

# Pacific Journal of Mathematics

**AN ABEL-MAXIMAL ERGODIC THEOREM FOR  
SEMI-GROUPS**

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## AN ABEL-MAXIMAL ERGODIC THEOREM FOR SEMI-GROUPS

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**The purpose of this paper is to prove a maximal ergodic theorem for Abel means of a strongly measurable semi-group  $\Gamma = \{T_t; t \geq 0\}$  of linear contractions on a complex  $L_1$ -space satisfying  $|T_t f| \leq c$  a.e. for any  $t \geq 0$  and any integrable  $f$  with  $|f| \leq c$  a.e. Applying the obtained maximal ergodic theorem, individual and dominated ergodic theorems for Abel means are also proved. These results extend results obtained by D. A. Edwards for sub-Markovian semi-groups.**

2. The maximal ergodic theorem. Let  $(X, \mathcal{B}, \mu)$  be a  $\sigma$ -finite measure space and  $L_p(X) = L_p(X, \mathcal{B}, \mu)$ ,  $1 \leq p \leq \infty$ , the usual (complex) Banach spaces. Let  $\Gamma = \{T_t; t \geq 0\}$  be a strongly measurable semi-group of linear contractions on  $L_1(X)$  with  $\|T_t f\|_\infty \leq \|f\|_\infty$  for any  $f \in L_1(X) \cap L_\infty(X)$  and any  $t \geq 0$ . By the Riesz convexity theorem  $\Gamma$  may be considered as a strongly measurable semi-group of linear contractions on  $L_p(X)$  for each  $p$  with  $1 \leq p < \infty$ . It is then known (cf. [4], p. 686) that for each  $f \in L_p(X)$  with  $1 \leq p < \infty$ , there exists a scalar function  $T_t f(x)$ , measurable with respect to the product of Lebesgue measure and  $\mu$ , such that for almost all  $t$ ,  $T_t f(x)$ , as a function of  $x$ , belongs to the equivalence class of  $T_t f$  and such a measurable representation is uniquely determined except for a set of the product-measure zero. Moreover, since the integral  $\int_0^\infty e^{-\lambda t} T_t f dt$  exists for any  $\lambda > 0$ , it follows from Theorem III.11.17 of [4] that there exists a  $\mu$ -null set  $E(f)$ , dependent on  $f$  but independent of  $\lambda$ , such that if  $x \notin E(f)$  then  $e^{-\lambda t} T_t f(x)$  is integrable on  $[0, \infty)$  for each  $\lambda > 0$  and the integral  $\int_0^\infty e^{-\lambda t} T_t f(x) dt$ , as a function of  $x$ , belongs to the equivalence class of  $\int_0^\infty e^{-\lambda t} T_t f dt$ . Thus if we denote the integral  $\int_0^\infty e^{-\lambda t} T_t f dt$  by  $R_\lambda f$  then  $\int_0^\infty e^{-\lambda t} T_t f(x) dt$  gives a representation of  $R_\lambda f$ , and hence, from now on, we shall write  $R_\lambda f(x)$  for  $\int_0^\infty e^{-\lambda t} T_t f(x) dt$ .

Let  $f \in L_p(X)$  and  $a > 0$ . Following Chacon [1], we define

$$f^{a-}(x) = [\text{sgn } f(x)] \min(a, |f(x)|),$$

$$f^{a+}(x) = [\text{sgn } f(x)] (|f(x)| - \min(a, |f(x)|)),$$

$$f^*(x) = \sup_{0 < \lambda < \infty} |\lambda R_\lambda f(x)|$$

and

$$E^*(a) = \{x; f^*(x) > a\},$$

where  $\operatorname{sgn} f(x) = f(x)/|f(x)|$  if  $f(x) \neq 0$  and  $\operatorname{sgn} f(x) = 0$  if  $f(x) = 0$ .

We are now in a position to state the main theorem of this paper.

**THEOREM 1.** *If  $f \in L_p(X)$ ,  $1 \leq p < \infty$ , then for any  $a > 0$  we have*

$$\int_{E^*(a)} (a - |f^{a-}|) d\mu \leq \int |f^{a+}| d\mu.$$

For the proof of Theorem 1 we shall need the following lemma, whose proof is given in [7].

