ such that if $M \in \mathcal{P}(S)$, then $\text{Co}(M) = \bigcup_{m \in M} T(m, m)$. If $\mathcal{C}(T)$ has property $R(k)$, then $\mathcal{C}(T)$ has property $R(k, n)$ with respect to $f(k, n) = (n - 1)k + 1$ for $n \geq 2$.

**Proof.** (We use induction on $n$.) Suppose $\mathcal{C}(T)$ has property $R(k)$, i.e., $\mathcal{C}(T)$ has property $R(k, 2)$. Suppose further that $\mathcal{C}(T)$ has property $R(k, m)$ for some $m \geq 2$, and let $M = \{x_1, x_2, \ldots, x_{mk+1}\}$ be an $[mk+1]$-set. Let $M_i = \{x_1, x_2, \ldots, x_{(m-1)k+1}\}$ be the subset of $M$ containing the first $(m-1)k + 1$ points of $M$. Then there exist $m$ points, $y_1, y_2, \ldots, y_m$, of $M_i$ and a point $p_i$ such that
Now choose \( M_2 = \{ x_{(m-1)k+1}, x_{(m-1)k+2} \} \sim \{ y_{(i)} \} \). Thus \( M_2 \) is an \( [(m-1)k+1] \)-set, and hence there exist \( m \) points, \( y_{(m-1)k+1}, y_{(m-1)k+2}, \ldots, y_{(m-1)k+2m} \), of \( M_2 \) and a point \( p_{(m-1)k+2} \in \bigcap_{i=1}^{m} \text{Co} \{ y_{(i)} \} \).

Continuing this process we get \( M_j = [M_{j-1} \cup \{ x_{(m-1)j+1}, x_{(m-1)j+2} \}] \sim \{ y_{j-1} \} \) for \( 3 \leq j \leq k+1 \), and each of the sets is an \( [(m-1)k+1] \)-set. Thus there exist \( m \) points, \( y_{j1}, y_{j2}, \ldots, y_{jm} \), of \( M_j \) and a point \( p_{j} \in \bigcap_{i=1}^{m} \text{Co} \{ y_{(i)} \} \).

Let \( K = \{ p_1, p_2, \ldots, p_{k+1} \} \). If \( p_i = p_j \) for some \( i \neq j \), the theorem is proved. Suppose \( p_i \neq p_j \) if \( i \neq j \). Since \( \mathcal{F}(T) \) has property \( R(k) \), there exist points, \( p_1 \) and \( p_j \), \( i < j \), in \( K \) and a point \( p_0 \in \text{Co} \{ p_i \} \cap \text{Co} \{ p_j \} \).

Since for each \( x \in S \), \( T(x, x) \) is convex, we have \( p_0 \in \text{Co} \{ y_{(i)} \} \cap \text{Co} \{ y_{(j)} \} \cap \cdots \cap \text{Co} \{ y_{(m)} \} \). Thus \( M \) is \((m+1)\)-divisible and \( \mathcal{F}(T) \) has property \( R(k, m+1) \). Therefore, \( \mathcal{F}(T) \) has property \( R(k, n) \) with respect to \( f(k, n) = (n-1)k + 1 \) for all \( n \geq 2 \).

**Example 3.1.** In \( \mathbb{R}^2 \) let \( l/P \) and \( \overline{l/P} \) denote, respectively, the open and the closed half planes determined by the line \( l \) and containing the point \( P \). \( PQ \) denotes the line determined by the points \( P \) and \( Q \), and \( P[m] \) denotes the line through \( P \) with slope \( m \). Let \( P_0 = (0, 0), P_1 = (1, 0), P_2 = (-1/2, \sqrt{3}/2), P_3 = (-1/2, -\sqrt{3}/2), P_4 = (1, 1), P_5 = (-1, 0), P_6 = (1, -1) \). Choose \( S = S_1 \cup S_2 \cup S_3 \) where \( S_1 = P_0 P_1/P_4 \cap P_5 P_6/P_2 \cap P_3 P_5/P_3 \), \( S_2 = P_0 P_2/P_4 \cap P_3 P_5/P_4 \), and \( S_3 = P_0 P_1/P_5 \cap P_3 P_5/P_3 \). We define an interval convexity \( T \) on \( S \) as follows:

\[
T(P, P) = \begin{cases} S_1 \cap \overline{P[-\sqrt{3}/P_6]} & \text{if } P \in S_1; \\ S_2 \cap \overline{P[\sqrt{3}/P_5]} & \text{if } P \in S_2; \\ S_3 \cap \overline{P[0]/P_6} & \text{if } P \in S_3. \end{cases}
\]

Thus if \( M \in \mathcal{F}(S) \), \( \text{Co} (M) = \bigcup_{m \in M} T(m, m) \). It is easily seen that \( \mathcal{F}(T) \) has property \( r(3) \). Thus if \( k \geq 3 \), \( \mathcal{F}(T) \) has property \( R(k, n) \) with respect to \( f(k, n) = (n-1)k + 1 \) for \( n \geq 2 \).

