CYCLIC MULTIPLICATION OPERATORS ON $L_p$-SPACES

H. A. SEID
 CYCLIC MULTIPLICATION OPERATORS
ON $L_p$-SPACES

H. A. SEID

Let $(X, \Sigma, \mu)$ be a measure space. Suppose $f$ is in $L_\infty(X, \Sigma, \mu)$. The operator $M_f$ on $L_p(X, \Sigma, \mu)$ defined by $M_f(g) = f \cdot g$, for $g$ in $L_p(X, \Sigma, \mu)$ is called a multiplication operator. The purpose of this paper is to characterize cyclic multiplication operators and to relate their structure to the properties of the measure space on which the underlying $L_p$-space is defined.

In $L_2(X, \Sigma, \mu)$ any maximal abelian self-adjoint algebra of bounded operators may be transformed isometrically to the algebra of all multiplication operators on some $L_2$-space (see, e.g. [9]). This is due to the fact that among the $L_p$-spaces, only $L_2$ possesses a sufficiently rich collection of orthogonal projections. In fact, if $p \neq 2$, the only “orthogonal” projections on $L_p$ are multiplications by characteristic functions (shown by Sullivan [10] for real $L_p$). As a consequence, isometries between $L_p$-spaces are related to $\sigma$-isomorphisms between the underlying measure spaces when $p \neq 2$. These relationships may be exploited to characterize certain multiplication operators on $L_p$-spaces where $1 \leq p < \infty$.

In §1, we present Sullivan’s theorem along with applications to direct sum decompositions of $L_p$-spaces and to surjective isometries between $L_p$-spaces.

Section 2 deals with the characterization of the operator $M_z$ on $L_p(\nu)$, where $\nu$ is a finite Borel measure with compact support in the plane, defined by

$$M_z f(z) = z f(z)$$

for $f$ in $L_p(\nu)$.

In §3 the concept of a normal measure space (introduced by Halmos and von Neumann [3]) is used to relate the structure of certain measure spaces $(X, \Sigma, \mu)$ to the structure of cyclic multiplication operators on $L_p(X, \Sigma, \mu)$.

We mention that throughout, all measure spaces are assumed to be $\sigma$-finite. For notational ease, we denote $L_p(X, \Sigma, \mu)$ by $L_p(\mu)$ when no confusion arises. The algebra of all multiplication operators on $L_p(\mu)$ is denoted by $\mathcal{M}_\mu$. Also if $f$ is a measurable function on $(X, \Sigma)$, then $\text{supp}(f) \equiv \{x \in X \mid |f(x)| > 0\}$ is called the support of $f$.

1. Structural and isometric properties of $L_p$-spaces.

**Definition 1.1.** A closed subspace $R$ of $L_p(\mu)$ is a $p$-direct
summand of $L_p(\mu)$ if there exists a closed subspace $S$ of $L_p(\mu)$ such that $L_p(\mu) = R \oplus S$ algebraically and if $r \in R$ and $s \in S$, then $\|r + s\|_p^p = \|r\|_p^p + \|s\|_p^p$. In this event we write $L_p(\mu) = R \oplus S$.

We recall a relation due to Hanner ([4], Theorem 1, p. 239). If $f$ and $g$ are in $L_p(X, \Sigma, \mu)$, $p \neq 2$, then

$$\|f + g\|_p^p + \|f - g\|_p^p = 2(\|f\|_p^p + \|g\|_p^p)$$

if and only if $f \cdot g = 0$ a.e.

THEOREM 1.1 (Sullivan). A closed subspace $R$ of $L_p(X, \Sigma, \mu)$ is a $p$-direct summand of $L_p(\mu)$ if and only if $R = M_{X(\sigma)}(L_p(\mu))$ for some $\sigma \in \Sigma$.

**Proof.** If $L_p(\mu) = R \oplus_p S$, then for $r \in R$ and $s \in S$, we have $\|r \pm s\|_p^p = \|r\|_p^p + \|s\|_p^p$. By Equation 1.1, it follows that $r \cdot s = 0$ a.e. There exists $f_0 \in L_p(\mu)$ such that $\supp(f_0) = X$ a.e. Let $f_0 = r_0 + s_0$ where $r_0 \in R$ and $s_0 \in S$. Then there exists $\sigma_0 \in \Sigma$ such that $r_0 = f_0\chi(\sigma_0)$ a.e. and $s_0 = f_0\chi(X\setminus\sigma_0)$ a.e. It follows that if $r \in R$ and $s \in S$, then we have $r = r\chi(\sigma_0)$ a.e. and $s = s\chi(X\setminus\sigma_0)$ a.e. We conclude that $R = M_{X(\sigma)}(L_p(\mu))$. The converse is immediate.

