

# Pacific Journal of Mathematics

**A CHARACTERIZATION OF QF – 3 RINGS**

EDGAR ANDREWS RUTTER

## A CHARACTERIZATION OF $QF$ -3 RINGS

E. A. RUTTER

Let  $R$  be a ring with minimum condition on left or right ideals. It is shown that  $R$  is a  $QF$ -3 ring if and only if each finitely generated submodule of the injective hull of  $R$ , regarded as a left  $R$ -module, is torsionless. The same approach yields a simplified proof that  $R$  is quasi-Frobenius if and only if every finitely generated left  $R$ -module is torsionless.

A ring with identity is called a *left  $QF$ -3 ring* if it has a (unique) minimal faithful left module, and a  $QF$ -3 ring means a ring which is both left and right  $QF$ -3. This class of rings originated with Thrall [9] as a generalization of quasi-Frobenius or  $QF$  algebras and has been studied extensively in recent years. Quasi-Frobenius rings have many interesting characterizations and in most instances there exists an analogous characterization of  $QF$ -3 rings at least in the case of rings with minimum condition and often for a much larger class of rings. It is well known that a ring with minimum condition on left or right ideals is a left  $QF$ -3 ring if and only if the injective hull  $E({}_R R)$  of the ring  $R$  regarded as a left  $R$ -module is projective. Moreover, in this case  $R$  is a  $QF$ -3 ring (cf. [6] and [8]). For semi-primary or perfect rings; however, the situation is somewhat different. Namely, a perfect ring is a left  $QF$ -3 ring if and only if  $E({}_R R)$  is torsionless. A module is called *torsionless* if it can be embedded in a direct product of copies of the ring regarded as a module over itself. In this case  $E({}_R R)$  need not be projective and  $R$  need not be right  $QF$ -3 (cf. [3] and [8]). However, a perfect ring is  $QF$ -3 if and only if both  $E({}_R R)$  and  $E(R_R)$  are projective (see [8]). In this note, it is shown that if  $R$  is left perfect ring,  $E({}_R R)$  is projective if and only if each finitely generated submodule of  $E({}_R R)$  can be embedded in a free  $R$ -module. For a ring with minimum condition on left or right ideals this latter condition is equivalent to each finitely generated submodule of  $E({}_R R)$  being torsionless. Thus in that case  $QF$ -3 rings may be characterized by this weaker condition. The technique of proof also yields a much simplified proof of a characterization of  $QF$  rings given by the present author in [7]. Namely, a ring with minimum condition on left or right ideals is  $QF$  if and only if each finitely generated left module is torsionless. Indeed, the characterization of  $QF$ -3 rings given here may be regarded as the analog of that result.

**THEOREM 1.** *Let  $R$  be a left perfect ring.  $E({}_R R)$  is projective if and only if each finitely generated submodule of  $E({}_R R)$  can be embedded*

in a free  $R$ -module.

Since flat modules over a left perfect ring are projective, this result is immediate from the following lemma. For a discussion of left perfect rings see [1].

**LEMMA 2.** *Let  $I$  be an injective left  $R$ -module. If each finitely generated submodule of  $I$  can be embedded in a flat  $R$ -module, then  $I$  is flat.*

*Proof.* By [2, Exercise 6, p. 123] it suffices to show that for any  $a_1, \dots, a_m \in I$  satisfying a linear relation  $\sum_{i=1}^m r_i a_i = 0$  with  $r_i \in R$ , there exists a positive integer  $n$  and elements  $b_j \in I, s_{ij} \in R$  such that for each  $1 \leq i \leq m$  and  $1 \leq j \leq n$

$$(*) \quad a_i = \sum_{j=1}^n s_{ij} b_j, \quad \sum_{i=1}^m r_i s_{ij} = 0.$$

Let  $A$  be the submodule of  $I$  generated by  $a_1, \dots, a_m$ . By hypothesis  $A$  is a submodule of a flat  $R$ -module  $F$  and so by [2, Exercise 6, p. 123] there exists an integer  $n$  and elements  $c_j \in F, s_{ij} \in R$  such that for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$

$$(**) \quad a_i = \sum_{j=1}^n s_{ij} c_j, \quad \sum_{i=1}^m r_i s_{ij} = 0.$$

Since  $I$  is injective the inclusion map of  $A$  into  $F$  can be extended to an  $R$ -homomorphism  $\alpha$  of  $F$  into  $I$  such that  $(a)\alpha = a$  for all  $a \in A$ . Setting  $b_j = (c_j)\alpha$  and applying  $\alpha$  to the first half of  $(**)$  shows that  $(*)$  can be satisfied for any such choice of  $a_1, \dots, a_m$ . Thus  $I$  is flat.

