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TTF CLASSES AND QUASI-GENERATORS

ROBERT WILMER MILLER

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Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for $_{\mathcal{A}}\mathcal{M}$, the category of left A-modules. In this paper the property that the torsionfree class \mathcal{F} be closed under homomorphic images is investigated. Particular attention is given to the case where the torsion class \mathcal{F} is torsion-torsionfree (TTF). Applications to projective quasi-generators are given.

When \mathscr{T} is a TTF class the question naturally arises as to when A_t , the \mathscr{T} -torsion submodule of A, is contained in a certain idempotent topologizing filter of right ideals of A. This condition is shown to be equivalent to the property that the torsionfree class \mathscr{F} be closed under homomorphic images. Our results generalize results of Jans [6] and Bernhardt [2] characterizing the property that the torsion theory $(\mathscr{T}, \mathscr{F})$ is centrally splitting. Dropping the assumption that \mathscr{T} is TTF, further investigation of the property that \mathscr{F} is closed under homomorphic images yields information as to when \mathscr{T} is TTF, generalizing a result due to Rutter [10]. Finally, our methods are applied to the TTF class $\mathscr{T} = \{_A X | P \otimes_A X = 0\}$ where P_A is a projective right A-module. The definition of P_A being a quasi-generator is given and characterizations are obtained.

Section 2 of this paper was taken from the author's doctoral dissertation, under the direction of Professor F. L. Sandomierski, at the University of Wisconsin. Section 1 provides a generalization of the material in §2 to arbitrary TTF classes. The author expresses his gratitude to Professor Sandomierski for his guidance and encouragement.

In this paper A will be an associative ring with unit and all modules will be unitary. The left (right) annihilator of I in X will be denoted by $l_X(I)$ $(r_X(I))$. It is easy to see that for a left A-module X and a two-sided ideal I of A, $r_X(I) \cong \text{Hom}_A(A/I, X)$.

Dickson [4] has defined a torsion theory for $_{\mathcal{A}}\mathcal{M}$ to be a pair $(\mathcal{T}, \mathcal{F})$ of classes of left A-modules satisfying

(1) $\mathcal{T} \cap \mathcal{F} = \{0\}.$

(2) \mathcal{T} is closed under homomorphic images.

(3) \mathcal{F} is closed under submodules.

(4) For each $X \in \mathcal{M}$ there exists a (unique) submodule $X_t \in \mathcal{T}$ such that $X/X_t \in \mathcal{F}$.

A class $\mathcal{T}(\mathcal{F})$ of left modules is called a torsion (torsionfree) class provided there is a (unique) class $\mathcal{F}(\mathcal{T})$ such that $(\mathcal{T}, \mathcal{F})$ is a torsion theory. A torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be hereditary if the torsion class \mathscr{T} is closed under submodules. For further information on torsion and torsionfree classes the reader is referred to [4].

Gabriel [5] has shown that for a ring A there is a one-to-one correspondence between the hereditary torsion classes of $_{\mathcal{A}}\mathcal{M}$ and the idempotent topologizing filters F of left ideals of A given by

$$\mathscr{T} \longrightarrow F(\mathscr{T}) = \{{}_{A}I \subseteq {}_{A}A \mid A/I \in \mathscr{T}\}.$$

The inverse correspondence is given by

$$F \longrightarrow \mathscr{T}(F) = \{X \in \mathscr{M} \mid l_A(x) \in F \text{ for all } x \in X\}$$
.

Jans [6] has called a torsion class \mathscr{T} which is also a torsionfree class for some torsion class \mathscr{C} , a torsion-torsionfree (TTF) class. In this case $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$ are called the torsion theories associated with \mathscr{T} . In [6] it is shown that \mathscr{T} is a TTF class if and only if $F(\mathscr{T})$ contains a unique minimal left ideal T. Furthermore, T is an idempotent two-sided ideal, $T = A_c$ (the \mathscr{C} -torsion submodule of A), and there is a one-to-one correspondence between the TTF classes \mathscr{T} of ${}_{\mathscr{A}}\mathscr{M}$ and the idempotent two-sided ideals of A given by

 $\mathscr{T} \longrightarrow T$.

