SUBALGEBRAS OF FINITE CODIMENSION IN THE ALGEBRA OF ANALYTIC FUNCTIONS ON A RIEMANN SURFACE

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Let \( R \) be a finite open Riemann surface with boundary \( \Gamma \). We set \( \overline{R} = R \cup \Gamma \) and let \( A(\overline{R}) \) denote the algebra of functions which are continuous on \( \overline{R} \) and analytic on \( R \). Suppose \( A \) is a uniform algebra contained in \( A(\overline{R}) \). The main result of this paper shows that if \( A \) contains a function \( F \) which is analytic in a neighborhood of \( \overline{R} \) and which maps \( \overline{R} \) in a \( n \)-to-one manner (counting multiplicity) onto \( \{ z : |z| \leq 1 \} \), then \( A \) has finite codimension in \( A(\overline{R}) \).

We say that \( A \) is a uniform algebra on \( \overline{R} \) if \( A \) is a uniformly closed subalgebra of the complex-valued continuous functions on \( \overline{R} \) which separates points of \( \overline{R} \) and contains the constant functions. If \( A \) is contained in \( A(\overline{R}) \), then we say \( A \) has finite codimension in \( A(\overline{R}) \) if \( A(\overline{R})/A \) is a finite dimensional vector space over \( \mathbb{C} \). A reference for uniform algebras is Gamelin [2].

Let \( U \) be the open unit disk in \( \mathbb{C} \). We call \( F \) an unimodular function if \( F \) is analytic in a neighborhood of \( \overline{R} \) and maps \( \overline{R} \) onto \( U \) so that \( F \) is \( \pi \)-to-one if we count the multiplicity of \( F \) where \( dF \) vanishes. If \( T \) is the unit circle, then \( F \) maps \( \Gamma \) onto \( T \). The existence of such a function was first proved by Ahlfors [1]. Later, Royden [4] gave another proof of this result.

1. Main results. Let \( A \) be a uniform algebra on \( \overline{R} \) which is contained in \( A(\overline{R}) \). If \( J = \{ f \in A(\overline{R}) : fA(\overline{R}) \subseteq A \} \), then \( J \) is a closed ideal in \( A(\overline{R}) \) and \( J \) is contained in \( A \).

**Lemma.** Let \( F \in A \) be an unimodular function of order \( n \). If \( \zeta_i \in \overline{R} \) is such that \( F^{-1}(F(\zeta_i)) \) consists of \( n \) distinct points, then there is \( G \in J \) such that \( G(\zeta_i) \neq 0 \).

**Proof.** Since \( A \) separates points on \( \overline{R} \), there is \( g \in A \) such that \( g \) separates \( F^{-1}(F(\zeta_i)) \). If \( z_i \in \overline{R} \), let \( F^{-1}(F(z_i)) = \{ z_i, z_2, \ldots, z_n \} \) (perhaps with repetitions) and let \( f \in A(\overline{R}) \).

Define \( Q(u) = f(z_i)[u - g(z_i)](u - g(z_2)) \cdots [u - g(z_n)] + f(z_2)[u - g(z_i)](u - g(z_2)) \cdots [u - g(z_n)] + \cdots + f(z_n)[u - g(z_2)] \cdots [u - g(z_n)] \) (cf. [5], p. 290). Then \( Q(u) \) is a polynomial in \( u \) of the form \( Q(u) = \alpha_{n-1}(z_i, \ldots, z_n)u^{n-1} + \alpha_{n-2}(z_i, \ldots, z_n)u^{n-2} + \cdots + \alpha_{0}(z_i, \ldots, z_n) \). The coefficients \( \alpha_j \) are symmetric functions in \( z_i, \ldots, z_n \). Hence, if
$w = F(z)$, then $a_j(w) = \alpha_j(z_1, \cdots, z_n)$ for $j = 0, \cdots, n - 1$ is well-defined on $\bar{U}$. Using Riemann’s removable singularity theorem, it follows that $a_j(w) \in A(U)$ for $j = 0, \cdots, n - 1$.

