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A CHARACTERIZATION OF QF – 3 RINGS

EDGAR ANDREWS RUTTER

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E. A. RUTTER

Let R be a ring with minimum condition on left or right ideals. It is shown that R is a QF -3 ring if and only if each finitely generated submodule of the injective hull of R , regarded as a left R -module, is torsionless. The same approach yields a simplified proof that R is quasi-Frobenius if and only if every finitely generated left R -module is torsionless.

A ring with identity is called a *left QF -3 ring* if it has a (unique) minimal faithful left module, and a QF -3 ring means a ring which is both left and right QF -3. This class of rings originated with Thrall [9] as a generalization of quasi-Frobenius or QF algebras and has been studied extensively in recent years. Quasi-Frobenius rings have many interesting characterizations and in most instances there exists an analogous characterization of QF -3 rings at least in the case of rings with minimum condition and often for a much larger class of rings. It is well known that a ring with minimum condition on left or right ideals is a left QF -3 ring if and only if the injective hull $E({}_R R)$ of the ring R regarded as a left R -module is projective. Moreover, in this case R is a QF -3 ring (cf. [6] and [8]). For semi-primary or perfect rings; however, the situation is somewhat different. Namely, a perfect ring is a left QF -3 ring if and only if $E({}_R R)$ is torsionless. A module is called *torsionless* if it can be embedded in a direct product of copies of the ring regarded as a module over itself. In this case $E({}_R R)$ need not be projective and R need not be right QF -3 (cf. [3] and [8]). However, a perfect ring is QF -3 if and only if both $E({}_R R)$ and $E(R_R)$ are projective (see [8]). In this note, it is shown that if R is left perfect ring, $E({}_R R)$ is projective if and only if each finitely generated submodule of $E({}_R R)$ can be embedded in a free R -module. For a ring with minimum condition on left or right ideals this latter condition is equivalent to each finitely generated submodule of $E({}_R R)$ being torsionless. Thus in that case QF -3 rings may be characterized by this weaker condition. The technique of proof also yields a much simplified proof of a characterization of QF rings given by the present author in [7]. Namely, a ring with minimum condition on left or right ideals is QF if and only if each finitely generated left module is torsionless. Indeed, the characterization of QF -3 rings given here may be regarded as the analog of that result.

THEOREM 1. *Let R be a left perfect ring. $E({}_R R)$ is projective if and only if each finitely generated submodule of $E({}_R R)$ can be embedded*

in a free R -module.

Since flat modules over a left perfect ring are projective, this result is immediate from the following lemma. For a discussion of left perfect rings see [1].

LEMMA 2. *Let I be an injective left R -module. If each finitely generated submodule of I can be embedded in a flat R -module, then I is flat.*

Proof. By [2, Exercise 6, p. 123] it suffices to show that for any $a_1, \dots, a_m \in I$ satisfying a linear relation $\sum_{i=1}^m r_i a_i = 0$ with $r_i \in R$, there exists a positive integer n and elements $b_j \in I$, $s_{ij} \in R$ such that for each $1 \leq i \leq m$ and $1 \leq j \leq n$

$$(*) \quad a_i = \sum_{j=1}^n s_{ij} b_j, \quad \sum_{i=1}^m r_i s_{ij} = 0.$$

Let A be the submodule of I generated by a_1, \dots, a_m . By hypothesis A is a submodule of a flat R -module F and so by [2, Exercise 6, p. 123] there exists an integer n and elements $c_j \in F$, $s_{ij} \in R$ such that for all $1 \leq i \leq m$ and $1 \leq j \leq n$

$$(**) \quad a_i = \sum_{j=1}^n s_{ij} c_j, \quad \sum_{i=1}^m r_i s_{ij} = 0.$$

Since I is injective the inclusion map of A into F can be extended to an R -homomorphism α of F into I such that $(a)\alpha = a$ for all $a \in A$. Setting $b_j = (c_j)\alpha$ and applying α to the first half of (**) shows that (*) can be satisfied for any such choice of a_1, \dots, a_m . Thus I is flat.

COROLLARY 3. *If R is a ring with minimum condition on left or right ideals, the following conditions are equivalent.*

- (a) R is a QF-3 ring.
- (b) Every finitely generated submodule of $E({}_R R)$ can be embedded in a free R -module.
- (c) Every finitely generated submodule of $E({}_R R)$ is torsionless.

Proof. In view of the introductory remarks and Theorem 1 it suffices to show that (c) implies (b). Let M be a finitely generated submodule of $E({}_R R)$ and $M^* = \text{Hom}_R(M, R)$. It suffices to find $f_1, \dots, f_n \in M^*$ such that $\bigcap_{i=1}^n \text{Ker } f_i = (0)$ since the map $f: M \rightarrow \bigoplus_{i=1}^n {}_R R$ via $m \rightarrow (f_1(m), \dots, f_n(m))$ will then give the desired embedding. Since M is torsionless, $(0) = \bigcap_f \text{Ker } f$ with $f \in M^*$. If R satisfies the minimum condition on left ideals such f_1, \dots, f_n exist since M being finitely generated satisfies the descending chain condition on

R -submodules. If R satisfies the minimum condition on right ideals then since M is finitely generated M^* is isomorphic to a submodule of a finitely generated free right R -module and hence is finitely generated. (See [5, p. 66].) If f_1, \dots, f_n generate M^* , they have the desired property since M is torsionless.

REMARK. Condition (b) does not imply condition (a) for rings with maximum condition since any commutative integral domain which is not a field satisfies (b) but is not QF-3.

COROLLARY 4. *If R is a left and right perfect ring then R is a QF-3 ring if and only if every finitely generated submodule of $E({}_R R)$ and $E(R_R)$ is isomorphic to a submodule of a free R -module.*

Proof. In view of the introductory remarks this result is immediate from Theorem 1 and its right hand analog.

The next theorem and its corollary were proved in [7].

THEOREM 5. *Let R be a left perfect ring. R is a quasi-Frobenius ring if and only if every finitely generated left R -module is isomorphic to a submodule of a free R -module.*

Proof. This result follows from Lemma 2 and the fact that QF rings are characterized by the property that every injective module is projective [4, Theorem 5.3].

The next corollary follows from Theorem 5 in exactly the same manner that Corollary 3 follows from Theorem 1.

COROLLARY 6. *If R is a ring with minimum condition on left or right ideals, the following conditions are equivalent.*

- (a) *R is quasi-Frobenius.*
- (b) *Every finitely generated left R -module is isomorphic to a submodule of a free R -module.*
- (c) *Every finitely generated left R -module is torsionless.*

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