

# Pacific Journal of Mathematics

**THE HADAMARD PRODUCT OF  $A$  AND  $A^*$**

CHARLES ROYAL JOHNSON

## THE HADAMARD PRODUCT OF $A$ AND $A^*$

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**Coefficient-wise multiplication was introduced by Hadamard and has been studied for certain square matrices by I. Schur and later authors. For  $A \in M_n(C)$ , the  $n$  by  $n$  complex matrices, this paper examines the Hadamard product of  $A$  and  $A^*$ . Upper estimates are given for the largest characteristic root of this necessarily Hermitian product, and three conditions on  $A$  sufficient for the product to be positive definite are presented.**

1. Preliminaries. If  $A = (a_{ij})$  and  $B = (b_{ij})$  are elements of  $M_n(C)$ , the *Hadamard product* [see 4, 5, 6] of  $A$  and  $B$  is the matrix  $A \circ B = (a_{ij}b_{ij}) \in M_n(C)$ . Let  $\Sigma_n$  denote the class of Hermitian positive definite elements of  $M_n(C)$ . I. Schur [7] showed that  $\Sigma_n$  is closed under Hadamard multiplication and this fact was further investigated in [5]. Fiedler [1] provided the result that  $A \in \Sigma_n$  implies  $A \circ A^{-1} \geq I$ .

Whereas the usual product of  $A$  and  $A^*$  is Hermitian and positive semidefinite, the Hadamard product  $A \circ A^* = f(A)$  is necessarily Hermitian but not necessarily positive semidefinite. We first develop several facts, some of which are of interest by themselves, with which to study  $f(A)$ . Theorem 1, for instance, generalizes Schur's result.

NOTATION 1. We shall adopt the following additional notational conveniences. For  $A \in M_n(C)$ ,  $H(A) = (A + A^*)/2$ , the *Hermitian part* and  $S(A) = (A - A^*)/2$ , the *skew-Hermitian part* of  $A$ , and let  $\Pi_n$  denote the class of  $A \in M_n(C)$  for which  $H(A) \in \Sigma_n$ . Also let  $F(A) = \{x^*Ax \mid x \in C^n, x^*x = 1\}$ , the *field of values* and  $F_{\text{ang}}(A) = \{x^*Ax \mid 0 \neq x \in C^n\}$ , the *angular field of values* of  $A$ . Starting with the upper right and proceeding counterclockwise, number the interiors of the *quadrants* of the complex plane  $Q_1, Q_2, Q_3, Q_4$ . If  $S$  and  $S_0$  are two sets in the complex plane their *sum*  $S + S_0 = \{x + x_0 \mid x \in S, x_0 \in S_0\}$  and their *product*  $SS_0 = \{xx_0 \mid x \in S, x_0 \in S_0\}$  and denote the closure of  $S$  with respect to the Euclidean norm by  $\bar{S}$ . Now it is clear that  $A \in \Pi_n$  if and only if  $F_{\text{ang}}(A) \subset \text{interior}(\bar{Q}_1 \cup \bar{Q}_4)$ . Denote by  $\sigma(A)$  the set of all characteristic roots of  $A \in M_n(C)$ , and for Hermitian  $A, B$  let  $A > B$  mean  $A - B \in \Sigma_n$ .  $X^{(m)}$  will denote the  $m$ th Hadamard power of  $X \in M_n(C)$  and  $J \in M_n(C)$  will be the *Hadamard identity*, the matrix of all ones.  $D$  will always be a diagonal matrix. It is well known that  $\sigma(A) \subseteq F(A) \subseteq F_{\text{ang}}(A)$  and the latter is a positive convex cone. Both  $F$  and  $F_{\text{ang}}$  are subadditive as set-valued functions of a matrix argument.

