

# Pacific Journal of Mathematics

**A COUNTEREXAMPLE TO A CONJECTURE ON AN  
INTEGRAL CONDITION FOR DETERMINING PEAK POINTS  
(COUNTEREXAMPLE CONCERNING PEAK POINTS)**

ALFRED PETER HALLSTROM

A COUNTEREXAMPLE TO A CONJECTURE  
 ON AN INTEGRAL CONDITION FOR  
 DETERMINING PEAK POINTS  
 (COUNTEREXAMPLE CONCERNING PEAK POINTS)

ALFRED P. HALLSTROM

Let  $X$  be a compact plane set. Denote by  $R(X)$  the uniform algebra generated by the rational functions with poles off  $X$  and by  $H(X)$  the space of functions harmonic in a neighborhood of  $X$  endowed with the sup norm. A point  $p \in \partial X$  is a peak point for  $R(X)$  if there exists a function  $f \in R(X)$  such that  $f(p) = 1$  and  $|f(x)| < 1$  if  $x \neq p$ . Moreover,  $p$  is a peak point for  $H(X)$  (consider  $\text{Re } f$ ) and hence, by a theorem of Keldysh,  $p$  is a regular point for the Dirichlet problem. Conditions which determine whether or not a point is a peak point for  $R(X)$  are thus of interest in harmonic analysis. Melnikov has given a necessary and sufficient condition that  $p$  be a peak point for  $R(X)$  in terms of analytic capacity,  $\gamma$ ; namely  $p$  is a peak point for  $R(X)$  if and only if

$$\sum_{n=0}^{\infty} 2^n \gamma(A_{np} \setminus X) = \infty. \quad A_{np} = \left\{ z: \frac{1}{2^{n+1}} \leq |z - p| \leq \frac{1}{2^n} \right\}.$$

Analytic capacity is generally difficult to compute, so it is desirable to obtain more computable types of conditions. Let  $X^c = C \setminus X$  and

$$I = \{t \in [0, 1]: z \in X^c \text{ and } |z| = t\}.$$

In this note the following conjecture, which can be found in Zalcman's Springer Lecture Notes and which is true for certain sets  $X$ , is shown to be false in general:

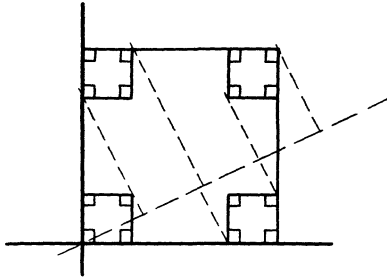
**Conjecture.** If  $\int_I t^{-1} dt = \infty$  then 0 is a peak point for  $R(X)$ .

Our counterexample uses Melnikov's theorem and the following lemma:

**LEMMA.** Given  $0 < a < b$  and  $\log b/a < 2\pi$  there exists a set  $K_{ab}$  such that  $K_{ab} \subset \{z: a \leq |z| \leq b\}$ ,  $\gamma(K_{ab}) = 0$  and  $\{t: z \in K_{ab} \text{ and } |z| = t\} = [a, b]$ .

The author is indebted to the referee for the following proof.

Garnett in [3] showed that the "Cantor corner square" set  $K$  constructed by removing all but the four corner squares of length 1/4 from the unit square, then removing all but the sixteen corner squares of length 1/16 from



these four squares, etc., has zero analytic capacity while the projection on the line  $y = x/2$  is full, i.e., is the same as the projection of the unit square on that line. Thus given  $0 < a < b$  and  $\log b/a < 2\pi$  there exists after a suitable rotation, expansion and translation a compact plane set  $L_{ab}$  such that  $\gamma(L_{ab}) = 0$ , the projection on the  $x$ -axis of  $L_{ab}$  is  $[\log a, \log b]$  and  $L_{ab} \subset \{z: \log a \leq x \leq \log b, 0 \leq y < 2\pi\}$ . Let  $K_{ab} = \{e^z: z \in L_{ab}\}$ . Let  $W$  be a small neighborhood of  $L_{ab}$  such that the exponential map is 1-1 on  $W$  and  $V = \{e^z: z \in w\}$ . If  $g$  is bounded and analytic on  $V \setminus K_{ab}$  then  $g(e^z)$  is bounded and analytic on  $W \setminus L_{ab}$ . Since  $\gamma(L_{ab}) = 0$ ,  $g(e^z)$  extends analytically to  $L_{ab}$  so  $g$  extends analytically to  $K_{ab}$ . Thus  $\gamma(K_{ab}) = 0$ . The other properties required of  $K_{ab}$  obviously hold.

To construct our counterexample we choose open sets  $U_n \subset A_{n_0}$  such that  $K_{6/5 \cdot 1/2^{2^n+1}, 5/6 \cdot 1/2^{2^n}} \subset U_n$ , and such that  $\gamma(U_n) < 1/4^n$ . Let  $X = \Delta(0, 1) \setminus (\bigcup_{n=0}^\infty U_n)$ . Then

$$\sum_{n=0}^\infty 2^n \gamma(A_{n_0} \setminus X) = \sum_{n=0}^\infty 2^n \gamma(U_n) < \infty$$

so 0 is not a peak point for  $R(X)$  by Melnikov's theorem. On the other hand,

$$\int_I t^{-1} dt \geq \sum_{n=0}^\infty \int_{6/5 \cdot 1/2^{2^n+1}}^{5/6 \cdot 1/2^{2^n}} t^{-1} dt = \sum_{n=0}^\infty \text{Ln} \frac{5}{6} \cdot \frac{1}{2^n} = \sum_{n=0}^\infty \text{Ln} \frac{50}{60} = \infty .$$

If we choose  $U_n$  such that  $\gamma(U_n) \leq 1/(2^n)^{2^n}$  then 0 supports bounded point derivations of all orders for  $R(X)$ , see [4], so 0 is indeed far from being a peak point for  $R(X)$ . The question remains whether 0 might be a regular point for the Dirichlet problem. We note that this could only happen if the set of representing measures  $M_x$  for  $R(K)$  at  $x \in X^0$  is not norm compact, see [1].