The inverse correspondence is given by

$$T \longrightarrow \{X \in \mathcal{M} \mid TX = 0\}$$

1. TTF classes. Let $\mathscr{T} \subseteq \mathscr{M}$ be a TTF class with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$. Let T be the minimal, idempotent, two-sided ideal in $F(\mathscr{T})$. One easily checks that

$$\mathscr{T} = \{_{\scriptscriptstyle A} X \, | \, TX = 0 \}$$
 , $\mathscr{F} = \{_{\scriptscriptstyle A} X \, | \, \mathrm{Hom} \, (A/T, \, X) = 0 \}$,

and

$$\mathscr{C} = \{_{\scriptscriptstyle A} X \, | \, A/T \otimes _{\scriptscriptstyle A} X = 0 \}$$
 .

Note that the \mathcal{T} -torsion submodule of $_{\mathcal{A}}X$ is $X_t = r_x(T)$ while the \mathscr{C} -torsion submodule is $X_c = TX$. Furthermore, let

$$\mathscr{H} = \{_{_{A}}X \,|\, A/T \bigotimes _{_{A}}M = 0 \,\, ext{for every submodule } M \,\, ext{of} \,\, X\}$$
 .

LEMMA 1.1. \mathscr{H} is a hereditary torsion class and $F(\mathscr{H}) = \{{}_{A}I \subseteq {}_{A}A \mid (I:a) + T = A \text{ for all } a \in A\}$ where $(I:a) = \{x \in A \mid x a \in I\}$.

Proof. It is left to the reader to check that ${}_{A}X \in \mathscr{H}$ if and only

if $x \in Tx$ for all $x \in X$. If ${}_{A}I \in F(\mathcal{H})$, then $A/I \in \mathcal{H}$. Hence $a + I \in T(a + I)$ for all $a \in A$. Thus given $a \in A$ there exists $t \in T$ such that $ta - a = (t - 1)a \in I$. Therefore, $t - 1 \in (I:a)$, which implies that $1 \in (I:a) + T$.

Conversely, for a left ideal I suppose that (I:a) + T = A for all $a \in A$. Hence given $a \in A$, we have that 1 = x + t where $x \in (I:a)$ and $t \in T$. Thus a + I = x(a + I) + t(a + I), and so $a + I \in T(a + I)$. Lemma 1.1 can also be deduced from [9, Lemma 1.1].

LEMMA 1.2. For the torsion class \mathcal{C} the following statements are equivalent.

(1) $(A/T)_A$ is flat.

(2) C is hereditary (i.e., $C = \mathcal{H}$).

(3) $\bigcap_{i=1}^{n} l_A(t_i) + T = A \text{ for all } t_i \in T.$

(4) $l_A(t) + T = A$ for all $t \in T$.

Furthermore, if \mathscr{C} is hereditary, then $F(\mathscr{C}) = \{{}_{A}I \subseteq {}_{A}A \mid I + T = A\}$.

Proof. (1) \rightarrow (2) Take $_{A}X \in \mathscr{C}$. Since $(A/T)_{A}$ is flat M/TM may be viewed as a submodule of X/TX. Thus M/TM = 0 as X/TX = 0.

(2) \rightarrow (3) Since $T \in \mathscr{C}$ we have that $l_A(t) \in F(\mathscr{C})$ for all $t \in T$. Thus $\bigcap_{i=1}^{n} l_A(t_i) \in F(\mathscr{C})$ for all $t_i \in T$ as $F(\mathscr{C})$ is closed under finite intersections. Hence $\bigcap_{i=1}^{n} l_A(t_i) + T = A$ for all $t_i \in T$ by Lemma 1.1.

 $(3) \rightarrow (4) \quad \text{Trivial.}$

 $(4) \rightarrow (1)$ For $t \in T$, by assumption we have that 1 = a + t' where $a \in l_A(t)$ and $t' \in T$. So $t = at + t't = t't \in Tt$. That $(A/T)_A$ is flat now follows by [8, Lemma 4.1].

If \mathscr{C} is hereditary then ${}_{A}I \in F(\mathscr{C})$ if and only if T(A/I) = A/I; i.e., if and only if I + T = A.