**COROLLARY 3.** *If  $R$  is a ring with minimum condition on left or right ideals, the following conditions are equivalent.*

- (a)  $R$  is a QF-3 ring.
- (b) Every finitely generated submodule of  $E({}_R R)$  can be embedded in a free  $R$ -module.
- (c) Every finitely generated submodule of  $E({}_R R)$  is torsionless.

*Proof.* In view of the introductory remarks and Theorem 1 it suffices to show that (c) implies (b). Let  $M$  be a finitely generated submodule of  $E({}_R R)$  and  $M^* = \text{Hom}_R(M, R)$ . It suffices to find  $f_1, \dots, f_n \in M^*$  such that  $\bigcap_{i=1}^n \text{Ker } f_i = (0)$  since the map  $f: M \rightarrow \bigoplus_{i=1}^n {}_R R$  via  $m \rightarrow (f_1(m), \dots, f_n(m))$  will then give the desired embedding. Since  $M$  is torsionless,  $(0) = \bigcap_f \text{Ker } f$  with  $f \in M^*$ . If  $R$  satisfies the minimum condition on left ideals such  $f_1, \dots, f_n$  exist since  $M$  being finitely generated satisfies the descending chain condition on

$R$ -submodules. If  $R$  satisfies the minimum condition on right ideals then since  $M$  is finitely generated  $M^*$  is isomorphic to a submodule of a finitely generated free right  $R$ -module and hence is finitely generated. (See [5, p. 66].) If  $f_1, \dots, f_n$  generate  $M^*$ , they have the desired property since  $M$  is torsionless.

REMARK. Condition (b) does not imply condition (a) for rings with maximum condition since any commutative integral domain which is not a field satisfies (b) but is not QF-3.

COROLLARY 4. *If  $R$  is a left and right perfect ring then  $R$  is a QF-3 ring if and only if every finitely generated submodule of  $E({}_R R)$  and  $E(R_R)$  is isomorphic to a submodule of a free  $R$ -module.*

*Proof.* In view of the introductory remarks this result is immediate from Theorem 1 and its right hand analog.

The next theorem and its corollary were proved in [7].

THEOREM 5. *Let  $R$  be a left perfect ring.  $R$  is a quasi-Frobenius ring if and only if every finitely generated left  $R$ -module is isomorphic to a submodule of a free  $R$ -module.*

*Proof.* This result follows from Lemma 2 and the fact that QF rings are characterized by the property that every injective module is projective [4, Theorem 5.3].

The next corollary follows from Theorem 5 in exactly the same manner that Corollary 3 follows from Theorem 1.

COROLLARY 6. *If  $R$  is a ring with minimum condition on left or right ideals, the following conditions are equivalent.*

- (a)  *$R$  is quasi-Frobenius.*
- (b) *Every finitely generated left  $R$ -module is isomorphic to a submodule of a free  $R$ -module.*
- (c) *Every finitely generated left  $R$ -module is torsionless.*

#### REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., **95** (1960), 466-488.
2. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton: Princeton University Press, 1956.
3. R. R. Colby and E. A. Rutter, *Semi-primary QF-3 rings*, Nagoya Math. J., **39** (1968), 253-258.
4. C. Faith and E. A. Walker, *Direct sum representations of injective modules*, J. Algebra, **5** (1967), 203-221.