**THEOREM 1.** *If  $H \in \Sigma_n$ ,  $A \in M_n(C)$ , then  $F_{\text{ang}}(H \circ A) \subseteq F_{\text{ang}}(A)$ .*

*Proof.* Since  $H \in \Sigma_n$  we may write  $H = B^*B$  where  $B$  is nonsingular. The  $i, j$ -entry of  $H \circ A$  is then  $\sum_{k=1}^n \bar{b}_{ki} b_{kj} a_{ij}$  so that we have

$$\begin{aligned} x^*(H \circ A)x &= \sum_{i,j,k=1}^n \bar{b}_{ki} b_{kj} a_{ij} \bar{x}_i x_j \\ &= \sum_{k=1}^n y_k^* A y_k \quad \text{where } y_k^* = (\bar{b}_{k1} \bar{x}_1, \dots, \bar{b}_{kn} \bar{x}_n). \end{aligned}$$

Since  $F_{\text{ang}}(A)$  is a positive convex cone and since  $B$  is nonsingular, the latter sum is in  $F_{\text{ang}}(A)$  when  $x \neq 0$ . We then conclude  $x^*(H \circ A)x \in F_{\text{ang}}(A)$  which completes the proof.

**COROLLARY 1.** *If  $A, B \in M_n(C)$  and  $F_{\text{ang}}(A) \subseteq Q_1$ , then*

$$F_{\text{ang}}(A \circ B) \subseteq F_{\text{ang}}(B) + iF_{\text{ang}}(B).$$

*Proof.*  $F_{\text{ang}}(A) \subseteq Q_1$  if and only if  $H(A) \in \Sigma_n$  and  $1/iS(A) = K \in \Sigma_n$ . Now  $A \circ B = H(A) \circ B + iK \circ B$  so that

$$F_{\text{ang}}(A \circ B) \subseteq F_{\text{ang}}(H(A) \circ B) + iF_{\text{ang}}(K \circ B)$$

because of the subadditivity of  $F_{\text{ang}}$ . By Theorem 1 it then follows that  $F_{\text{ang}}(A \circ B) \subseteq F_{\text{ang}}(B) + iF_{\text{ang}}(B)$  as the corollary asserts.

**COROLLARY 2.** *If  $A, B \in M_n(C)$  and  $F_{\text{ang}}(A) \subseteq Q_1$  and  $F_{\text{ang}}(B^*) \subseteq Q_1$ , then  $A \circ B \in \Pi_n$ .*

*Proof.* Since  $F_{\text{ang}}(B^*) \subseteq Q_1$ ,  $F_{\text{ang}}(B) \subseteq Q_4$  and since  $F_{\text{ang}}(A) \subseteq Q_1$ , we have by Corollary 1 that  $F_{\text{ang}}(A \circ B) \subseteq F_{\text{ang}}(B) + iF_{\text{ang}}(B) \subseteq Q_4 + iQ_4 = Q_4 + Q_1 \subseteq \text{interior}(\bar{Q}_1 \cup \bar{Q}_4)$ . That  $F_{\text{ang}}(A \circ B) \subseteq \text{interior}(\bar{Q}_1 \cup \bar{Q}_4)$  means  $A \circ B \in \Pi_n$  and completes the proof.

**REMARK.**  $A \circ B \in \Pi_n$  if and only if  $H(A) \circ H(B) + S(A) \circ S(B) > 0$  and thus  $f(A) \in \Sigma_n$  if and only if  $H(A)^{(2)} > S(A)^{(2)}$ .

*Proof.* An easy computation shows that  $H(A \circ B) = H(A) \circ H(B) + S(A) \circ S(B)$  so that the first part of the remark follows. The second portion then follows by taking  $B = A^*$  and thus  $S(B) = -S(A)$ .

**THEOREM 2.** *Suppose  $A, D \in M_n(C)$  and  $D$  is a nonsingular diagonal matrix. Then  $f(A) \in \Sigma_n$  if and only if  $f(DA) \in \Sigma_n$ .*

*Proof.* Since  $\Sigma_n$  is closed under congruence, the statement of the theorem follows from the observation that  $f(DA) = DA \circ A^* D^* = D(A \circ A^*) D^* = Df(A) D^*$ .

