

# Pacific Journal of Mathematics

**SUBALGEBRAS OF FINITE CODIMENSION IN THE ALGEBRA  
OF ANALYTIC FUNCTIONS ON A RIEMANN SURFACE**

BRUCE STEPHEN LUND

## SUBALGEBRAS OF FINITE CODIMENSION IN THE ALGEBRA OF ANALYTIC FUNCTIONS ON A RIEMANN SURFACE

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Let  $R$  be a finite open Riemann surface with boundary  $\Gamma$ . We set  $\bar{R} = R \cup \Gamma$  and let  $A(R)$  denote the algebra of functions which are continuous on  $\bar{R}$  and analytic on  $R$ . Suppose  $A$  is a uniform algebra contained in  $A(R)$ . The main result of this paper shows that if  $A$  contains a function  $F$  which is analytic in a neighborhood of  $\bar{R}$  and which maps  $\bar{R}$  in a  $n$ -to-one manner (counting multiplicity) onto  $\{z: |z| \leq 1\}$ , then  $A$  has finite codimension in  $A(R)$ .

We say that  $A$  is a uniform algebra on  $\bar{R}$  if  $A$  is a uniformly closed subalgebra of the complex-valued continuous functions on  $\bar{R}$  which separates points of  $\bar{R}$  and contains the constant functions. If  $A$  is contained in  $A(R)$ , then we say  $A$  has finite codimension in  $A(R)$  if  $A(R)/A$  is a finite dimensional vector space over  $C$ . A reference for uniform algebras is Gamelin [2].

Let  $U$  be the open unit disk in  $C$ . We call  $F$  an unimodular function if  $F$  is analytic in a neighborhood of  $\bar{R}$  and maps  $\bar{R}$  onto  $\bar{U}$  so that  $F$  is  $n$ -to-one if we count the multiplicity of  $F$  where  $dF$  vanishes. If  $T$  is the unit circle, then  $F$  maps  $\Gamma$  onto  $T$ . The existence of such a function was first proved by Ahlfors [1]. Later, Royden [4] gave another proof of this result.

1. Main results. Let  $A$  be a uniform algebra on  $\bar{R}$  which is contained in  $A(R)$ . If  $J = \{f \in A(R): fA(R) \subset A\}$ , then  $J$  is a closed ideal in  $A(R)$  and  $J$  is contained in  $A$ .

LEMMA. Let  $F \in A$  be an unimodular function of order  $n$ . If  $\zeta_1 \in \bar{R}$  is such that  $F^{-1}(F(\zeta_1))$  consists of  $n$  distinct points, then there is  $G \in J$  such that  $G(\zeta_1) \neq 0$ .

*Proof.* Since  $A$  separates points on  $\bar{R}$ , there is  $g \in A$  such that  $g$  separates  $F^{-1}(F(\zeta_1))$ . If  $z_1 \in \bar{R}$ , let  $F^{-1}(F(z_1)) = \{z_1, z_2, \dots, z_n\}$  (perhaps with repetitions) and let  $f \in A(R)$ .

Define  $Q(u) = f(z_1)\{u - g(z_2)\}\{u - g(z_3)\} \cdots \{u - g(z_n)\} + f(z_2)\{u - g(z_1)\}\{u - g(z_3)\} \cdots \{u - g(z_n)\} + \cdots + f(z_n)\{u - g(z_1)\}\{u - g(z_2)\} \cdots \{u - g(z_{n-1})\}$  (cf. [5], p. 290). Then  $Q(u)$  is a polynomial in  $u$  of the form  $Q(u) = \alpha_{n-1}(z_1, \dots, z_n)u^{n-1} + \alpha_{n-2}(z_1, \dots, z_n)u^{n-2} + \cdots + \alpha_0(z_1, \dots, z_n)$ . The coefficients  $\alpha_j$  are symmetric functions in  $z_1, \dots, z_n$ . Hence, if

$w = F(z_1)$ , then  $a_j(w) = \alpha_j(z_1, \dots, z_n)$  for  $j = 0, \dots, n-1$  is well-defined on  $\bar{U}$ . Using Riemann's removable singularity theorem, it follows that  $a_j(w) \in A(U)$  for  $j = 0, \dots, n-1$ .

Since  $a_j(w) \in A(U)$  for each  $j$ , there are polynomials  $\{p_k^j(w)\}_{k=1}^\infty$  such that the  $p_k^j$ 's converge uniformly to  $a_j$  on  $\bar{U}$ . Then  $p_k^j(F(z)) \in A$  for each  $k$ , and we conclude that  $a_j(F(z)) \in A$ . Letting  $z = z_1$  and setting  $u = g(z)$ , we obtain  $Q(g(z)) = a_{n-1}(F(z))g(z)^{n-1} + a_{n-2}(F(z))g(z)^{n-2} + \dots + a_0(F(z)) = f(z) \prod_{i=2}^n \{g(z) - g(z_i)\} \in A$ . Let  $G(z) = \prod_{i=2}^n \{g(z) - g(z_i)\}$ . Then  $G(\zeta_i) \neq 0$  and we have shown that  $fG \in A$  for any  $f \in A(R)$ . Therefore,  $G \in J$ .