5. J. P. Jans, *Rings and Homology*, New York: Holt, Rinehart and Winston Inc., 1964.
6. K. Morita, *Duality in QF-3 rings*, Math. Z., **108** (1969), 237-252.
7. E. A. Rutter, *Two characterizations of quasi-Frobenius rings*, Pacific J. Math., **30** (1969), 777-784.
8. H. Tachikawa, *On left QF-3 rings*, Pacific J. Math., **32** (1970), 255-268.
9. R. M. Thrall, *Some generalizations of quasi-Frobenius algebras*, Trans. Amer. Math. Soc., **64** (1948), 173-183.

Received February 1, 1973.

VIRGINIA POLYTECHNIC INSTITUTE  
AND STATE UNIVERSITY

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI\*  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

\* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Copyright © 1973 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

Robert F. V. Anderson, <i>Laplace transform methods in multivariate spectral theory</i> .....	339
William George Bade, <i>Two properties of the Sorgenfrey plane</i> .....	349
John Robert Baxter and Rafael Van Severen Chacon, <i>Functionals on continuous functions</i> .....	355
Phillip Wayne Bean, <i>Helly and Radon-type theorems in interval convexity spaces</i> .....	363
James Robert Boone, <i>On <math>k</math>-quotient mappings</i> .....	369
Ronald P. Brown, <i>Extended prime spots and quadratic forms</i> .....	379
William Hugh Cornish, <i>Crawley's completion of a conditionally upper continuous lattice</i> .....	397
Robert S. Cunningham, <i>On finite left localizations</i> .....	407
Robert Jay Daverman, <i>Approximating polyhedra in codimension one spheres embedded in <math>S^n</math> by tame polyhedra</i> .....	417
Burton I. Fein, <i>Minimal splitting fields for group representations</i> .....	427
Peter Fletcher and Robert Allen McCoy, <i>Conditions under which a connected representable space is locally connected</i> .....	433
Jonathan Samuel Golan, <i>Topologies on the torsion-theoretic spectrum of a noncommutative ring</i> .....	439
Manfred Gordon and Edward Martin Wilkinson, <i>Determinants of Petrie matrices</i> .....	451
Alfred Peter Hallstrom, <i>A counterexample to a conjecture on an integral condition for determining peak points (counterexample concerning peak points)</i> .....	455
E. R. Heal and Michael Windham, <i>Finitely generated <math>F</math>-algebras with applications to Stein manifolds</i> .....	459
Denton Elwood Hewgill, <i>On the eigenvalues of a second order elliptic operator in an unbounded domain</i> .....	467
Charles Royal Johnson, <i>The Hadamard product of <math>A</math> and <math>A^*</math></i> .....	477
Darrell Conley Kent and Gary Douglas Richardson, <i>Regular completions of Cauchy spaces</i> .....	483
Alan Greenwell Law and Ann L. McKerracher, <i>Sharpened polynomial approximation</i> .....	491
Bruce Stephen Lund, <i>Subalgebras of finite codimension in the algebra of analytic functions on a Riemann surface</i> .....	495
Robert Wilmer Miller, <i>TTF classes and quasi-generators</i> .....	499
Roberta Mura and Akbar H. Rhemtulla, <i>Solvable groups in which every maximal partial order is isolated</i> .....	509
Isaac Namioka, <i>Separate continuity and joint continuity</i> .....	515
Edgar Andrews Rutter, <i>A characterization of QF - 3 rings</i> .....	533
Alan Saleski, <i>Entropy of self-homeomorphisms of statistical pseudo-metric spaces</i> .....	537
Ryōtarō Satō, <i>An Abel-maximal ergodic theorem for semi-groups</i> .....	543
H. A. Seid, <i>Cyclic multiplication operators on <math>L_p</math>-spaces</i> .....	549
H. B. Skerry, <i>On matrix maps of entire sequences</i> .....	563
John Brendan Sullivan, <i>A proof of the finite generation of invariants of a normal subgroup</i> .....	571
John Griggs Thompson, <i>Nonsolvable finite groups all of whose local subgroups are solvable, VI</i> .....	573
Ronson Joseph Warne, <i>Generalized <math>\omega</math> - <math>\mathcal{L}</math>-unipotent bisimple semigroups</i> .....	631
Toshihiko Yamada, <i>On a splitting field of representations of a finite group</i> .....	649