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**ENTROPY OF SELF-HOMEOMORPHISMS OF STATISTICAL
PSEUDO-METRIC SPACES**

ALAN SALESKI

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A pseudo-Menger space is a set X together with a function $\theta: X \times X \rightarrow \mathcal{D}$, the set of distribution functions, satisfying certain natural axioms similar to those of a pseudo-metric space. Let $T: X \rightarrow X$ be a bijection and let θ_T denote the topology generated by $\{T^i U(p, \varepsilon, \lambda): i \in \mathbb{Z}, p \in X, \varepsilon > 0, \lambda > 0\}$ where $U(p, \varepsilon, \lambda) = \{q: \theta(p, q)(\varepsilon) > 1 - \lambda\}$. Assume that θ_T is compact. Let $h(T, \theta)$ denote the topological entropy of T with respect to the θ_T topology. The purpose of this note is to show that if one is given a sequence $\{\theta_n\}$ of pseudo-Menger structures on X satisfying $\theta_n(p, q) \geq \theta(p, q)$ and $\theta_n(p, q) \rightarrow \theta(p, q)$ in distribution for all $p, q \in X$ then $h(T, \theta_n) \rightarrow h(T, \theta)$. A counterexample is then given to show that, in general, the condition $\theta_n(p, q) \geq \theta(p, q)$ cannot be removed.

1. The investigation of statistical metric spaces was undertaken by Karl Menger [5] in 1942. Essentially these are spaces in which the "distance" between any two points is given by a probability distribution function. Our purpose is to investigate the behavior of the topological entropy of a self-homeomorphism of a compact Menger space under perturbations of these distribution functions. We proceed to give precise definitions.

2. Preliminaries. Let I denote the closed unit interval, \mathbb{Q}^+ the positive rationals, \mathbb{Z}^+ the positive integers, and \mathcal{D} the set of all left-continuous monotone increasing functions $F: \mathbb{R} \rightarrow I$ satisfying $F(0) = 0$ and $\sup F(x) = 1$. Let H be the function defined by: $H(t) = 0$ for $t \leq 0$ and $H(t) = 1$ for $t > 0$.

Throughout our discussion, X will be a fixed set. Let \mathcal{F} denote the collection of all functions $\theta: X \times X \rightarrow \mathcal{D}$. For convenience we shall often write θ_{pq} in place of $\theta(p, q)$. A *statistical pseudo-metric space* is an ordered pair (X, θ) where $\theta \in \mathcal{F}$ satisfies

- (a) $\theta_{pq} = \theta_{qp}$ for all $p, q \in X$.
- (b) $\theta_{pq}(a + b) = 1$ whenever $\theta_{pr}(a) = \theta_{rq}(b) = 1$.
- (c) $\theta_{pp} = H$ for all $p \in X$.

If, in addition, θ satisfies

- (d) $\theta_{pq} = H$ only if $p = q$

then (X, θ) is a *statistical metric space*.

Let \mathcal{S} denote the collection of all θ for which (X, θ) is a statistical pseudo-metric space.

A *triangular norm* is a function $\Delta: I \times I \rightarrow I$ which is associative,

commutative, nondecreasing in each variable and satisfies $\Delta(y, 1) = y$ for each $y \in I$. A continuous Menger space [pseudo-Menger space] is a statistical metric space [statistical pseudo-metric space] (X, θ) for which there exists a continuous triangular norm Δ satisfying:

(e) $\theta_{pr}(a + b) \geq \Delta(\theta_{pq}(a), \theta_{qr}(b))$ for all $p, q, r \in X$ and all $a, b \in \mathbf{R}$.

Let \mathcal{M} denote the set of all θ for which (X, θ) is a continuous pseudo-Menger space.

If $\theta \in \mathcal{F}$ then one defines a topology on X , $\tau(\theta)$, in the following manner: If $p \in X$, $\varepsilon, \lambda > 0$, let $U(p, \varepsilon, \lambda, \theta) = \{q \in X: \theta_{pq}(\varepsilon) > 1 - \lambda\}$. These sets are a subbasis for $\tau(\theta)$. It was proven in [9] that if (X, θ) is a continuous Menger space then $\tau(\theta)$ is metrizable.