**Theorem 3.2.** Suppose \( \leq \) is a partial order on \( S \) such that \( \mathcal{F}(\leq) \) has property \( R(k) \). Then \( \mathcal{F}(\leq) \) has property \( R(k, n) \) with respect to \( f(k, n) = (2n-3)k + 1 \) for all \( n \geq 2 \).

**Proof.** (The proof is a slight modification of the proof of Theorem 3.1.) The statement is true for \( n = 2 \) since property \( R(k) \) is the same as property \( R(k, 2) \). Now suppose the statement is true for
n = m, and let $M = \{x_1, x_2, \ldots, x_{(2m-3)k+1}, \ldots, x_{(2m-1)k+1}\}$ be a $[(2m-1)k+1]$-set. (Note that $[(2m-3)k+1] - [(2m-3)k+1] = 2k$.) Let $K_0 = \{x_1, x_2, \ldots, x_{(2m-3)k+1}\}$ be the subset of $M$ containing the first $(2m-3)k+1$ points of $M$. Thus there exist $m$ mutually exclusive subsets, $K_{01}, K_{02}, \ldots, K_{0m}$, of $K_0$ and a point $y_0 \in \bigcap_{i=1}^{m} \text{Co}(K_{0i})$. It follows then that there exist points $s_0$ and $t_0$ in $K_0$ such that $s_0 < y_0 < t_0$.

Now let $K$ be the $[(2m-3)k+1]$-set $[K_0 \sim \{s_0, t_0\}] \cup \{x_{(2m-3)k+2}, x_{(2m-3)k+3}\}$. Again there exist $m$ mutually exclusive subsets, $K_{11}, K_{12}, \ldots, K_{1m}$, of $K$ such that $\bigcap_{i=1}^{m} \text{Co}(K_{1i}) \neq \emptyset$. If $y_0 \in \bigcap_{i=1}^{m} \text{Co}(K_{1i})$, the theorem is proved.

Suppose $y_0 \in \bigcap_{i=1}^{m} \text{Co}(K_{1i})$. Let $y_1 \in \bigcap_{i=1}^{m} \text{Co}(K_{1i})$. Then there exist points $s_1$ and $t_1$ in $K$ such that $s_1 < y_1 < t_1$.

Continuing this process for $2 \leq i \leq k$, we obtain $K_i = [K_{i-1} \sim \{s_{i-1}, t_{i-1}\}] \cup \{x_{(2m-3)k+2i}, x_{(2m-3)k+2i+1}\}$ and correspondingly $m$ mutually exclusive subsets, $K_{i1}, K_{i2}, \ldots, K_{im}$, of $K_i$ such that $\bigcap_{i=1}^{m} \text{Co}(K_{ip})$ contains a point $y_i$. Now if for some $j$ and $i$, $0 \leq j < i \leq k$, $y_j \in \bigcap_{i=1}^{m} \text{Co}(K_{ip})$, the theorem is proved.

Suppose that if $0 \leq j < i \leq k$, $y_j \in \bigcap_{i=1}^{m} \text{Co}(K_{ip})$. Then the $(k+1)$-set $C = \{y_0, y_1, \ldots, y_k\}$ is 2-divisible. Let $C_1$ and $C_2$ be mutually exclusive subsets of $C$ such that $\text{Co}(C_1) \cap \text{Co}(C_2) \neq \emptyset$. It can be shown that if $w \in \text{Co}(C_1) \cap \text{Co}(C_2)$ then there are $m+1$ mutually exclusive subsets, $M_1, M_2, \ldots, M_{m+1}$, of $M$ such that $w \in \bigcap_{i=1}^{m+1} \text{Co}(M_i)$. Hence $M$ is $m+1$ divisible. Therefore, $\mathcal{E}(\subseteq)$ has property $R(k, n)$ with respect to $f(k, n) = (2n-3)k+1$ for all $n \geq 2$.

REFERENCES


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