Since the minimal idempotent ideal T in $F(\mathcal{T})$ is two-sided let

 $\mathcal{H}' = \{X_A \mid M \otimes_A A/T = 0 \text{ for every submodule } M \text{ of } X\},\$

a hereditary torsion class in \mathcal{M}_A . Let *I* be a right ideal of *A* contained in $F(\mathcal{H}')$. Then I + T = A. If $x \in A_t = r_A(T)$, then $Ax = Ix + Tx = Ix \subseteq I$. That is, A_t is contained in every right ideal in the filter $F(\mathcal{H}')$. Hence the question arises as to when A_t is itself in $F(\mathcal{H}')$?

THEOREM 1.3. Let $\mathscr{T} \subseteq \mathscr{M}$ be a TTF class with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$. Using the above notation the following statements are equivalent.

- (1) $A_t \in F(\mathscr{H}')$.
- $(2) \quad A = A_t + A_c.$
- (3) $X = X_t + X_c$ for all $X \in {}_{A}\mathcal{M}$.
- (4) $_{A}(A/A_{c})$ is projective.

(5) \mathscr{F} is closed under homomorphic images. (6) $\mathscr{F} = \mathscr{H}$. (7) $\mathscr{F} \subseteq \mathscr{C}$.

Proof. $(1) \rightarrow (2)$ This is immediate from Lemma 1.1.

(2) \leftrightarrow (3) If $A = A_t + A_c$, then for every $_AX$ we have $_AX = A_tX + A_cX \subseteq X_t + X_c$. The converse is trivial.

(2) \rightarrow (4) This is Lemma 4.10 of [8].

(4) \rightarrow (5) Let $_{A}X \in \mathscr{F}$ and let $\alpha: _{A}X \rightarrow _{A}Y$ be an epimorphism. Since $_{A}(A/T)$ is projective we have an epimorphism $\alpha^{*}: \operatorname{Hom}_{A}(A/T, X) \rightarrow \operatorname{Hom}_{A}(A/T, Y)$. Thus $\operatorname{Hom}_{A}(A/T, Y) = 0$; hence $Y \in \mathscr{F}$.

(5) \rightarrow (6) For $_{A}X \in \mathscr{H}$ consider $X_{t} \subseteq X$. Now $TX_{t} = 0$; but also $TX_{t} = X_{t}$ since $X \in \mathscr{H}$. Thus $X \in \mathscr{F}$ as $X_{t} = 0$.

Conversely, let $_{A}X \in \mathscr{F}$. For $x \in X$, $Ax \in \mathscr{F}$ since \mathscr{F} is closed under submodules. Thus $Ax/Tx \in \mathscr{F}$ by assumption. But $Ax/Tx \in \mathscr{F}$ since T(Ax/Tx) = 0. Hence Tx = Ax for all $x \in X$, which implies that $X \in \mathscr{H}$.

(6) \rightarrow (7) Trivial.

(7) \rightarrow (2) By assumption $A/A_t \in \mathscr{C}$. Thus $A/A_t = T(A/A_t) = T + A_t/A_t$; i.e., $A_t + A_c = A$.

(2) \rightarrow (1) For all $t \in T$, $r_A(t) + T = A$ since $A_t \subseteq r_A(t)$. Thus by Lemma 1.2 $F(\mathscr{H}') = \{I_A \subseteq A_A \mid I + T = A\}$. Hence $A_t \in F(\mathscr{H}')$.

REMARK. The equivalence of (2), (3), and (7) was recently shown by Kurata in [7]. He also shows that $(5) \rightarrow (7)$.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is centrally splitting if \mathcal{T} is a TTF class (with associated torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{T})$) and $X = X_t \bigoplus X_c$ for all $X \in {}_{\mathcal{A}}\mathcal{M}$. The following corollary characterizing centrally splitting torsion theories is due to Jans [6] and Bernhardt [2].

COROLLARY 1.4. Let $\mathscr{T} \subseteq {}_{\mathbb{A}}\mathscr{M}$ be a TTF class with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$. The following statements are equivalent.

(1) $A = A_t \bigoplus A_c$ (ring direct sum).

(2) $X = X_t \bigoplus X_c$ for all $X \in A_{\mathcal{M}}$.

 $(3) \quad \mathcal{F} = \mathcal{C}.$

(4) \mathcal{T} is stable and A_c is a direct summand of A.

(5) \mathscr{F} is closed under homomorphic images and A_t is a direct summand of A.

(6) A_c is a ring direct summand of A.

