

# Pacific Journal of Mathematics

## **DETERMINANTS OF PETRIE MATRICES**

MANFRED GORDON AND EDWARD MARTIN WILKINSON

## DETERMINANTS OF PETRIE MATRICES

MANFRED GORDON AND E. MARTIN WILKINSON

**The number of nonsingular square Petrie matrices is calculated by setting up a correspondence with spanning trees and then using Cayley's Theorem to count these trees. This work has applications in the 'excluded volume' problem in Polymer Science.**

A Petrie matrix is a finite matrix whose elements are either zeros or ones such that the ones in each column occur consecutively. Such matrices have been studied in a molecular biological situation by Fulkerson and Gross [2] and in archaeology by Kendall [4 and 5] and Wilkinson [7]. The problem here treated has arisen from a combinatorial analysis [3] of the theory of the 'excluded volume' of a polymer chain [1], in which Petrie matrices play an important, though hitherto unrecognized, role.

The Fulkerson and Gross paper quotes the interesting fact that Petrie matrices are unimodular, that is all square sub-matrices have a determinant which is either  $-1$ ,  $0$ , or  $1$ . The object of this paper is to investigate square Petrie matrices with the view of counting the number with nonzero determinant. We propose to do this by constructing a correspondence between  $n$  by  $n$  Petrie matrices and graphs on  $n + 1$  vertices with at most  $n$  edges, and then to show that precisely those graphs that are spanning trees correspond to nonsingular Petrie matrices. A well known theorem by Cayley can then be used to count up the number of distinct labelled nonrooted trees.

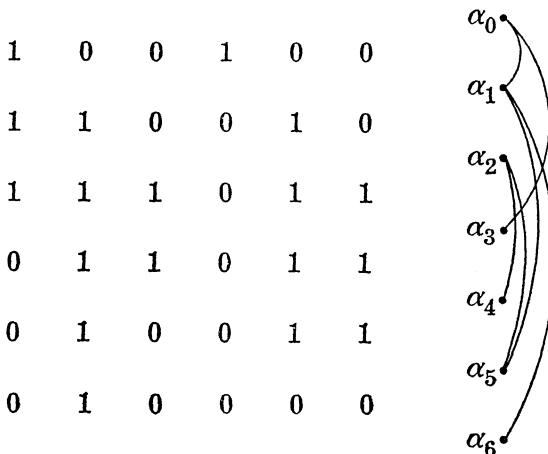


FIGURE 1

For an  $n$  by  $n$  Petrie matrix  $A$ , we construct a graph on  $n + 1$  points, which we label  $\alpha_0 \cdots \alpha_n$ . If a column of  $A$  has ones from row  $i$  to row  $j$  then we insert an edge between  $\alpha_{i-1}$  and  $\alpha_j$ . Figure 1 illustrates the construction of a tree from a Petrie matrix.

If the graph has  $m$  nonzero columns, there will be  $m$  edges in the graph. So we have constructed a mapping from the set of  $n$  by  $n$  Petrie matrices to the graphs on  $n + 1$  points with at most  $n$  edges. This mapping is onto, for given any such graph, we can trivially construct a Petrie matrix that maps onto it. If the graphs of two matrices are the same, then one is a column permutation of the other. So we can construct a one-to-one correspondence between the equivalence classes of Petrie matrices with respect to column permutations and the set of graphs.

**THEOREM 1.** *A square Petrie matrix is nonsingular if and only if its associated graph is a spanning tree. The determinant is either 0, 1 or  $-1$ .*

*Proof.* If the graph is not a spanning tree then either there is a circuit, or the graph is disconnected. If there is no circuit and the graph is disconnected then there are less than  $n$  edges and hence at least one of the columns of  $A$  must be all zeros. If there is a circuit then we can construct a linear dependence between the columns. Consider a journey round a circuit. When going from  $\alpha_i$  to  $\alpha_j$  along an edge, add the corresponding column of  $A$  if  $i > j$  and subtract if  $j > i$ . On completing the circuit the sum of these columns will be zero, since one has traversed each row as many times upwards as downwards.

If, on the other hand, the graph is a spanning tree, we prove by induction on  $n$  that the determinant of the Petrie matrix is plus or minus one. If  $n = 1$  the result is true. Assume the hypothesis is true for all matrices of size  $n - 1$ . There is at least one edge incident on  $\alpha_n$ . Consider those columns corresponding to edges incident on  $\alpha_n$ . There is just one ( $X$ , say) with fewest ones, since otherwise there would be a circuit in the tree. We subtract this column  $X$  from the others, leaving a new Petrie matrix with a single one in the last row. The determinant of the matrix is the cofactor of the single one in the last row. The graph corresponding to this determinant of order  $(n - 1)$  is obtained from that of the original determinant of order  $n$  by elementary contraction of the edge corresponding to  $X$ , and thus is again a spanning tree.

**THEOREM 2.** *There are  $n!(n + 1)^{n-1}$  distinct nonsingular  $n$  by  $n$  Petrie matrices.*

*Proof.* By Cayley's theorem, there are  $(n + 1)^{n-1}$  distinct vertex-labelled trees on  $n + 1$  vertices. Each tree corresponds to a class of Petrie matrices. The columns are necessarily distinct and so each class contains  $n!$  distinct matrices.