**THEOREM.** *Let  $A$  be a uniform algebra on  $\bar{R}$  which is contained in  $A(R)$ . If  $A$  contains an unimodular function, then  $A$  has finite codimension in  $A(R)$ .*

*Proof.* Suppose  $F \in A$  is an unimodular function of order  $n$ . Let  $\text{hull } J = \{z \in \bar{R} : f(z) = 0 \text{ for all } f \in J\}$ . If  $\zeta \in \Gamma$ , then  $dF(\zeta) \neq 0$  ([7], p. 367) and consequently  $F^{-1}(F(\zeta))$  consists of  $n$  distinct points. By the lemma,  $\text{hull } J \subset R$ . It follows that  $\text{hull } J$  is a finite set. By applying [6], Theorem 1 and [3], Lemma 2.5, we conclude that  $A(R)/J$  is finite dimensional. Hence,  $A$  has finite codimension in  $A(R)$ .

Let  $R = \{z \in \mathbb{C} : 1 < |z| < 2\}$ . Again let  $J = \{f \in A(R) : fA(R) \subset A\}$  where  $A$  is a uniform algebra on  $\bar{R}$ . Using the same technique we prove the proposition below.

**PROPOSITION.** *Let  $A$  be a uniform algebra on  $\bar{R}$  which is contained in  $A(R)$ . If  $A$  contains  $z^n$  and  $z^{-m}$  for some positive integers  $n$  and  $m$ , then  $A = A(R)$ .*

*Proof.* Let  $N$  be the least common multiple of  $n$  and  $m$ . Then  $z^N$  and  $z^{-N} \in A$ . Also,  $z^N$  is an  $N$ -to-one map of  $\bar{R}$  onto  $\bar{R}$  without branch points. For any  $\zeta_1 \in \bar{R}$  there are  $N$  distinct points  $\{\zeta_1, \zeta_2, \dots, \zeta_N\}$  which satisfy  $\zeta_i^N = \zeta_1^N$ . Fix  $\zeta_1 \in \bar{R}$  and let  $g \in A$  separate  $\{\zeta_1, \zeta_2, \dots, \zeta_N\}$ . Let  $f \in A(R)$ .

Letting  $z^N$  take the role of  $F$  and using  $g$  and  $f$ , we form  $Q(u)$  just as in the proof of the lemma. The coefficients  $a_j(w)$  of  $Q(u)$  belong to  $A(R)$ . Hence there are polynomials in  $w$  and  $w^{-1}$  which converge uniformly to  $a_j(w)$  on  $\bar{R}$ . Since  $z^N$  and  $z^{-N}$  belong to  $A$ , it follows that  $a_j(z^N)$  is in  $A$ .

Consequently,  $Q(g(z)) = f(z) \prod_{i=2}^N \{g(z) - g(z_i)\} \in A$  for all  $f \in A(R)$ . Let  $G(z) = \prod_{i=2}^N \{g(z) - g(z_i)\}$ . Then  $G \in J$  and  $G(\zeta_i) \neq 0$ . Therefore,  $\text{hull } J = \phi$ . This implies  $A = A(R)$ .

2. Question. The theorem of this paper gives an affirmative

answer to a special case of the following question. Suppose  $A$  is a uniform algebra on  $\bar{R}$  and  $A$  is contained in  $A(R)$ . If  $A$  contains a nonconstant function which is analytic in a neighborhood of  $\bar{R}$ , does it follow that  $A$  has finite codimension in  $A(R)$ ?

## REFERENCES

1. L. V. Ahlfors, *Open Riemann surfaces and extremal problems on compact subregions*, Comment. Math. Helv., **24** (1950), 100-134.
2. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.
3. A. Read, *A converse of Cauchy's theorem and application to extremal problems*, Acta Math., **100** (1958), 1-22.
4. H. L. Royden, *The boundary values of analytic and harmonic functions*, Math. Z., **78** (1962), 1-24.
5. G. Springer, *Introduction to Riemann Surfaces*, Addison-Wesley, Reading, Mass., 1957.
6. C. M. Stanton, *The closed ideals of a function algebra*, Trans. Amer. Math. Soc., **154** (1971), 289-300.
7. E. L. Stout, *On some algebras of analytic functions on finite open Riemann surfaces*, Math. Z., **92** (1966), 366-379.

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