We shall be concerned with studying a bijective map $T: X \rightarrow X$. If τ is a topology on X then τ^T will denote the topology generated by $\{T^i A: A \in \tau, i \in \mathbf{Z}\}$. We will say that a map $\theta \in \mathcal{F}$ is *T-admissible* if $\tau(\theta)^T$ is compact and $\theta(T^i x, T^i y) = H$ ($i \in \mathbf{Z}$) whenever $\theta_{xy} = H$.

Let $\mathcal{F}_0, \mathcal{S}_0, \mathcal{M}_0$ denote the *T-admissible* maps belonging to $\mathcal{F}, \mathcal{S}, \mathcal{M}$ respectively. Of course $\mathcal{M}_0 \subset \mathcal{S}_0 \subset \mathcal{F}_0$. We define a partial order on \mathcal{F}_0 as follows: If $\theta, \Psi \in \mathcal{F}_0$ then $\theta \leq \Psi$ if and only if $\theta_{pq}(t) \leq \Psi_{pq}(t)$ for all $t \in \mathbf{R}$ and $p, q \in X$. If $\{\theta^n: n \in \mathbf{Z}^+\}$ and Ψ belong to \mathcal{F} then $\theta^n \xrightarrow{D} \Psi$ will mean that for every $(p, q) \in X \times X$, $\theta^n_{pq}(y) \rightarrow \Psi_{pq}(y)$ for each $y \in X$ at which Ψ_{pq} is continuous.

For definitions and properties of measure theoretic entropy the reader is referred to [2] or [6]. If Σ is a σ -algebra of subsets of X , μ an invariant measure on Σ , $T\Sigma = \Sigma \pmod{0}$ and Γ a sub- σ -algebra of Σ then $h_\mu(T, \Gamma)$ will denote the measure theoretic entropy of $(X, \bigvee_{-\infty}^{\infty} T^i \Gamma, \mu, T)$.

For definitions and properties of topological entropy we refer the reader to [1] and [3]. If τ is a compact topology on X for which $T^{-1}(\tau) \subset \tau$ then $h(T, \tau)$ will denote the topological entropy of (X, τ, T) . In case $\theta \in \mathcal{M}_0$ we shall write $h(T, \theta)$ in place of $h(T, \tau(\theta)^T)$.

If τ is a topology on X then $\sigma(\tau)$ will denote the Borel σ -algebra generated by τ . If A and B are subsets of X let $A + B$ represent $(A \cup B) \sim (A \cap B)$.

3. A convergence theorem. If $\theta \in \mathcal{M}_0$ one can define the following relation on $X: x \sim y$ if and only if $\theta_{xy} = H$. This is an equivalence relation on X due to condition (b) of \mathcal{S} above. Note that θ induces the structure of a continuous Menger space on \tilde{X} . Let $\pi: X \rightarrow X/\sim$ be the projection map. Since X/\sim is metrizable, the topology on X is induced by a pseudo-metric. Consequently π is a continuous open and closed surjection. Let \tilde{T} be the self-homeomorphism of X/\sim defined by $\tilde{T}\pi = \pi T$.

LEMMA 1. *If $\theta \in \mathcal{M}_0$ then*
 $h(T) = h(\tilde{T})$ and
 $h(T) = \sup \{h_\mu(T) : \mu \text{ is a } T\text{-invariant regular probability measure on the Borel sets of } X\}$.

Proof. π induces a Boolean isomorphism $\hat{\pi}$ between the Borel σ -algebra of X and that of X/\sim satisfying $\tilde{T}\hat{\pi} = \hat{\pi}T$. Since X/\sim is compact Hausdorff, we can apply the theorem of Goodwyn-Dinaburg-Goodman [3].

LEMMA 2. *Let T be a bijective map of a set X onto itself and let $\theta \in \mathcal{M}_0$. Suppose $\{\tau_i\}_1^\infty$ is a sequence of sub-topologies of $\tau = \tau(\theta)$. Denote $\sigma(\tau^i)$ by Σ and $\sigma(\tau_i)$ by Σ_i . In addition, assume that:*

(*) *Given any regular T -invariant probability measure μ on Σ , $A \in \tau$ and $\xi > 0$ then there is a positive integer N such that for each $i \geq N$ there exists a set $A_i \in \Sigma_i$ satisfying $\mu(A + A_i) < \xi$.*

Then $h(T, \tau_i^T) \rightarrow h(T, \tau^T)$.