REFERENCES

1. S. Fisher, *Norm compact sets of representing measures*, Proc. Amer. Math. Soc., **19** (1968), 1063-1068.

2. J. Garnett, *Analytic capacity and measure*, University of California, preprint p. 87-95.
3. ———, *Positive length but zero analytic capacity*, Proc. Amer. Math. Soc., **21** (1970), 696-699.
4. A. P. Hallstrom, *On bounded point derivations and analytic capacity*, J. Functional Analysis, **4** (1969).
5. M. V. Keldysh, *On the solvability and stability of the Dirichlet problem*, Uspehi Mat. Nauk., **8** (1941), 171-231.
6. M. S. Melnikov, *Analytic capacity and the Cauchy integral*, Soviet Math. Dokl., **8** (1967), 20-23.
7. L. Zalcman, *Analytic capacity and rational approximation*, Lecture Notes in Mathematics 50, Berlin, 1968, p. 130.

Received August 18, 1972. This work was partially supported by NSF Grant No. GP-616508.

UNIVERSITY OF WASHINGTON



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI\*  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

# Pacific Journal of Mathematics

Vol. 51, No. 2

December, 1974

Robert F. V. Anderson, <i>Laplace transform methods in multivariate spectral theory</i> .....	339
William George Bade, <i>Two properties of the Sorgenfrey plane</i> .....	349
John Robert Baxter and Rafael Van Severen Chacon, <i>Functionals on continuous functions</i> .....	355
Phillip Wayne Bean, <i>Helly and Radon-type theorems in interval convexity spaces</i> .....	363
James Robert Boone, <i>On <math>k</math>-quotient mappings</i> .....	369
Ronald P. Brown, <i>Extended prime spots and quadratic forms</i> .....	379
William Hugh Cornish, <i>Crawley's completion of a conditionally upper continuous lattice</i> .....	397
Robert S. Cunningham, <i>On finite left localizations</i> .....	407
Robert Jay Daverman, <i>Approximating polyhedra in codimension one spheres embedded in <math>S^n</math> by tame polyhedra</i> .....	417
Burton I. Fein, <i>Minimal splitting fields for group representations</i> .....	427
Peter Fletcher and Robert Allen McCoy, <i>Conditions under which a connected representable space is locally connected</i> .....	433
Jonathan Samuel Golan, <i>Topologies on the torsion-theoretic spectrum of a noncommutative ring</i> .....	439
Manfred Gordon and Edward Martin Wilkinson, <i>Determinants of Petrie matrices</i> .....	451
Alfred Peter Hallstrom, <i>A counterexample to a conjecture on an integral condition for determining peak points (counterexample concerning peak points)</i> .....	455
E. R. Heal and Michael Windham, <i>Finitely generated <math>F</math>-algebras with applications to Stein manifolds</i> .....	459
Denton Elwood Hewgill, <i>On the eigenvalues of a second order elliptic operator in an unbounded domain</i> .....	467
Charles Royal Johnson, <i>The Hadamard product of <math>A</math> and <math>A^*</math></i> .....	477
Darrell Conley Kent and Gary Douglas Richardson, <i>Regular completions of Cauchy spaces</i> .....	483
Alan Greenwell Law and Ann L. McKerracher, <i>Sharpened polynomial approximation</i> .....	491
Bruce Stephen Lund, <i>Subalgebras of finite codimension in the algebra of analytic functions on a Riemann surface</i> .....	495
Robert Wilmer Miller, <i>TTF classes and quasi-generators</i> .....	499
Roberta Mura and Akbar H. Rhemtulla, <i>Solvable groups in which every maximal partial order is isolated</i> .....	509
Isaac Namioka, <i>Separate continuity and joint continuity</i> .....	515
Edgar Andrews Rutter, <i>A characterization of QF - 3 rings</i> .....	533
Alan Saleski, <i>Entropy of self-homeomorphisms of statistical pseudo-metric spaces</i> .....	537
Ryōtarō Satō, <i>An Abel-maximal ergodic theorem for semi-groups</i> .....	543
H. A. Seid, <i>Cyclic multiplication operators on <math>L_p</math>-spaces</i> .....	549
H. B. Skerry, <i>On matrix maps of entire sequences</i> .....	563
John Brendan Sullivan, <i>A proof of the finite generation of invariants of a normal subgroup</i> .....	571
John Griggs Thompson, <i>Nonsolvable finite groups all of whose local subgroups are solvable, VI</i> .....	573
Ronson Joseph Warne, <i>Generalized <math>\omega - \mathcal{L}</math>-unipotent bisimple semigroups</i> .....	631
Toshihiko Yamada, <i>On a splitting field of representations of a finite group</i> .....	649