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HELLY AND RADON-TYPE THEOREMS IN INTERVAL CONVEXITY SPACES

PHILLIP WAYNE BEAN

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The notion of interval convexity T on a point set S is defined. If T is an interval convexity defined on S , $\mathcal{C}(T)$ will denote the collection of nonempty T -convex subsets of S . Properties k , $H(k)$ (a Helly property), and $R(k, n)$ (a Radon property) are defined on $\mathcal{C}(T)$, and relationships between these properties are investigated.

A partial order convexity \leq on a point set S is a special type of interval convexity. Some sufficient conditions are imposed on \leq and $\mathcal{C}(\leq)$ to insure the existence of certain Radon-type properties.

1. Introduction. Suppose S is a point set, and $\mathcal{P}(S)$ is the collection of nonempty subsets of S . The statement that T is an interval convexity on S means that T is a transformation from $S \times S$ into $\mathcal{P}(S)$. A subset M of S is said to be T -convex provided that $T(x, y)$ is a subset of M for every x and y in M . Let $\mathcal{C}(T)$ denote the collection of all nonempty T -convex subsets of S . For each $M \in \mathcal{P}(S)$, the convex hull of M relative to T , denoted by $\text{Co}(M)$, is the intersection of the elements of $\mathcal{C}(T)$ which contain M . We assume that if each of x and y is in S , then $T(x, y)$ is T -convex, $T(x, y)$ contains x and y , and $T(x, y) = T(y, x)$.

Let m -set mean a set of m points of S . A subset M of S is said to be n -divisible provided it may be partitioned into n mutually exclusive subsets whose T -convex hulls have a common point of S . In this paper we consider the relationship of the following Helly and Radon-type properties on a set S with an interval convexity T . $\mathcal{C}(T)$ has property $R(k)$ if each $(k + 1)$ -set of S is 2-divisible, and more generally, $\mathcal{C}(T)$ has property $R(k, n)$ with respect to some integer valued function f if each $[f(k, n)]$ -set is n -divisible. We say that $\mathcal{C}(T)$ has property $r(k)$ if k is the smallest integer for which $\mathcal{C}(T)$ has property $R(k)$. $\mathcal{C}(T)$ is said to have property $H(k)$ provided that if \mathcal{G} is a finite subcollection of $\mathcal{C}(T)$ containing at least k elements, then the following two statements are equivalent:

- (a) Each k elements of \mathcal{G} have a common point.
- (b) The elements of \mathcal{G} have a common point.

In (2) we give sufficient conditions for property $R(k)$ to be equivalent to property $H(k)$. We also consider in (2) the existence of sets with property $R(k)$ in partially ordered spaces and more generally, in (3) the existence of sets with property $R(k, n)$.

2. Theorems concerning properties k , $R(k)$, and $H(k)$. From a theorem of Levi [7] we have that property $R(k)$ implies property $H(k)$. In [1] Calder introduces the following property: $\mathcal{C}(T)$ has property k provided that if M is a finite point set containing at least $k + 1$ points, then there exists a point p such that $p \in \text{Co}[M \sim \{m\}]$ for each m in M . He proves that property k is equivalent to property $H(k)$ and then proves that property $R(k)$ is equivalent to property $H(k)$ in a partially ordered space. It should be noted that the partial order does not have to be antisymmetric. Calder also gives an example of an interval convexity T such that property $H(k)$ is not equivalent to property $R(k)$ in $\mathcal{C}(T)$. In the first two theorems of this section we give sufficient conditions on T for properties $H(k)$ and $R(k)$ to be equivalent.

If each of A and B is in $\mathcal{P}(S)$, then $A * B$ denotes the set

$$\bigcup_{a \in A, b \in B} T(a, b).$$

THEOREM 2.1. *Let T be an interval convexity on S such that for each M in $\mathcal{P}(S)$, $\text{Co}(M) = M * M$; and if a, b, c , and d are four distinct points such that d is in $T(a, b)$ and $T(a, c)$, then b is in $T(a, c)$, or c is in $T(a, b)$. Then property $H(k) \Leftrightarrow$ property $R(k)$ in $\mathcal{C}(T)$.*

THEOREM 2.2. *Let T be an interval convexity on S such that for each M in $\mathcal{P}(S)$, $\text{Co}(M) = \bigcup_{m \in M} T(m, m)$. Then property $H(k) \Leftrightarrow$ property $R(k)$ in $\mathcal{C}(T)$.*

The proofs of Theorems 2.1 and 2.2 are easy modifications of the proof of Theorem 3.2 of Calder [1].