Proof. We shall begin by assuming $h(T, \tau^T) < \infty$. Let $\varepsilon > 0$ be given. Lemma 1 allows us to choose a regular T -invariant probability measure μ on Σ such that $h_\mu(T, \Sigma) \geq h(T, \tau^T) - \varepsilon/3$. Let $P = \{P^1, \dots, P^s\} \subset \Sigma$ be a finite partition of X such that $h_\mu(T, P) \geq h_\mu(T, \Sigma) - \varepsilon/3$. Choose $\delta > 0$ such that if $Q \subset \Sigma$ is a finite partition of X into s sets then $|h_\mu(T, P) - h_\mu(T, Q)| < \varepsilon/3$ provided $\sum_1^s \mu(P^i + Q^i) < 3\delta$. Since μ is regular, there exist sets $A^i \in \tau^T$ ($1 \leq i \leq s$) satisfying $\mu(P^i + A^i) < \delta/3s$. Choose $K > 0$ and $B^1, \dots, B^s \in \mathbf{V}_{i=-K}^K T^i \tau$ such that $\mu(A^i + B^i) < \delta/3s$. This can be achieved since finite intersections of $\{T^i A : A \in \tau, i \in \mathbf{Z}\}$ constitute a basis for τ^T . Applying condition (*) of the hypothesis, there exists an integer $N > 0$ such that for each $j \geq N$ one can find sets $C^1, \dots, C^s \in \mathbf{V}_{i=-K}^K T^i \Sigma_j$ satisfying $\mu(B^i + C^i) < \delta/3s$ ($1 \leq i \leq s$). Applying the triangle inequality one obtains $\mu(P^i + C^i) < \delta/s$. Now since $\Sigma \mu(C^i) \leq \Sigma(\mu(P^i) + \delta/s) = 1 + \delta$ one can construct a partition $\bar{C} = \{\bar{C}^1, \dots, \bar{C}^s\} \subset \mathbf{V}_{i=-K}^K T^i \Sigma_j$ of X satisfying $\sum_{i=1}^s \mu(P^i + \bar{C}^i) \leq s(\delta/s) + 2\delta = 3\delta$. Therefore, $|h_\mu(T, \bar{C}) - h_\mu(T, P)| < \varepsilon/3$. Consequently, for all $j \geq N$, $h(T, \tau_j^T) \geq h_\mu(T, \sigma(\tau_j)) \geq h_\mu(T, \bar{C}) \geq h_\mu(T, P) - \varepsilon/3 \geq h_\mu(T, \Sigma) - 2/3\varepsilon \geq h(T, \Sigma) - \varepsilon$. It remains only to note that since $\tau_j \subset \tau$, $h(T, \tau_j^T) \leq h(T, \tau^T)$.

In case $h(T, \tau^T) = \infty$ a similar argument may be used.

LEMMA 3. *If $\theta \in \mathcal{M}_0$ and $p \in X$ then $\{U(p, \varepsilon, \lambda, \theta) : \varepsilon > 0, \lambda > 0\}$ is a local basis for $\tau(\theta)$ at p .*

Proof. Since $\hat{\pi}$ is an isomorphism of the Borel σ -algebra of X

with that of \tilde{X} such that $\hat{\pi}(U(p, \varepsilon, \lambda)) = \tilde{U}(\pi(p), \varepsilon, \lambda)$, it suffices to prove the lemma under the additional hypothesis that (X, θ) is a Menger space.