2. The largest eigenvalue of  $A \circ A^*$ . Since  $f(A)$  is Hermitian,  $\sigma(f(A))$  is real. Employing a result of [4] we next estimate the largest member of  $\sigma(f(A))$  which is necessarily nonnegative.

NOTATION 2. Denote the numerical radius of  $A \in M_n(C)$  by  $r(A) = \max_{t \in F(A)} |t|$ . If  $\sigma(A)$  is real, let  $\lambda_M(A) = \max_{\lambda \in \sigma(A)} \lambda$  and  $\lambda_m(A) = \min_{\lambda \in \sigma(A)} \lambda$ . In case  $A$  is Hermitian,  $r(A) = \max \{ \lambda_M(A), |\lambda_m(A)| \}$ .

LEMMA 1. [4]. If  $A, N \in M_n(C)$  and  $N$  is normal, then

$$r(N \circ A) \leq r(N)r(A).$$

THEOREM 3. For  $A \in M_n(C)$ , we have

$$r(A \circ A^*) \leq r(H(A))^2 + r(S(A))^2.$$

*Proof.* Since  $f(A) = H(A)^{(2)} - S(A)^{(2)}$ , it follows that  $r(f(A)) = r(H(A)^{(2)} - S(A)^{(2)}) \leq r(H(A)^{(2)}) + r(-S(A)^{(2)}) \leq r(H(A))^2 + r(S(A))^2$ . The latter inequality is from the lemma and completes the proof.

COROLLARY 3. For  $A \in M_n(C)$ ,

$$\lambda_M(A \circ A^*) \leq \lambda_M(H(A)^2) - \lambda_m(S(A)^2).$$

*Proof.* Since  $\lambda_M(f(A)) \leq r(f(A))$ ,  $r(H(A))^2 = \lambda_M(H(A)^2)$ , and

$$r(S(A))^2 = -\lambda_m(S(A)^2),$$

this follows directly from Theorem 3.

EXAMPLE. The estimates of Theorem 3 and Corollary 3 are sharp. Equality may be attained even for nonHermitian matrices. Let  $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ ; then  $F(A)$  is the unit closed circular disk and thus  $r(A) = r(H(A)) = r(S(A)) = 1$ . Also  $f(A) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$  so that  $r(f(A)) = \lambda_M(f(A)) = 2 = r(H(A))^2 + r(S(A))^2 = \lambda_M(H(A)^2) - \lambda_m(S(A)^2)$ .

Although we will not do so here, estimates for  $\lambda_m(A \circ A^*)$  may straightforwardly be obtained from the results of the next section.

3. Conditions sufficient for  $A \circ A^* \in \Sigma_n$ . We next study three rather different sufficient conditions (Theorems 4, 5, and 6) for the Hermitian matrix  $f(A)$  to be positive definite.

NOTATION 3. If  $X \in M_n(C)$  denote the union of the Gersgorin circles [3] obtained from the rows of  $X$  by  $G_r(X)$  and the union of the Gersgorin circles obtained from the columns of  $X$  by  $G_c(X)$ .

Let  $G(X) = G_r(X) \cap G_c(X)$ . Then  $\sigma(X) \subseteq G(X)$ , [3], and  $0 \notin G_r(X)$  is the assumption of *row diagonal dominance* while  $0 \notin G_c(X)$  is *column diagonal dominance*. We shall call a matrix  $T = (t_{ij}) \in M_n(C)$  *combinatorially triangular* if for all pairs  $i \neq j$  either of  $t_j$  or  $t_{ji}$  is 0.

**THEOREM 4.** *If  $A \in M_n(C)$  and there is a diagonal matrix  $D \in M_n(C)$  such that  $F(DA) \subseteq Q_1$ , then  $f(A) \in \Sigma_n$ .*

*Proof.* If there is such a  $D$ , then it must be nonsingular and by Theorem 2 it suffices to prove the statement of this theorem for  $D = I$ . By letting  $B = A^*$ , the hypothesis of Corollary 2 is satisfied in our case and we may conclude  $f(A) = A \circ A^* \in \Pi_n$ . But since  $f(A)$  is Hermitian it is then in  $\Sigma_n$  which completes the proof.