Proof. \mathcal{T} being stable (closed under injective hulls) is equivalent to \mathcal{C} being hereditary (see e.g. [2]). Using Lemma 1.2 and Theorem 1.3 conditions (1), (2), (3), (4), and (6) are easily seen to be equivalent

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to the condition that $_{A}(A/A_{c})$ is projective and $(A/A_{c})_{A}$ is flat. That (1) is equivalent to (5) is easy using Theorem 1.3.

An example of a TTF class \mathcal{T} for which \mathcal{F} is closed under homomorphic images, but $(\mathcal{T}, \mathcal{F})$ is not centrally splitting is given in §2. See Example 2.5 (i).

In the remainder of this section we investigate the condition that the torsionfree class of a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ be closed under homomorphic images, dropping the condition that \mathcal{T} be a TTF class. Assuming \mathcal{F} is closed under homomorphic images, \mathcal{F} is now a torsion class; i.e., a TTF class. $(\mathcal{T}, \mathcal{F})$ is one torsion theory associated with \mathcal{F} . Let $(\mathcal{F}, \mathcal{L})$ be the other. The \mathcal{T} -torsion submodule of A, A_t , is in $F(\mathcal{F})$ since $A/A_t \in \mathcal{F}$. Furthermore, if ${}_{\mathcal{A}}I \in$ $F(\mathcal{F})$, then $A_t \cap I \in F(\mathcal{F})$, which means that $A/A_t \cap I \in \mathcal{F}$. Thus $A_t/A_t \cap I \in \mathcal{F}$. Since $A_t/A_t \cap I$ is also in \mathcal{T} we have that $A_t/A_t \cap$ I = 0; i.e., $A_t \subseteq I$. Hence A_t is the minimal element of the filter $F(\mathcal{F})$. As before we have that

$$\mathscr{F} = \{_{\scriptscriptstyle A} X \, | \, A_t X = 0 \} \; ,$$
 $\mathscr{L} = \{_{\scriptscriptstyle A} X \, | \operatorname{Hom}_{\scriptscriptstyle A} \left(A / A_t, \; X
ight) = 0 \} \; ,$

and

$$\mathscr{T} = \{{}_{\scriptscriptstyle A} X \mid A / A_t \otimes {}_{\scriptscriptstyle A} X = 0\}$$
 .

(By Theorem 1.3 if \mathscr{T} is a TTF class then \mathscr{L} is the torsionfree class associated with \mathscr{H} .) Since \mathscr{T} is hereditary, $(A/A_t)_A$ is flat by Lemma 1.2. Furthermore, $F(\mathscr{T}) = \{{}_A I \subseteq {}_A A \mid I + A_t = A\}$.

If in fact $(A/A_t)_A$ is projective, then $A = l_A(A_t) + A_t$ by [8, Lemma 4.10]. Thus $l_A(A_t) \in F(\mathcal{T})$. Furthermore, $l_A(A_t)$ is the minimal element of $F(\mathcal{T})$ by an argument similar to the one used prior to Theorem 1.3. Hence \mathcal{T} is a TTF class.

THEOREM 1.5. For a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ the following statements are equivalent.

(1) \mathscr{F} is closed under homomorphic images and $(A/A_t)_A$ has a projective cover.

(2) \mathscr{F} is closed under homomorphic images and A_t is finitely generated as a right A-module.

(3) \mathscr{T} is a TTF class where $A_c = l_A(A_t)$ and $(A/A_t)_A$ is projective.

Proof. By our above argument (3) will follow from (1) or (2) if we can show that the flat module $(A/A_t)_A$ is projective. In (1) this follows since $(A/A_t)_A$ has a projective cover [8, Lemma 1.2]. In (2) this follows by [3, corollary to Proposition 2.2].

By Theorem 1.3 (1) and (2) will follow from (3) if we show that $A = A_t + A_c$. Since $(A/A_t)_A$ is projective, $A = A_t + l_A(A_t)$ by [8, Lemma 4.10]. Thus $A = A_t + A_c$ by assumption.

COROLLARY 1.6. (Rutter [10, Proposition 2].) Let A be a semiperfect ring and $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in $_{\mathcal{A}}\mathcal{M}$. If \mathcal{F} is closed under homomorphic images, then \mathcal{T} is a TTF class where $A_c = l_A(A_t)$.

COROLLARY 1.7. Let A be a right Noetherian ring and $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in $_{\mathcal{A}}\mathcal{M}$. If \mathcal{F} is closed under homomorphic images, then \mathcal{T} is a TTF class where $A_c = l_{\mathcal{A}}(A_t)$.