Definition 1.1 extends in a natural way to any collection of closed subspaces $\{L_\alpha\}_{\alpha \in A}$ of $L_p(\mu)$ and we say that $L_p(\mu)$ is the $p$-direct sum of $\{L_\alpha\}_{\alpha \in A}$ (written $\bigoplus_{\alpha \in A} L_\alpha$) if $L_p(\mu)$ is algebraically and $\|\sum_{\alpha \in A} f_\alpha\|_p^p = \sum_{\alpha \in A} \|f_\alpha\|_p^p$, where $f_\alpha \in L_\alpha$. Clearly in this event, each $L_\alpha$ is a $p$-direct summand and for each sum $\sum_{\alpha \in A} f_\alpha$ at most a countable number of summands are nonzero.

COROLLARY 1.1. If $\{L_\alpha\}_{\alpha \in A}$ is a collection of nontrivial closed subspaces of $L_p(\mu)$, $p \neq 2$, such that $L_p(\mu) = \bigoplus_{\alpha \in A} L_\alpha$, then card $(A) \leq \aleph_0$.

For a measure space $(X, \Sigma, \mu)$ we set $\Sigma'$ equal to $\Sigma/N_\mu$ where $N_\mu$ is the collection of all sets of measure zero in $\Sigma$. Then $[\sigma] \in \Sigma'$ will be the class of all sets $\tau$ in $\Sigma$ such that $\tau \Delta \sigma = \emptyset$ a.e.

For $[\sigma]$ and $[\tau]$ in $\Sigma'$, let $[\sigma] \cup [\tau] \equiv [\sigma \cup \tau]$, $[\sigma] \cap [\tau] \equiv [\sigma \cap \tau]$, and $[\sigma] \setminus [\tau] \equiv [\sigma \setminus \tau] = [\sigma] \cap [X \setminus \tau]$. These operations are all well defined. We say that $[\tau] \subset [\sigma]$ if $\tau \subset \sigma$ a.e.

Let $(X, \Sigma, \mu)$ and $(Y, \Phi, \nu)$ be measure spaces.

**DEFINITION 1.2.** A map $\Gamma: \Sigma' \to \Phi'$ is $\sigma$-isomorphism if

(i) $\Gamma$ is a bijection;

(ii) $\Gamma([\sigma] \cap [\tau]) = \Gamma([\sigma]) \cap \Gamma([\tau])$ for $[\sigma]$ and $[\tau]$ in $\Sigma'$;

(iii) if $\{[\sigma_i]\}_{i=1}^\infty$ is a sequence of elements of $\Sigma'$, then

$$\Gamma\left(\bigcup_{i=1}^\infty [\sigma_i]\right) = \bigcup_{i=1}^\infty \Gamma([\sigma_i]).$$
It is convenient to regard $\Gamma$ as a map from $\Sigma$ to $\Phi$ by merely setting $\Gamma(\sigma) = \varphi$ where $\sigma \in [\sigma]$ and $\varphi$ is a fixed representative of $\Gamma([\sigma])$. Then this mapping is a $\sigma$-isomorphism if we identify sets equal almost everywhere.

**Definition 1.3.** A surjective isometry $J: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$ induces a $\sigma$-isomorphism $\Gamma': \Sigma' \to \Phi'$ if $[\text{supp}(J(f))] = \Gamma([\sigma])$ whenever $f$ is in $L_p(\mu)$ and $\text{supp}(f) = \sigma$ a.e. $\mu$.

We now give a generalization of part of a theorem due to Lamperti ([7], Theorem 3.1, p. 361).

**Theorem 1.2.** Let $J: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$, $p \neq 2$, be a surjective isometry. Then $J$ induces a $\sigma$-isomorphism $\Gamma': \Sigma' \to \Phi'$ and $\Gamma'$ preserves atoms.