**LEMMA.** *Let  $\tau$  be a positive linear contraction on  $L_1(X)$  satisfying  $\|\tau f\|_\infty \leq \|f\|_\infty$  for any  $f \in L_1(X) \cap L_\infty(X)$ , let  $f \in L_p(X)$  with  $1 \leq p < \infty$  and let  $a > 0$ . Define*

$$e^*(a) = \left\{ x; \sup_{0 < r < 1} \left| (1-r) \sum_{k=0}^{\infty} r^k \tau^k f(x) \right| > a \right\}.$$

Then we have

$$\int_{e^*(a)} (a - |f^{a-}|) d\mu \leq \int |f^{a+}| d\mu.$$

*Proof of Theorem 1.* For each  $\lambda > 0$  and each positive integer  $n$ , define

$$R_\lambda^{(n)} f = \frac{1}{n} \sum_{k=0}^{\infty} e^{-\lambda k/n} T_{k/n} f.$$

We shall first prove that for any fixed  $\lambda > 0$ ,

$$(1) \quad \lim_n \|R_\lambda f - R_\lambda^{(n)} f\|_p = 0.$$

In fact, if  $\varepsilon > 0$  then choose a positive real number  $a$  such that

$$(2) \quad \int_a^\infty \|e^{-\lambda t} T_t f\|_p dt < \varepsilon \quad \text{and} \quad \frac{e^{-\lambda a}}{\lambda} < \varepsilon.$$

Let  $k(n)$  be the positive integer such that

$$(3) \quad \frac{k(n)}{n} \leq a < \frac{k(n)+1}{n}.$$

Then

$$\begin{aligned} \|R_\lambda f - R_\lambda^{(n)} f\|_p &\leq \left\| \int_0^a e^{-\lambda t} T_t f dt - \frac{1}{n} \sum_{k=0}^{k(n)} e^{-\lambda k/n} T_{k/n} f \right\|_p \\ &\quad + \varepsilon + \frac{1}{n} \sum_{k=k(n)+1}^{\infty} e^{-\lambda k/n}. \end{aligned}$$

Since  $1/n \sum_{k=k(n)+1}^{\infty} e^{-\lambda k/n} \leq 1/n e^{-\lambda} / (1 - e^{-\lambda/n})$  by (3) and  $\lim_n n(1 - e^{-\lambda/n}) = \lambda$ , it follows from (2) that for  $N_0$  sufficiently large enough and  $n \geq N_0$  we have

$$(4) \quad \frac{1}{n} \sum_{k=k(n)+1}^{\infty} e^{-\lambda k/n} < \varepsilon.$$

Let

$$g_n(t) = e^{-\lambda k/n} \quad \text{for} \quad \frac{k}{n} \leq t < \frac{k+1}{n}$$

and

$$T_t^{(n)}f = T_{k/n}f \quad \text{for} \quad \frac{k}{n} \leq t < \frac{k+1}{n}.$$

Since  $\Gamma = \{T_t; t \geq 0\}$  is strongly continuous on  $(0, \infty)$  (cf. [4], Lemma VIII.1.3),  $\lim_n \|g_n(t)T_t^{(n)}f - e^{-\lambda t}T_t f\|_p = 0$ , from which it follows that  $\lim_n \int_0^a \|e^{-\lambda t}T_t f - g_n(t)T_t^{(n)}f\|_p dt = 0$ , and hence (1) follows.

Since  $\lim_n n(1 - e^{-\lambda/n}) = \lambda$ , (1) implies at once that  $\lim_n \|\lambda R_\lambda f - (1 - e^{-\lambda/n}) \sum_{k=0}^{\infty} e^{-\lambda k/n} T_{k/n} f\|_p = 0$ . Let  $Q$  be the set of all positive rational numbers. By the Cantor diagonal argument there exists a subsequence  $\{n_i\}$  such that for any  $\lambda \in Q$ ,

$$\lambda R_\lambda f(x) = \lim_i (1 - e^{-\lambda/n_i}) \sum_{k=0}^{\infty} e^{-\lambda k/n_i} T_{k/n_i} f(x) \quad \text{a.e.}$$