**REMARK.** It is an easy observation that  $f(e^{i\theta}A) = f(A)$ . By Theorem 3 this means that if  $F_{\text{ang}}(A) \subseteq Q$ , where  $Q$  is any rotation of  $Q_1$ , then  $f(A) \in \Sigma_n$ .

**LEMMA 2.** *If  $0 \notin G_r(A) \cup G_c(A)$ , then  $0 \notin G(f(A))$ .*

*Proof.* Since  $f(A)$  is Hermitian,  $G(f(A)) = G_r(f(A)) = G_c(f(A))$ . Since  $0 \notin G_r(A) \cup G_c(A)$ ,  $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$  and  $|a_{ii}| > \sum_{j \neq i} |a_{ji}|$ , for all  $i = 1, \dots, n$ . Thus

$$a_{ii}\overline{a_{ii}} = |a_{ii}|^2 > \left( \sum_{j \neq i} |a_{ij}| \right) \left( \sum_{j \neq i} |a_{ji}| \right) \geq \sum_{j \neq i} |a_{ij}| |a_{ji}| = \sum_{j \neq i} |a_{ij}\overline{a_{ji}}|$$

which means that  $0 \notin G(f(A))$ .

**LEMMA 3.** *If  $0 \notin G_r(A)$ , there is a positive diagonal matrix  $D$  such that  $0 \notin G_r(DA) \cup G_c(DA)$ .*

*Proof.* Since  $D$  diagonal and invertible and  $0 \notin G_r(A)$  imply  $0 \notin G_r(DA)$ , it suffices to show that under the assumption a  $D$  may be found such that  $0 \notin G_c(DA)$ . This may be done by an  $M$ -matrix argument [2]. Without loss of generality we may assume  $A$  is real with positive diagonal entries and nonpositive off-diagonal entries. Our assumption,  $0 \notin G_r(A)$ , then implies that  $A$  and thus  $A^*$  are  $M$ -matrices. By [2, Theorem 4.3] this implies the existence of a positive diagonal  $D$  such that  $0 \notin G_r(A^*D) = G_c(DA)$ . For this  $D$ , then,  $0 \notin G_r(DA) \cup G_c(DA)$  as desired.

**THEOREM 5.** *If  $A \in M_n(C)$  and there is a diagonal matrix  $D \in M_n(C)$  such that  $0 \notin G(DA)$ , then  $f(A) \in \Sigma_n$ .*

*Proof.* Again by Theorem 2 it suffices to prove the weaker statement that  $0 \notin G(A)$  implies  $f(A) \in \Sigma_n$ , and since  $f(A) = f(A^*)$  we may assume without loss of generality that  $0 \notin G_r(A)$ . Then by Lemma 3, there is a positive diagonal matrix  $D$  such that  $0 \notin G_r(DA) \cup G_i(DA)$ . According to Lemma 2 this implies  $0 \notin G(f(DA))$ . Since  $f(DA)$  is Hermitian with nonnegative diagonal entries,  $0 \notin G(f(DA))$  implies  $G(f(DA)) \subseteq \text{interior}(\bar{Q}_1 \cup \bar{Q}_4)$  and that all eigenvalues of  $f(DA)$  are positive. This means that  $f(DA) \in \Sigma_n$  and by Theorem 2 that  $f(A) \in \Sigma_n$  which completes the proof.

**THEOREM 6.** *If  $A = (a_{ij}) \in M_n(C)$  is combinatorially triangular and  $a_{ii} \neq 0$ ,  $i = 1, \dots, n$ , then  $f(A) \in \Sigma_n$ .*

*Proof.* Under the hypothesis  $a_{ij}\bar{a}_{ji}$  is 0 if  $i \neq j$  and positive if  $i = j$ . This means  $f(A)$  is a positive diagonal matrix and, therefore, a member of  $\Sigma_n$ .

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