EXAMPLE 2.1. Let M be a subset of a linear space S . A subset K of M is said to be *extremal* provided that if k is an element of K , and there exist elements x and y in M such that $k = tx + (1 - t)y$ for some $t \in (0, 1)$, then x and y are elements of K . Obviously, the union and intersection of any collection of extremal subsets of M are extremal.

We define an interval convexity T on M as follows: If each of x and y is an element of M , $T(x, y)$ is the intersection of the extremal subsets of M which contain $\{x, y\}$.

For each subset K of M , $K \subset \bigcup_{k \in K} T(k, k)$. Since $\bigcup_{k \in K} T(k, k)$ is convex, $\text{Co}(K) \subset \bigcup_{k \in K} T(k, k)$. However, $\bigcup_{k \in K} T(k, k) \subset \text{Co}(K)$. Thus $\text{Co}(K) = \bigcup_{k \in K} T(k, k)$, and hence property $H(k) \Leftrightarrow$ property $R(k)$ in $\mathcal{C}(T)$.

Let \leq be a partial order on the set S . If each of x and y is a point of S , $[x, y] = \{p \mid p = x, \text{ or } p = y, \text{ or } x < p < y, \text{ or } y < p < x\}$.

A subset M of S is said to be \leq -convex if for all elements x and y of M , $[x, y]$ is a subset of M . The collection of all \leq -convex subsets of S is denoted by $\mathcal{C}(\leq)$. In [5], Franklin shows that $\text{Co}(M) = M * M$ for any M in $\mathcal{P}(S)$.

THEOREM 2.3. *Suppose \leq is a partial order on S , and S is the union of n linearly ordered sets, S_1, S_2, \dots, S_n . Then $\mathcal{C}(\leq)$ has property $R(2n)$.*

Proof. Suppose $M = \{x_1, x_2, \dots, x_{2n+1}\}$ is a $(2n + 1)$ -set. Then for some $i, 1 \leq i \leq n$, S_i contains at least three points, z_1, z_2, z_3 , of M such that $z_1 < z_2 < z_3$. Thus $\text{Co}\{z_2\}$ and $\text{Co}\{z_1, z_3\}$ have a common point, and therefore $\mathcal{C}(\leq)$ has property $R(2n)$.

It is easy to show that $\mathcal{C}(\leq)$ has property $r(2)$ if and only if \leq linearly orders S . Suppose \leq is a partial order on S which does not linearly order S . Under these conditions on \leq , does $\mathcal{C}(\leq)$ have property $r(3)$ if and only if S is union of two mutually exclusive, linearly ordered subsets S_1 and S_2 ? The following example shows the answer to this question is no.

EXAMPLE 2.2. Let $S = \{(x, y) \in R^2 \mid y = 0 \text{ or } y = 1\}$. Define \leq on S as follows: $(x_1, y_1) \leq (x_2, y_2)$ if $y_1 = y_2$ and $x_1 \leq x_2$. Thus \leq is a partial order on S which does not linearly order S . However, \leq does linearly order $S_1 = \{(x, 1) \in R^2\}$ and $S_2 = \{(x, 0) \in R^2\}$, and $S = S_1 \cup S_2$. To show that $\mathcal{C}(\leq)$ does not have property $r(3)$ we choose $M = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Obviously M is not 2-divisible.

3. Property $R(k, n)$. Tverberg shows in [11] that the collection on convex sets in R^{k-1} has property $R(k, n)$ with respect to $f(k, n) = (n - 1)k + 1$ for $n, k \geq 2$. By putting suitable restrictions on T , we have the following:

THEOREM 3.1. *Suppose T is an interval convexity on S such that if $M \in \mathcal{P}(S)$, then $\text{Co}(M) = \bigcup_{m \in M} T(m, m)$. If $\mathcal{C}(T)$ has property $R(k)$, then $\mathcal{C}(T)$ has property $R(k, n)$ with respect to $f(k, n) = (n - 1)k + 1$ for $n \geq 2$.*