**Proof.** Since $J$ is a surjective isometry, it preserves $p$-direct summands.

Define $\Gamma: \Sigma \to \Phi$ by $\Gamma(\sigma) = \varphi$ where $J(M_{\chi(\sigma)}(L_p(\mu))) = M_{\chi(\varphi)}(L_p(\nu))$. Then $\Gamma$ is a $\sigma$-isomorphism and hence $\Gamma$ preserves atoms.

Suppose $f \in L_p(\mu)$ and $\text{supp}(f) = \sigma$ a.e. Clearly $\text{supp}(J(f)) \subseteq \Gamma(\sigma)$ a.e. $\nu$. If $\text{supp}(J(f)) \neq \Gamma(\sigma)$ a.e., then there exists $h \neq 0$ in $M_{\chi(\sigma)}(L_p(\nu))$ such that $h \cdot J(f) = 0$ a.e. $\nu$. By equation 1.1 it follows that $J^{-1}(h) \cdot f = 0$ a.e. $\mu$ and $J^{-1}(h) \in M_{\chi(\sigma)}(L_p(\mu))$. This is contradiction.

**Remark 1.1.** Suppose $(X, \Sigma, \mu)$ and $(\Sigma, \Phi, \nu)$ are such that there exists a $\sigma$-isomorphism $\Gamma: \Sigma' \to \Phi'$. The measure $\omega$ defined by $\omega(\varphi) = \mu(\Gamma^{-1}(\varphi))$ for $\varphi \in \Phi$ is equivalent to $\nu$ and $h = d\omega/d\nu$. Define $J_1: L_p(\omega) \to L_p(\nu)$ by $J_1(f) = h^{1/p} \cdot f$. Let $J_2: L_p(\mu) \to L_p(\omega)$ be defined on characteristic functions by $J_2(\chi(\sigma)) = \chi(\Gamma(\sigma))$. Extend $J_2$ by linearity and continuity to all of $L_p(\mu)$. The map $J_\Gamma = J_1 J_2$ is called the canonical surjective isometry inducing $\Gamma'.

**Definition 1.4.** A bounded operator $T$ on $L_p(X, \Sigma, \mu)$ corresponds to a bounded operator $U$ on $L_p(Y, \Phi, \nu)$ if there exists a surjective isometry $J: L_p(\mu) \to L_p(\nu)$ such that $T = J^{-1} U J$.

**Theorem 1.3.** If $J: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$ is a surjective isometry inducing a $\sigma$-isomorphism $\Gamma: \Sigma' \to \Phi'$, then the algebra of multiplication operators $\mathcal{M}_\mu$ on $L_p(\mu)$ corresponds bijectively to the algebra of multiplication operators $\mathcal{M}_\nu$ on $L_p(\nu)$.

**Proof.** Since $J$ induces $\Gamma$, if $\sigma \in \Sigma$, then $M_{\chi(\sigma)} = J^{-1} M_{\chi(\Gamma(\sigma))} J$. Since $J$ is linear, if $s \in L_p(\mu)$ is a simple function, say $s = \sum_{i=1}^n \tau_i \chi(\sigma_i)$, then $M_s = J^{-1} M_s J$ where $r = \sum_{i=1}^n \tau_i \chi(\Gamma(\sigma_i))$ and $\|M_s\| = \|M_r\|$. But the multiplications by simple functions form dense subsets in both $\mathcal{M}_\mu$...
and $\mathcal{M}_\mu$. By continuity, it follows that $\mathcal{M}_\mu$ corresponds bijectively and isometrically to $\mathcal{M}_\nu$ under $J$.

**Definition 1.5.** An algebra $\mathcal{A}$ of bounded operators on $L_p(X, \Sigma, \mu)$ is maximal abelian if

(i) $\mathcal{A}$ is commutative;

(ii) if $T$ is a bounded operator which commutes with all the elements of $\mathcal{A}$, then $T$ is in $\mathcal{A}$.

**Theorem 1.4.** A bounded operator $T$ on $L_p(\mu), p \neq 2$, is in $\mathcal{M}_\mu$ if and only if each $p$-direct summand of $L_p(\mu)$ is invariant for $T$.

**Proof.** Let $R = M_{X(\sigma)}(L_p(\mu))$ be an arbitrary $p$-direct summand of $L_p(\mu)$. Suppose $T$ is in $\mathcal{M}_\mu$. Then we have $TM_{X(\sigma)}(L_p(\mu)) = M_{X(\sigma)}TM_{X(\sigma)}(L_p(\mu))$, i.e., $TM_{X(\sigma)} = M_{X(\sigma)}TM_{X(\sigma)}$ and thus $R$ is invariant for $T$.