It was proven in Theorem 7.2 of [8] that $\{U(p, \varepsilon, \lambda): p \in X, \varepsilon > 0, \lambda > 0\}$ is a basis for $\tau(\theta)$. Let $z \in U(p, \varepsilon, \lambda)$ be given. We must show that there exist $\bar{\varepsilon} > 0, \bar{\lambda} > 0$ such that $U(z, \bar{\varepsilon}, \bar{\lambda}) \subset U(p, \varepsilon, \lambda)$. Suppose that for each positive integer j there exists $y_j \in U(p, 1/j, 1/j) \sim U(p, \varepsilon, \lambda)$. By compactness there exists a $y \in X$ and a subsequence $\{y_{j_n}\}$ such that $y_{j_n} \rightarrow y$. Now Theorem 8.1 of [8] yields $\liminf_{n \rightarrow \infty} \theta(y_{j_n}, z) = \theta(y, z)$. But $\theta(y_{j_n}, z)(1/m) > 1 - 1/m$ for $j_n \geq m$. Hence $\theta(y, z) = H$ which implies $y = z$ contradicting the fact that $y \in X \sim U(p, \varepsilon, \lambda)$.

LEMMA 4. *If $\theta \in \mathcal{M}_0, \Gamma \in \mathcal{S}_0$, and $\theta \geq \Gamma$ then $\tau(\theta) \subset \tau(\Gamma)$.*

Proof. This follows at once from Lemma 3 together with the observation that $U(p, \varepsilon, \lambda, \theta) \supset U(p, \varepsilon, \lambda, \Gamma)$.

LEMMA 5. *Given $\theta \in \mathcal{M}_0$ and a countable dense subset $\{y_i\}_1^\infty$ of $(X, \tau(\theta))$ then $\{U(y_i, \varepsilon, \lambda): i \in \mathbb{Z}^+, \varepsilon, \lambda \in \mathbb{Q}^+\}$ is a countable basis for $\tau(\theta)$.*

Proof. This follows from an argument similar to the one given in the proof of Lemma 3.

LEMMA 6. *Suppose $F_n, F \in \mathcal{D}$ and $F_n \geq F$ for all n . Let $F_n(x+)$ denote the right limit of F_n at x . If $F_n \xrightarrow{D} F$ then $F_n(x+) \rightarrow F(x+)$ for all x .*

Proof. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Choose y to be a continuity point of F such that $y > x$ and $F(y) - F(x+) < \varepsilon$. Since $F(x+) \leq F_n(x+) \leq F_n(y)$ we have $F(x+) \leq \liminf_{n \rightarrow \infty} F_n(x+) \leq \limsup_{n \rightarrow \infty} F_n(x+) \leq F(y)$. The desired conclusion now easily follows.

THEOREM. *Given $\{\theta^i \in \mathcal{M}_0: i \in \mathbb{Z}^+\}$ and $\Psi \in \mathcal{M}_0$ satisfying $\theta^i \geq \Psi$ for each i and $\theta^i \xrightarrow{D} \Psi$ then $h(T, \theta^i) \rightarrow h(T, \Psi)$.*

Proof. Let $A \in \tau(\Psi)$ and $\varepsilon > 0$ be given and let μ be a T -invariant probability measure on $\sigma(\tau(\Psi)^T)$. Since $(X, \tau(\Psi))$ is separable, Lemma 5 implies the existence of sequences $\{p_i \in A\}, \{\varepsilon_i > 0\}$ and $\{\lambda_i > 0\}$ satisfying $\bigcup_{i=1}^\infty U(p_i, \varepsilon_i, \lambda_i, \Psi) = A$. Since μ is finite, there exists a positive integer N_1 such that $\mu(A \sim \bigcup_{i=1}^{N_1} U(p_i, \varepsilon_i, \lambda_i, \Psi)) < \varepsilon/4$. Using the fact that $\bigcup_{j=1}^\infty U(p_i, \varepsilon_i, \lambda_i(1 - 2^{-j}), \Psi) = U(p_i, \varepsilon_i, \lambda_i, \Psi)$ we can select $0 < \bar{\lambda}_i < \lambda_i$ for $1 \leq i \leq N_1$ such that $\mu(A \sim \bigcup_{i=1}^{N_1} U(p_i, \varepsilon_i, \bar{\lambda}_i, \Psi)) < \varepsilon/3$. Furthermore, since Ψ_{pq} is left continuous we may choose $0 <$