It is clear that there are  $((n(n + 1))/2 + 1)^n$  distinct Petrie matrices, so the proportion that are nonsingular tends to zero as  $n$  tends to infinity. This is interesting because for the general class of  $n$  by  $n$   $(0, 1)$ -matrices, of which Petrie matrices form a sub-class, Komlós [6] has proved the opposite kind of asymptotic behavior: for them, the fraction of nonsingular ones tends to unity.

*Acknowledgement.* M. Gordon wishes to thank Professors D. G. Kendall and P. Whittle for hospitality at the Statistical Laboratory and Corpus Christi College for a Visiting Scholarship. E. M. Wilkinson wishes to thank the S.R.C. for a Research Studentship.

#### REFERENCES

1. M. Fixman, *Excluded volume in polymer chains*, J. Chem. Phys., **23** (1955), 1656.
2. D. R. Fulkerson and O. A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math., **15**, 3, (1965), 835-855.
3. M. Gordon, S. B. Ross-Murphy, and H. Suzuki, to be published.
4. D. G. Kendall, *Incidence matrices, interval graphs, and seriation in archaeology*, Pacific J. Math., **28**, 3, (1969), 565-570.
5. ———, *Seriation from Abundance Matrices*, Mathematics in the Archaeological and Historical Sciences, Edinburgh University Press, (1971), 215-252.
6. J. Komlós, *On the determinant of  $(0, 1)$  matrices*, Studia Sci. Math. Hung., **2** (1967), 7.
7. E. M. Wilkinson, *Archaeological Seriation and the Travelling Salesman Problem*, Mathematics in the Archaeological and Historical Sciences, Edinburgh University Press, (1971), 276-287.

Received October 4, 1972.

UNIVERSITY OF CAMBRIDGE

*Permanent address:* Chemistry Department, University of Essex, Colchester, England  
AND

*Present address:* Van den Berghs & Jurgens Ltd., Cheshire L62 3NU, England



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI\*  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

Robert F. V. Anderson, <i>Laplace transform methods in multivariate spectral theory</i> .....	339
William George Bade, <i>Two properties of the Sorgenfrey plane</i> .....	349
John Robert Baxter and Rafael Van Severen Chacon, <i>Functionals on continuous functions</i> .....	355
Phillip Wayne Bean, <i>Helly and Radon-type theorems in interval convexity spaces</i> .....	363
James Robert Boone, <i>On <math>k</math>-quotient mappings</i> .....	369
Ronald P. Brown, <i>Extended prime spots and quadratic forms</i> .....	379
William Hugh Cornish, <i>Crawley's completion of a conditionally upper continuous lattice</i> .....	397
Robert S. Cunningham, <i>On finite left localizations</i> .....	407
Robert Jay Daverman, <i>Approximating polyhedra in codimension one spheres embedded in <math>S^n</math> by tame polyhedra</i> .....	417
Burton I. Fein, <i>Minimal splitting fields for group representations</i> .....	427
Peter Fletcher and Robert Allen McCoy, <i>Conditions under which a connected representable space is locally connected</i> .....	433
Jonathan Samuel Golan, <i>Topologies on the torsion-theoretic spectrum of a noncommutative ring</i> .....	439
Manfred Gordon and Edward Martin Wilkinson, <i>Determinants of Petrie matrices</i> .....	451
Alfred Peter Hallstrom, <i>A counterexample to a conjecture on an integral condition for determining peak points (counterexample concerning peak points)</i> .....	455
E. R. Heal and Michael Windham, <i>Finitely generated <math>F</math>-algebras with applications to Stein manifolds</i> .....	459
Denton Elwood Hewgill, <i>On the eigenvalues of a second order elliptic operator in an unbounded domain</i> .....	467
Charles Royal Johnson, <i>The Hadamard product of <math>A</math> and <math>A^*</math></i> .....	477
Darrell Conley Kent and Gary Douglas Richardson, <i>Regular completions of Cauchy spaces</i> .....	483
Alan Greenwell Law and Ann L. McKerracher, <i>Sharpened polynomial approximation</i> .....	491
Bruce Stephen Lund, <i>Subalgebras of finite codimension in the algebra of analytic functions on a Riemann surface</i> .....	495
Robert Wilmer Miller, <i>TTF classes and quasi-generators</i> .....	499
Roberta Mura and Akbar H. Rhemtulla, <i>Solvable groups in which every maximal partial order is isolated</i> .....	509
Isaac Namioka, <i>Separate continuity and joint continuity</i> .....	515
Edgar Andrews Rutter, <i>A characterization of QF - 3 rings</i> .....	533
Alan Saleski, <i>Entropy of self-homeomorphisms of statistical pseudo-metric spaces</i> .....	537
Ryōtarō Satō, <i>An Abel-maximal ergodic theorem for semi-groups</i> .....	543
H. A. Seid, <i>Cyclic multiplication operators on <math>L_p</math>-spaces</i> .....	549
H. B. Skerry, <i>On matrix maps of entire sequences</i> .....	563
John Brendan Sullivan, <i>A proof of the finite generation of invariants of a normal subgroup</i> .....	571
John Griggs Thompson, <i>Nonsolvable finite groups all of whose local subgroups are solvable, VI</i> .....	573
Ronson Joseph Warne, <i>Generalized <math>\omega</math> - <math>\mathcal{L}</math>-unipotent bisimple semigroups</i> .....	631
Toshihiko Yamada, <i>On a splitting field of representations of a finite group</i> .....	649