Conversely suppose that each $R = M_{X(\sigma)}(L_p(\mu))$ is invariant for $T$. Let $S = (I - M_{X(\sigma)})(L_p(\mu)) = M_{X(\sigma)}(L_p(\mu))$. Then

\[(1.2) \quad M_{X(\sigma)}TM_{X(\sigma)} = TM_{X(\sigma)}\]

and

\[(1.3) \quad (I - M_{X(\sigma)})T(I - M_{X(\sigma)}) = T(I - M_{X(\sigma)}).\]

Equations (1.2) and (1.3) imply that $TM_{X(\sigma)} = M_{X(\sigma)}T$. Thus $T$ commutes with $\mathcal{P} = \{M_{X(\sigma)} \mid \sigma \in \Sigma\}$. Since $\mathcal{P}$ generates $\mathcal{M}_\mu$ which is maximal abelian, we conclude that $T \in \mathcal{M}_\mu$.

**Theorem 1.5.** Let $J_i : L_p(X, \Sigma, \mu) \rightarrow L_p(Y, \Phi, \nu)$, $i = 1, 2$, be two surjective isometries which induce the same $\sigma$-isomorphism $\Gamma : \Sigma \rightarrow \Phi$. Then there exists $f \in L_p(\nu)$ such that $|f| = 1$ a.e. $\nu$ and $J_2 = M_fJ_1$.

If in addition $p \neq 2$, then the converse is also true.

**Proof.** First assume $(X, \Sigma, \mu)$ is a finite measure space. Let $J_1(\chi(X)) = g$ and $J_2(\chi(X)) = h$. For $\sigma \in \Sigma$, it follows that $J_1(\chi(\sigma)) = g\chi(\Gamma(\sigma))$ a.e. $\nu$ and $J_2(\chi(\sigma)) = h\chi(\Gamma(\sigma))$ a.e. $\nu$. We have \[\int_{\phi} |g|^p d\nu = \int_{\phi} |h|^p d\nu\] for all $\phi \in \Phi$. Thus $|g| = |h|$ a.e. $\nu$. Let $f = h/g$. Since $\text{supp}(g) = \text{supp}(h) = Y$ a.e. $\nu$, we have $|f| = 1$ a.e. $\nu$, and if $s$ is a simple function in $L_p(\mu)$, then $J_2(s) = M_fJ_1(s)$. By continuity, we conclude that $J_2 = M_fJ_1$.

In the standard way the theorem holds when $(X, \Sigma, \mu)$ is $\sigma$-finite.

If $p \neq 2$, then employing Theorem 1.2, the converse follows immediately.
2. Cyclicity and extended cyclicity.

**Definition 2.1.** A bounded operator $T$ on $L_p(X, \Sigma, \mu)$ is cyclic if there exists a function $f_0 \in L_p(\mu)$ such that \{\(p(T)(f_0) \mid p \text{ is a polynomial in the variable } z\)\} is norm-dense in $L_p(\mu)$. The function $f_0$ is called a cyclic function.

**Definition 2.2.** A multiplication operator $M_f$ on $L_\infty(\mu)$ is extended cyclic if there exists a function $g_0 \in L_\infty(\mu)$ such that \{\(p(M_f, M_f)(g_0) \mid p(z, \bar{z}) \text{ is a polynomial in the variables } z \text{ and } \bar{z}\)\} is norm-dense in $L_p(\mu)$. The function $g_0$ is called an extended cyclic function. (Note that Definition 2.2 is merely a special case of the usual definition of the cyclicity of a normal operator on $L_2$.)

Throughout the sequel, the triple $(S, \mathscr{B}(S), \nu)$ shall be a finite measure space where $S$ is a compact subset of $C$ with Borel sets $\mathscr{B}(S)$ and $\nu$ is a finite measure (hence regular) on $\mathscr{B}(S)$.

**Remark 2.1.** Let $z : S \to S$ be the identity function. The Stone-Weierstrass theorem implies that $M_z$ is extended cyclic on $L_\infty(S, \mathscr{B}(S), \nu)$ with extended cyclic function $\chi(S)$.