Since $a_j(w) \in A(U)$ for each $j$, there are polynomials $\{p_i(w)\}_{i=1}^\infty$ such that the $p_i$’s converge uniformly to $a_j$ on $\bar{U}$. Then $p_j(F(z)) \in A$ for each $k$, and we conclude that $a_j(F(z)) \in A$. Letting $z = z$, and setting $u = g(z)$, we obtain $Q(g(z)) = a_{n-1}(F(z))g(z)^{n-1} + a_{n-2}(F(z))g(z)^{n-2} + \cdots + a_0(F(z)) = f(z) \prod_{i=2}^n [g(z) - g(z_i)] \in A$. Let $G(z) = \prod_{i=2}^n [g(z) - g(z_i)]$. Then $G(z_i) \neq 0$ and we have shown that $fG \in A$ for any $f \in A(R)$. Therefore, $G \in J$.

**Theorem.** Let $A$ be a uniform algebra on $\bar{R}$ which is contained in $A(R)$. If $A$ contains an unimodular function, then $A$ has finite codimension in $A(R)$.

**Proof.** Suppose $F \in A$ is an unimodular function of order $n$. Let hull $J = \{z \in \bar{R} : f(z) = 0 \text{ for all } f \in J\}$. If $\zeta \in J$, then $dF(\zeta) \neq 0$ ([7], p. 367) and consequently $F^{-1}(F(\zeta))$ consists of $n$ distinct points. By the lemma, hull $J \subset R$. It follows that hull $J$ is a finite set. By applying [6], Theorem 1 and [3], Lemma 2.5, we conclude that $A(R)/J$ is finite dimensional. Hence, $A$ has finite codimension in $A(R)$.

Let $R = \{z \in C : 1 < |z| < 2\}$. Again let $J = \{f \in A(R) : fA(R) \subset A\}$ where $A$ is a uniform algebra on $\bar{R}$. Using the same technique we prove the proposition below.

**Proposition.** Let $A$ be a uniform algebra on $\bar{R}$ which is contained in $A(R)$. If $A$ contains $z^n$ and $z^{-n}$ for some positive integers $n$ and $m$, then $A = A(R)$.

**Proof.** Let $N$ be the least common multiple of $n$ and $m$. Then $z^n$ and $z^{-n} \in A$. Also, $z^n$ is an $N$-to-one map of $\bar{R}$ onto $\bar{R}$ without branch points. For any $\zeta_i \in \bar{R}$ there are $N$ distinct points $\{\zeta_{i1}, \zeta_{i2}, \cdots, \zeta_{iN}\}$ which satisfy $\zeta_i^n = \zeta_{iN}$. Fix $\zeta_i \in \bar{R}$ and let $g \in A$ separate $\{\zeta_i, \zeta_{i2}, \cdots, \zeta_{iN}\}$. Let $f \in A(R)$.

Letting $z^n$ take the role of $F$ and using $g$ and $f$, we form $Q(u)$ just as in the proof of the lemma. The coefficients $a_j(w)$ of $Q(u)$ belong to $A(R)$. Hence there are polynomials in $w$ and $w^{-1}$ which converge uniformly to $a_j(w)$ on $\bar{R}$. Since $z^n$ and $z^{-n}$ belong to $A$, it follows that $a_j(z^n)$ is in $A$.

Consequently, $Q(g(z)) = f(z) \prod_{i=2}^n [g(z) - g(z_i)] \in A$ for all $f \in A(R)$. Let $G(z) = \prod_{i=2}^n [g(z) - g(z_i)]$. Then $G \in J$ and $G(\zeta_i) \neq 0$. Therefore, hull $J = \phi$. This implies $A = A(R)$.

2. Question. The theorem of this paper gives an affirmative
answer to a special case of the following question. Suppose \( A \) is a uniform algebra on \( \tilde{R} \) and \( A \) is contained in \( A(R) \). If \( A \) contains a nonconstant function which is analytic in a neighborhood of \( \tilde{R} \), does it follow that \( A \) has finite codimension in \( A(R) \)?

**References**


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