Hence if we let

$$f_i^*(x) = \sup_{0 < \lambda < \infty} (1 - e^{-\lambda/n_i}) \sum_{k=0}^{\infty} e^{-\lambda k/n_i} \tau_i^k |f|(x),$$

where  $\tau_i$  denotes the linear modulus [2] of  $T_{1/n_i}$ , then  $|\lambda R_\lambda f(x)| \leq \liminf_i f_i^*(x)$  a.e. for any  $\lambda \in Q$ . Since the mapping  $\lambda \rightarrow \lambda \int_0^\infty e^{-\lambda t} T_t f(x) dt$  is continuous for almost all  $x \in X$ , it follows that  $\sup_{0 < \lambda < \infty} |\lambda R_\lambda f(x)| = \sup_{\lambda \in Q} \left| \lambda \int_0^\infty e^{-\lambda t} T_t f(x) dt \right|$  a.e., and thus

$$f^*(x) \leq \liminf_i f_i^*(x) \quad \text{a.e.}$$

Let  $e_i^*(a) = \{x; f_i^*(x) > a\}$ . It is clear that  $E^*(a) \subset \liminf_i e_i^*(a)$ , and hence Fatou's lemma and the above lemma imply that

$$\int_{E^*(a)} (a - |f^{a-}|) d\mu \leq \liminf_i \int_{e_i^*(a)} (a - |f^{a-}|) d\mu \leq \int |f^{a+}| d\mu,$$

and the theorem is proved.

**3. Applications.** It is known (cf. [3]) that (i) if  $1 < p < \infty$  and  $f \in L_p(X)$ , then the function  $*f$  defined by

$$*f(x) = \sup_{0 < b < \infty} \left| \frac{1}{b} \int_0^b T_t f(x) dt \right|$$

is in  $L_p(X)$  and  $\|*f\|_p \leq p/(p-1)\|f\|_p$ ; (ii) for every  $f \in L_p(X)$  with  $1 \leq p < \infty$ , the limit

$$\lim_{b \uparrow \infty} \frac{1}{b} \int_0^b T_t f(x) dt$$

exists and is finite a.e. In this section we shall prove the exact analogues for Abel means.

**THEOREM 2.** *If  $1 \leq p < \infty$  and  $f \in L_p(X)$ , then  $f^* < \infty$  a.e. In particular if  $1 < p < \infty$ , then  $f^*$  is in  $L_p(X)$  and*

$$\|f^*\|_p \leq \frac{p}{p-1} \|f\|_p.$$

*Proof.* It follows easily from Theorem 1 that for any  $a > 0$ ,

$$\mu(E^*(a)) \leq \frac{1}{a} \int_{E^*(a)} |f| d\mu < \infty,$$

from which we observe that  $f^* < \infty$  a.e. The second half of the theorem follows from Theorem 2.2.3 of [6]. The proof is complete.

**THEOREM 3.** *For any  $f \in L_p(X)$  with  $1 \leq p < \infty$ , the limit*

$$(5) \quad \lim_{\lambda \downarrow 0} \lambda R_\lambda f(x)$$

*exists and is finite a.e.*

Before the proof we note that if the semi-group  $\Gamma = \{T_t; t \geq 0\}$  is sub-Markovian (for definition, see [5]) and of type  $C_1$ , then the above theorem has been proved by Edwards [5].

*Proof.* For  $1 < p < \infty$ ,  $L_p(X)$  is reflexive and thus it follows from Corollary VIII.7.2 of [4] that the functions  $f$  of the form

$$f = h + \sum_{i=1}^n (I - T_{t_i})g_i,$$

where  $T_t h = h$  for all  $t \geq 0$ , is dense in  $L_p(X)$  in the norm topology. Since

$$\begin{aligned} \lambda \int_0^\infty e^{-\lambda t} T_t (I - T_{t_i})g_i(x) dt &= \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T_t g_i(x) dt \\ &+ \lambda(1 - e^{\lambda t_i}) \int_0^\infty e^{-\lambda t} T_t g_i(x) dt \text{ a.e.} \end{aligned}$$

for each  $i$ , and

$$\lim_{\lambda \downarrow 0} \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T_t f(x) dt = 0 \text{ a.e.}$$

for each  $i$ , it follows from Theorem 2 that

$$\lim_{\lambda \downarrow 0} \lambda \int_0^{\infty} e^{-\lambda t} T_t (I - T_{t_i}) g_i(x) dt = 0 \text{ a.e.}$$

for each  $i$ . Thus we observe that the limit (5) exists and is finite a.e. for any function  $f$  in a dense subset of  $L_p(X)$  in the norm topology. Hence the Banach convergence theorem [3] and Theorem 2 imply that the limit (5) exists and is finite a.e. for any  $f \in L_p(X)$ . Since  $L_p(X) \cap L_1(X)$  is dense in  $L_1(X)$  in the norm topology, the Banach theorem and Theorem 2 are also sufficient to prove that the limit (5) exists and is finite a.e. for any  $f \in L_1(X)$ . This completes the proof of Theorem 3.

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