2. Projective quasi-generators. In this section P_A will be a projective right A-module with trace ideal $T = \sum_{f \in \operatorname{Hom}(P,A)} \operatorname{im} f$ and A-endomorphism ring B. For further information on projective modules and the idempotent trace ideal the reader is referred to [13]. Let $\mathscr{T} = \{{}_{A}X | P \otimes_{A}X = 0\}$, a hereditary torsion class in ${}_{A}\mathscr{M}$. It is easily seen that $\mathscr{T} = \{{}_{A}X | TX = 0\}$ and $F(\mathscr{T}) = \{{}_{A}I \subseteq {}_{A}A | T \subseteq I\}$. Hence \mathscr{T} is a TTF class, and the notation of §1 applies.

Sandomierski [12] has defined an A-module $_{A}X(X_{A})$ to be T-accessible if TX = X(XT = X). Note that X_{A} is T-accessible if and only if X_{A} is a homomorphic image of a direct sum of copies of P_{A} [12]. Define an A-module X to be strongly T-accessible if every submodule of X is T-accessible. The class of T-accessible (strongly T-accessible) left A-modules is our class $\mathscr{C}(\mathscr{H})$. From Lemma 1.1 we see that $_{A}X$ is strongly T-accessible if and only if $x \in Tx$ for all $x \in X$.

We shall call P_A a quasi-generator if every *T*-accessible right *A*module is strongly *T*-accessible. That is, if every submodule of a homomorphic image of a direct sum of copies of P_A is itself a homomorphic image of a direct sum of copies of P_A . This definition is dual to the definition of self-cogenerator given in [11].

THEOREM 2.1. For P_A projective with trace ideal T the following statements are equivalent.

- (1) P_A is a quasi-generator.
- (2) P_A is strongly T-accessible.
- (3) T_A is strongly T-accessible.
- $(4) _{A}(A/T)$ is flat.
- (5) $r_A(p) + T = A$ for all $p \in P$.
- (6) $r_A(t) + T = A$ for all $t \in T$.

Proof. By definition P_A is a quasi-generator if and only if the torsion class $\mathscr{C}' = \{X_A \mid X \bigotimes_A A/T = 0\}$ is hereditary. Thus the equiv-

alence of (1), (4), and (6) follows by Lemma 1.2. As $\mathscr{H}' = \{X_A \mid M \otimes_A A/T = 0 \text{ for every submodule } M \text{ of } X\}$ is the class of strongly *T*-accessible right *A*-modules, the equivalence of (2) and (5), and the equivalence of (3) and (6) follows from Lemma 1.1. That (1) implies (2) is by definition.

(2) \rightarrow (1) Let X_A be *T*-accessible. Then X_A is a homomorphic image of P_A^I , a direct sum of copies of P_A . Now $P_A^I \in \mathscr{H}'$ as $P_A \in \mathscr{H}'$ by assumption and \mathscr{H}' is closed under direct sums. Thus $X \in \mathscr{H}'$ as \mathscr{H}' is closed under homomorphic images.

Ware [13] has called a projective module regular if every homomorphic image is flat.

COROLLARY 2.2. Let P_A be a regular module. Then P_A is a quasi-generator. Hence over a regular ring, every projective is a quasi-generator.

Proof. Let $x \in P_A$ and consider the exact sequence

$$0 \longrightarrow xA \longrightarrow P \longrightarrow P/xA \longrightarrow 0 .$$

Since P/xA is flat there exists a map $\theta: P_A \to xA$ such that $\theta(x) = x$ [13, Lemma 2.2]. Letting T be the trace ideal of P_A , we have that $x = \sum_{i=1}^{n} p_i t_i$ where $p_i \in P$, $t_i \in T$, $i = 1, 2, \dots, n$. Hence $x = \theta(x) = \theta(\sum_{i=1}^{n} p_i t_i) = \sum_{i=1}^{n} \theta(p_i) t_i \in xAT = xT$. Thus P_A is strongly T-accessible, hence a quasi-generator by Theorem 2.1.

The remainder of the corollary follows from the fact that over a regular ring every projective is regular [13, Example 1, page 238].