Proof. (We use induction on n .) Suppose $\mathcal{C}(T)$ has property $R(k)$, i.e., $\mathcal{C}(T)$ has property $R(k, 2)$. Suppose further that $\mathcal{C}(T)$ has property $R(k, m)$ for some $m \geq 2$, and let $M = \{x_1, x_2, \dots, x_{mk+1}\}$ be an $[mk + 1]$ -set. Let $M_1 = \{x_1, x_2, \dots, x_{(m-1)k+1}\}$ be the subset of M containing the first $(m - 1)k + 1$ points of M . Then there exist m points, $y_{11}, y_{12}, \dots, y_{1m}$, of M_1 and a point p_1 such that

$$p_1 \in \bigcap_{i=1}^m \text{Co} \{y_{1i}\} .$$

Now choose $M_2 = \{x_1, x_2, \dots, x_{(m-1)k+1}, x_{(m-1)k+2}\} \sim \{y_{11}\}$. Thus M_2 is an $[(m-1)k+1]$ -set, and hence there exist m points, $y_{21}, y_{22}, \dots, y_{2m}$, of M_2 and a point $p_2 \in \bigcap_{i=1}^m \text{Co} \{y_{2i}\}$.

Continuing this process we get $M_j = [M_{j-1} \cup \{x_{(m-1)k+j}\}] \sim \{y_{j-11}\}$ for $3 \leq j \leq k+1$, and each of the sets is an $[(m-1)k+1]$ -set. Thus there exist m points, $y_{j1}, y_{j2}, \dots, y_{jm}$, of M_j and a point $p_j \in \bigcap_{i=1}^m \text{Co} \{y_{ji}\}$.

Let $K = \{p_1, p_2, \dots, p_{k+1}\}$. If $p_i = p_j$ for some $i \neq j$, the theorem is proved. Suppose $p_i \neq p_j$ if $i \neq j$. Since $\mathcal{C}(T)$ has property $R(k)$, there exist points, p_i and $p_j, i < j$, in K and a point

$$p_0 \in \text{Co} \{p_i\} \cap \text{Co} \{p_j\} .$$

Since for each $x \in S, T(x, x)$ is convex, we have $p_0 \in \text{Co} \{y_{i1}\} \cap \text{Co} \{y_{j1}\} \cap \dots \cap \text{Co} \{y_{jm}\}$. Thus M is $(m+1)$ -divisible and $\mathcal{C}(T)$ has property $R(k, m+1)$. Therefore, $\mathcal{C}(T)$ has property $R(k, n)$ with respect to $f(k, n) = (n-1)k+1$ for all $n \geq 2$.

EXAMPLE 3.1. In R^2 let l/P and $\overline{l/P}$ denote, respectively, the open and the closed half planes determined by the line l and containing the point P . PQ denotes the line determined by the points P and Q , and $P[m]$ denotes the line through P with slope m . Let $P_0 = (0, 0), P_1 = (1, 0), P_2 = (-1/2, \sqrt{3}/2), P_3 = (-1/2, -\sqrt{3}/2), P_4 = (1, 1), P_5 = (-1, 0), P_6 = (1, -1)$. Choose $S = S_1 \cup S_2 \cup S_3$, where $S_1 = P_0P_1/P_4 \cap P_0P_2/P_4, S_2 = P_0P_2/P_5 \cap P_0P_3/P_5$, and $S_3 = P_0P_1/P_6 \cap P_0P_3/P_6$. We define an interval convexity T on S as follows:

$$(a) \quad T(P, P) = \begin{cases} S_1 \cap \overline{P[-\sqrt{3}]/P_0} & \text{if } P \in S_1; \\ S_2 \cap \overline{P[\sqrt{3}]/P_0} & \text{if } P \in S_2; \\ S_3 \cap \overline{P[0]/P_0} & \text{if } P \in S_3. \end{cases}$$

$$(b) \quad T(P, Q) = T(P, P) \cup T(Q, Q) .$$

Thus if $M \in \mathcal{S}(S), \text{Co}(M) = \bigcup_{m \in M} T(m, m)$. It is easily seen that $\mathcal{C}(T)$ has property $r(3)$. Thus if $k \geq 3, \mathcal{C}(T)$ has property $R(k, n)$ with respect to $f(k, n) = (n-1)k+1$ for $n \geq 2$.

THEOREM 3.2. Suppose \leq is a partial order on S such that $\mathcal{C}(\leq)$ has property $R(k)$. Then $\mathcal{C}(\leq)$ has property $R(k, n)$ with respect to $f(k, n) = (2n-3)k+1$ for all $n \geq 2$.