The essential range of a function $f \in L_\infty(X, \Sigma, \mu)$ (written $S_f$) is defined in the usual way. It is easy to show that if $M_f$ is a multiplication operator on $L_p(X, \Sigma, \mu)$, then $S_f$ is the spectrum of $M_f$. Also, since $C$ is a Lindelöf space, there exists $X_0 \in \Sigma$, with $\mu(X_0) = 0$, such that $f(X \setminus X_0) \subset S_f$.

**Remark 2.2.** Let $(X, \Sigma, \mu)$ and $(Y, \Phi, \nu)$ be measure spaces. Let $\Gamma : \Sigma' \to \Phi'$ be a $\sigma$-isomorphism. If there is a set $Y_0 \in \Phi$ with $\nu(Y_0) = 0$ and a measurable point mapping $\theta : Y \setminus Y_0 \to X$ such that $[\theta^{-1}(\sigma)] = \Gamma'([\sigma])$ for $\sigma \in \Sigma$, then $\theta^{-1}$ induces $\Gamma$. In this event, $J_\Gamma : L_p(\mu) \to L_p(\nu)$ has the form $J_\Gamma(g)(y) = h|^{1/p}(y)(g \circ \theta)(y) \text{ a.e. } \nu$ on $Y \setminus Y_0$ for $g \in L_p(\nu)$ and $h$ as in Remark 1.1.

**Theorem 2.1.** An operator $M_f$ of $L_p(X, \Sigma, \mu)$, $p \neq 2$, corresponds to $M_z$ on $L_p(S, \mathscr{B}(S), \nu)$ if and only if $f^{-1}$ induces a $\sigma$-isomorphism $\Gamma : \mathscr{B}'(S) \to \Sigma'$.

**Proof.** Suppose there exists a surjective isometry $J : L_p(\nu) \to L_p(\mu)$ such that $M_fJ = JM_z$ where $J$ induces a $\sigma$-isomorphism $\Gamma : \mathscr{B}'(S) \to \Sigma'$. There exists $k \in L_\infty(\mu)$ such that $|k| = 1 \text{ a.e. } \mu$ and $J = M_kJ_\Gamma$. Also, there exists $X_0 \in \Sigma$, with $\mu(X_0) = 0$, and a measurable point mapping $\theta : X \setminus X_0 \to S$ such that $\theta^{-1}$ induces $\Gamma$, (see, e.g., [8] Corollary 11, p. 272). By constructing $J_\Gamma$ as in Remark 2.2 and noting that $\chi(S) \in L_\infty(\nu)$ we find that $M_fJ(\chi(S)) = JM_z(\chi(S))$ implies that $f = \theta$ a.e. $\mu$ on $X \setminus X_0$. 
Conversely if \( f^{-1} \) induces that \( \sigma \)-isomorphism \( I' \), it follows directly that if we construct \( J_r \) as in Remark 2.2 then \( J_r M_z = M_r J_r \).

**Theorem 2.2.** Let \( \mu \) and \( \nu \) be two \( \sigma \)-finite measures on the Borel sets of \( S \), a compact subset of \( C \). Then \( M_z \) on \( L_p(\mu) \) corresponds to \( M_z \) on \( L_p(\nu) \) if and only if \( \mu \) is equivalent to \( \nu \).

**Proof.** If \( p = 2 \), this result is known and follows from the uniqueness of resolutions of the identity (see, e.g. [2], Theorem 1, p. 65).

Suppose \( p \neq 2 \) and \( M_z \) on \( L_p(\mu) \) corresponds to \( M_z \) on \( L_p(\nu) \) under a surjective isometry \( J : L_p(\nu) \to L_p(\mu) \), inducing a \( \sigma \)-isomorphism \( I' : \mathcal{B}(S)/N_\nu \to \mathcal{B}(S)/N_\mu \). Then by Theorem 2.1, \( \mu \) and \( \nu \) are equivalent.

Conversely if \( \mu \) and \( \nu \) are equivalent, then the identity mapping \( z : S \to S \) induces the identity \( \sigma \)-isomorphism \( I : \mathcal{B}(S)/N_\nu \to \mathcal{B}(S)/N_\mu \). Constructing \( J : L_p(\mu) \to L_p(\nu) \) as in Remark 2.2, we find that \( M_z J_r = J_r M_z \).

**Lemma 2.1.** An operator \( M_f \) on \( L_p(X, \Sigma, \mu) \) corresponds to \( M_z \) on \( L_p(S, \mathcal{B}(S), \nu) \) if and only if \( M_f \) is extended cyclic.