$\bar{\varepsilon}_i < \varepsilon_i$ for $1 \leq i \leq N_1$ such that $\mu(A \sim \bigcup_{i=1}^{N_1} U(p_i, \bar{\varepsilon}_i, \bar{\lambda}_i, \Psi)) < \varepsilon/2$. Applying Lemma 6 and Egoroff's theorem, there exists a set $G \subset A$ such that $\mu(A \sim G) < \varepsilon/2$ and $\theta_{p_i q}^n(\bar{\varepsilon}_i +) \rightarrow \Psi_{p_i q}(\bar{\varepsilon}_i +)$ uniformly in $q \in G$ as $n \rightarrow \infty$ for $1 \leq i \leq N_1$. As a consequence of the uniform convergence of the functions $\theta_{p_i q}^n(\bar{\varepsilon}_i +)$ on G there exists a positive integer N_2 such that:

$$\begin{aligned} U(p_i, \bar{\varepsilon}_i, \bar{\lambda}_i, \Psi) \cap G &\subset \{q: \theta_{p_i q}^n(\bar{\varepsilon}_i +) > 1 - \bar{\lambda}_i\} \cap G \\ &\subset \{q: \Psi_{p_i q}(\bar{\varepsilon}_i +) > 1 - \lambda_i\} \cap G \subset U(p_i, \varepsilon_i, \lambda_i, \Psi) \subset A \end{aligned}$$

for $1 \leq i \leq N_1$ and all $n \geq N_2$. Then for $n \geq N_2$, $\mu(A + \bigcup_{i=1}^{N_1} U(p_i, \varepsilon_i, \lambda_i, \theta^n)) < \varepsilon$. Lemma 4 implies that the $\{\tau(\theta^i)\}$ are subtopologies of $\tau(\Psi)$. An application of Lemma 2 completes the proof.

As a consequence of the theorem we obtain:

COROLLARY. Let $\{d, d_n: n \in \mathbb{Z}^+\}$ be pseudo-metrics on the space X and $T: X \rightarrow X$ be a bijection satisfying

- (a) $d_n(x, y) \leq d(x, y)$ for all $x, y \in X$.
- (b) $d_n(x, y) \rightarrow d(x, y)$ as $n \rightarrow \infty$.
- (c) T is a self-homeomorphism of (X, d) and of (X, d_n) for each n .
- (d) (X, d) is compact.

Let T_n denote the self-homeomorphism T of the topological space (X, d_n) .

Then $h(T_n) \rightarrow h(T)$.

Proof. The proof follows from Theorem 5.1 of [8] where it is shown that every metric space is a continuous Menger space. Since T_n is a homeomorphism it is easy to see that $d_n(T^i x, T^i y) = 0$ whenever $d_n(x, y) = 0$. Now the theorem may be applied.

We present two examples to illustrate the use of the theorem and to show that the theorem is no longer valid if one removes the condition that $\theta_n \geq \theta$.

EXAMPLE A. Let $Y = \{0, 1/n: n \in \mathbb{Z}^+\}$ be endowed with the induced topology of I , $X = Y^{\mathbb{Z}}$ and $T: X \rightarrow X$ be the shift defined by $T(\{y_i\}) = \{y_{i+1}\}$. The space X is metrized by:

$$d(\{y_i\}, \{z_i\}) = \sum_{i=-\infty}^{\infty} \frac{|y_i - z_i|}{2^{|i|}}.$$

Subtopologies of (X, d) are determined by pseudo-metrics d_n defined by:

$$d_n(\{y_j\}, \{z_j\}) = \sum_{j=-\infty}^{\infty} \frac{|y_j \omega_n(y_j) - z_j \omega_n(z_j)|}{2^{|j|}}$$

where ω_n is the characteristic function of $[1/n, 1]$. Let T_n denote the self-homeomorphism T of (X, d_n) . Then $d_n \leq d$, $d_n \rightarrow d$, $h(T_n) = \ln n$ and $h(T) = \infty$.