Proof. (The proof is a slight modification of the proof of Theorem 3.1.) The statement is true for $n = 2$ since property $R(k)$ is the same as property $R(k, 2)$. Now suppose the statement is true for

$n = m$, and let $M = \{x_1, x_2, \dots, x_{(2m-3)k+1}, \dots, x_{(2m-1)k+1}\}$ be a $[(2m-1)k + 1]$ -set. (Note that $[(2m-1)k + 1] - [(2m-3)k + 1] = 2k$.) Let $K_0 = \{x_1, x_2, \dots, x_{(2m-3)k+1}\}$ be the subset of M containing the first $(2m-3)k + 1$ points of M . Thus there exist m mutually exclusive subsets, $K_{01}, K_{02}, \dots, K_{0m}$, of K_0 and a point $y_0 \in \bigcap_{i=1}^m \text{Co}(K_{0i})$. It follows then that there exist points s_0 and t_0 in K_0 such that $s_0 < y_0 < t_0$. Now let K_1 be the $[(2m-3)k + 1]$ -set $[K_0 \sim \{s_0, t_0\}] \cup \{x_{(2m-3)k+2}, x_{(2m-3)k+3}\}$. Again there exist m mutually exclusive subsets, $K_{11}, K_{12}, \dots, K_{1m}$, of K_1 such that $\bigcap_{i=1}^m \text{Co}(K_{1i}) \neq \emptyset$. If $y_0 \in \bigcap_{i=1}^m \text{Co}(K_{1i})$, the theorem is proved.

Suppose $y_0 \notin \bigcap_{i=1}^m \text{Co}(K_{1i})$. Let $y_1 \in \bigcap_{i=1}^m \text{Co}(K_{1i})$. Then there exist points s_1 and t_1 in K_1 such that $s_1 < y_1 < t_1$.

Continuing this process for $2 \leq i \leq k$, we obtain $K_i = [K_{i-1} \sim \{s_{i-1}, t_{i-1}\}] \cup \{x_{(2m-3)k+2i}, x_{(2m-3)k+(2i+1)}\}$ and correspondingly m mutually exclusive subsets, $K_{i1}, K_{i2}, \dots, K_{im}$, of K_i such that $\bigcap_{p=1}^m \text{Co}(K_{ip})$ contains a point y_i . Now if for some j and i , $0 \leq j < i \leq k$, $y_j \in \bigcap_{p=1}^m \text{Co}(K_{ip})$, the theorem is proved.

Suppose that if $0 \leq j < i \leq k$, $y_j \notin \bigcap_{p=1}^m \text{Co}(K_{ip})$. Then the $(k + 1)$ -set $C = \{y_0, y_1, \dots, y_k\}$ is 2-divisible. Let C_1 and C_2 be mutually exclusive subsets of C such that $\text{Co}(C_1) \cap \text{Co}(C_2) \neq \emptyset$. It can be shown that if $w \in \text{Co}(C_1) \cap \text{Co}(C_2)$ then there are $m + 1$ mutually exclusive subsets, M_1, M_2, \dots, M_{m+1} , of M such that $w \in \bigcap_{i=1}^{m+1} \text{Co}(M_i)$. Hence M is $m + 1$ divisible. Therefore, $\mathcal{C}(\leq)$ has property $R(k, n)$ with respect to $f(k, n) = (2n - 3)k + 1$ for all $n \geq 2$.

REFERENCES

1. J. R. Calder, *Some elementary properties of interval convexities*, J. London Math. Soc., **3** (1971), 422-428.
2. L. Danzer, N. Grümbaum, and V. Klee, *Helly's theorem and its relatives*, Proceedings of Symposia in Pure Mathematics, **7** (1963), Convexity, 101-177.
3. J. Eckhoff, *Der Satz von Radon in konvexen Produktstrukturen I - II*, Monatsh. Math., **72** (1968), 303-314, and **73** (1969), 7-30.
4. J. W. Ellis, *A general set-separation theorem*, Duke Math. J., **19** (1952), 417-421.
5. S. P. Franklin, *Some results on order convexity*, Amer. Math. Monthly, **69** (1962), 357-359.
6. David Kay and E. W. Womble, *Axiomatic convexity theory and relationships between the Caratheodory, Helly and Radon numbers*, Pacific J. Math., **38** (1971), 471-485.
7. F. W. Levi, *On Helly's theorem and the axioms of convexity*, J. Indian Mathematical Society, (N. S.) Part A, **15** (1951), 65-76.
8. T. S. Motzkin, *Linear Inequalities*, Mimeographed lecture notes, University of California, Los Angeles, 1951.
9. J. R. Reay, *Caratheodory theorems in convex product structure*, Pacific J. Math., **35** (1970), 227-230.
10. ———, *An extension of Radon's theorem*, Illinois J. Math., **12** (1968), 184-189.

11. H. Tverberg, *A generalization of Radon's theorem*, J. London Math. Soc., **41** (1966), 123-128.

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