In our present framework the question that we asked prior to Theorem 1.3 becomes the following: When is $r_{\mathcal{A}}(T) = r_{\mathcal{A}}(P)$ strongly *T*-accessible as a right *A*-module?

THEOREM 2.3. Let P_A be projective with trace ideal T and B =End (P_A) . The following statements are equivalent.

- (1) $r_A(P)$ is strongly T-accessible as a right A-module.
- (2) $r_A(P) + T = A$.
- (3) $_{A}(A/T)$ is projective.
- (4) $P_{\overline{A}}$ is a generator where $\overline{A} = A/r_A(P)$.
- (5) P_A is a quasi-generator and $_AT$ is finitely generated.
- (6) P_A is a quasi-generator and $_BP$ is finitely generated.

Proof. The equivalence of (1), (2), and (3) follows from Theorem 1.3. That (3) is equivalent to (5) follows since P_A is a quasi-generator if and only if $_{A}(A/T)$ is flat.

(2) \leftrightarrow (4) $P_{\overline{A}}$ is projective with trace ideal $\overline{T} = T + r_A(T)/r_A(T)$.

Thus $r_A(P) + T = A$ if and only if $\overline{T} = \overline{A}$; i.e., if and only if $P_{\overline{A}}$ is a generator.

(4) \rightarrow (6) Since $B = \text{End}(P_{\overline{A}})$, $_{B}P$ is finitely generated. Thus (6) follows by (3).

(6) \rightarrow (1) Since $_{B}P$ is finitely generated we may write $P = Bp_{1} + \cdots + Bp_{n}$ where $p_{1}, \dots, p_{n} \in P$. Since P_{A} is a quasi-generator $r_{A}(p_{i}) \in F(\mathscr{H}')$ for $i = 1, \dots, n$. Thus $r_{A}(T) = r_{A}(P) = \bigcap_{i=1}^{n} r_{A}(p_{i}) \in F(\mathscr{H}')$ since $F(\mathscr{H}')$ is closed under finite intersections.

REMARK. While $r_A(T) \in F(\mathcal{H}')$ implies that $F(\mathcal{H}')$ has a minimal element, it is possible for $F(\mathcal{H}')$ to have a minimal element without $r_A(T)$ being contained in $F(\mathcal{H}')$. Let A be a left perfect ring and let P_A be a faithful projective that is not a generator. $F(\mathcal{H}')$ has a minimal element L [1, Corollary 1.6], but $L \neq r_A(T)$. Otherwise L = 0, which implies that every right A-module is torsion; i.e., P_A is a generator.

COROLLARY 2.4. Let P_A be projective with trace ideal T and $B = \text{End}(P_A)$. The following statements are equivalent.

(1) P_A is a generator (progenerator).

(2) P_A is a (finitely generated) faithful quasi-generator and $_{_B}P$ is finitely generated.

(3) P_A is (finitely generated) faithful and $_A(A/T)$ is projective.

EXAMPLE 2.5. (i) Let A be the ring of all 2 by 2 upper triangular matrices over a field K. Let

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
 .

Then $P_A = eA$ is a projective quasi-generator. $_BP$ is finitely generated since $_BP = Be$ where B = eAe. However, P_A is not faithful, so P_A is not a generator.

Also, T = AeA is a left direct summand; i.e., $_A(A/T)$ is projective. However, $(A/T)_A$ is not flat (see Corollary 1.4).

(ii) Let $A = \prod_{i \in I} K_i$ where $K_i = K$ a field and the index set I is countable infinite. Let $P_A = \bigoplus_{i \in I} K_i$. We may write $P_A = \bigoplus_{i \in I} e_i A$ where $e_i^s = e_i \in A$ and $e_i A \cong K_i$ for $i \in I$. Then P_A is projective with trace ideal $T = \bigoplus_{i \in I} e_i A$. Since A is a regular ring $_A(A/T)$ is flat, and thus P_A is a quasi-generator. Furthermore, P_A is faithful, but $_BP$ is not finitely generated as $_AT$ is not finitely generated. Hence P_A is not a generator.

REMARK. By Corollary 2.4 (3) if A is a semiperfect ring, then a projective module P_A is a generator if and only if P_A is a faithful

quasi-generator. Also, by Corollary 2.4 (2) if A is a regular ring, then a projective module P_A is a generator if and only if P_A is faithful and $_{B}P$ is finitely generated.

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