EXAMPLE B. Let T be the shift on X as above. For $n \geq 1$ let u_n be the pseudo-metric on X given by:

$$u_n(\{y_j\}, \{z_j\}) = \sum_{j=-\infty}^{\infty} \frac{|t_n(y_j) - t_n(z_j)|}{2^{j_1}}$$

where

$$t_n(s) = \begin{cases} \frac{1}{n} & \text{if } s \geq \frac{1}{n} \\ s & \text{otherwise.} \end{cases}$$

Then $u_n \rightarrow u$ as $n \rightarrow \infty$, where $u(x, y) = 0$ for all $x, y \in X$. Now $h(T, u_n) = \infty$ but $h(T, u) = 0$.

We conclude with the following speculation:

Conjecture. Given $\{\theta^i \in \mathcal{M}_0; i \in \mathbb{Z}^+\}$ and $\Psi \in \mathcal{M}_0$ satisfying $\theta^i \xrightarrow{D} \Psi$ then

$$\liminf_{i \rightarrow \infty} h(T, \theta^i) \geq h(T, \Psi).$$

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December, 1974

Robert F. V. Anderson, <i>Laplace transform methods in multivariate spectral theory</i>	339
William George Bade, <i>Two properties of the Sorgenfrey plane</i>	349
John Robert Baxter and Rafael Van Severen Chacon, <i>Functionals on continuous functions</i>	355
Phillip Wayne Bean, <i>Helly and Radon-type theorems in interval convexity spaces</i>	363
James Robert Boone, <i>On k-quotient mappings</i>	369
Ronald P. Brown, <i>Extended prime spots and quadratic forms</i>	379
William Hugh Cornish, <i>Crawley's completion of a conditionally upper continuous lattice</i>	397
Robert S. Cunningham, <i>On finite left localizations</i>	407
Robert Jay Daverman, <i>Approximating polyhedra in codimension one spheres embedded in S^n by tame polyhedra</i>	417
Burton I. Fein, <i>Minimal splitting fields for group representations</i>	427
Peter Fletcher and Robert Allen McCoy, <i>Conditions under which a connected representable space is locally connected</i>	433
Jonathan Samuel Golan, <i>Topologies on the torsion-theoretic spectrum of a noncommutative ring</i>	439
Manfred Gordon and Edward Martin Wilkinson, <i>Determinants of Petrie matrices</i>	451
Alfred Peter Hallstrom, <i>A counterexample to a conjecture on an integral condition for determining peak points (counterexample concerning peak points)</i>	455
E. R. Heal and Michael Windham, <i>Finitely generated F-algebras with applications to Stein manifolds</i>	459
Denton Elwood Hewgill, <i>On the eigenvalues of a second order elliptic operator in an unbounded domain</i>	467
Charles Royal Johnson, <i>The Hadamard product of A and A^*</i>	477
Darrell Conley Kent and Gary Douglas Richardson, <i>Regular completions of Cauchy spaces</i>	483
Alan Greenwell Law and Ann L. McKerracher, <i>Sharpened polynomial approximation</i>	491
Bruce Stephen Lund, <i>Subalgebras of finite codimension in the algebra of analytic functions on a Riemann surface</i>	495
Robert Wilmer Miller, <i>TTF classes and quasi-generators</i>	499
Roberta Mura and Akbar H. Rhemtulla, <i>Solvable groups in which every maximal partial order is isolated</i>	509
Isaac Namioka, <i>Separate continuity and joint continuity</i>	515
Edgar Andrews Rutter, <i>A characterization of QF - 3 rings</i>	533
Alan Saleski, <i>Entropy of self-homeomorphisms of statistical pseudo-metric spaces</i>	537
Ryōtarō Satō, <i>An Abel-maximal ergodic theorem for semi-groups</i>	543
H. A. Seid, <i>Cyclic multiplication operators on L_p-spaces</i>	549
H. B. Skerry, <i>On matrix maps of entire sequences</i>	563
John Brendan Sullivan, <i>A proof of the finite generation of invariants of a normal subgroup</i>	571
John Griggs Thompson, <i>Nonsolvable finite groups all of whose local subgroups are solvable, VI</i>	573
Ronson Joseph Warne, <i>Generalized $\omega - \mathcal{L}$-unipotent bisimple semigroups</i>	631
Toshihiko Yamada, <i>On a splitting field of representations of a finite group</i>	649