**Proof.** If \( p = 2 \), this is a special case of a well known theorem for cyclic normal operators ([2], Theorem 1, p. 95).

If \( p \neq 2 \) and if \( M_f \) corresponds to \( M_z \), then there exists a surjective isometry \( J : L_p(\nu) \to L_p(\mu) \) such that \( M_f J = J M_z \) and \( J \) induces a \( \sigma \)-isomorphism \( I' : \mathcal{B}'(S) \to \Sigma' \). Let \( \{s_n\} \) be a sequence of simple functions in \( L_\infty(\mu) \) such that \( s_n \to f \) as \( n \to \infty \) in \( L_\infty(\mu) \). Then \( \{M_z s_n\} \) converges to \( M_f \) in \( M_\mu \) and \( \{J^{-1} M_z s_n J\} \) converges to \( M_z \). Therefore, the sequence \( \{J^{-1} M_z s_n J\} \) converges to \( M_z \), i.e., \( M_z J = J M_z \).

If \( g \) is a polynomial in the variables \( z \) and \( \bar{z} \), then \( J^{-1} q(M_f, M_z) J \) equals \( q(M_z, M_z) \). Let \( g = J(\mathcal{G}(S)) \). Then it follows that

\[
J q(M_z, M_z)(J^{-1}(g)) = q(M_f, M_f)(g) .
\]

Thus \( M_f \) is extended cyclic with extended cyclic function \( g \).

Conversely let \( A = \{p(M_f, M_z) = M_p(z, \bar{z}) \mid p(z, \bar{z}) \text{ is a polynomial in } z \text{ and } \bar{z} \} \). Let \( \mathcal{F} \) be the operator norm closure of \( A \). Then it follows that under the involution induced by complex-conjugation, \( \mathcal{F} \) is isometrically \( * \)-isomorphic to \( C(S_f) \), the continuous functions on \( S_f \), under an algebra isomorphism \( \tau : \mathcal{F} \to C(S_f) \) such that \( \tau(I) = 1 \), \( \tau(M_f) = z \), and \( \tau(M_z) = \bar{z} \) where \( z \) is the identity mapping on \( S_f \). Let \( g \in L_p(\mu) \) be an extended cyclic function for \( M_f \). Define a linear functional \( L \) on \( \mathcal{P} = \{q(z, \bar{z}) \mid q \text{ is a polynomial in the variables } z \text{ and } \bar{z} \} \) by

\[
L(q) = q(M_f, M_f) .
\]
L(q) = \int q(M_f, M_\gamma)(g) | g |^{-1} \text{sgn}(g) \, d\mu = \int q(f, \bar{f}) | g |^p \, d\mu.

It follows from Hölder's inequality that the integral exists and \(|L(q)| \leq \|q\|_\infty \|g\|_p^p\). We extend \(L\) to all of \(C(S_f)\) by continuity and observe that \(L\) is a positive linear functional. There exists a regular positive Borel measure \(\nu \in C^*(S_f)\) such that \(L(h) = \int hd\nu\) for \(h \in C(S_f)\). Since \(|h|^p \in C(S_f)\), it follows that

\[
L(|h|^p) = \int |h|^p \, d\nu = \|h\|_p^p = \int |h(f) \cdot g|^p \, d\mu = \|h(f) \cdot g\|_p^p.
\]

Let \(W = \{q(M_f, M_\gamma)(g) \mid q(M_f, M_\gamma) \in A\}\). Then \(L_p(\mu)\) is the \(p\)-norm closure of \(W\). Define \(J: W \to L_p(\nu)\) by \(J(q(M_f, M_\gamma)(g)) = q(z, \bar{z}) \in \mathcal{P}\). \(J\) is an isometry on \(W\). Extend \(J\) by continuity to a surjective isometry on all of \(L_p(\mu)\). We conclude that \(M_z J = JM_z\) since this is true on \(W\).

The proof of this lemma follows that employed by Kalisch [6] in his proof of the above mentioned \(p = 2\) cyclicity theorem of Halmos.

**Remark 2.3.** By standard methods, one can show that a function \(g \in L_p(\mu)\) is an extended cyclic function for an extended cyclic multiplication operator \(M_z\) on \(L_p(\mu)\) if and only if \(|g| > 0\) a.e.

Bram ([1], Theorem 6, p. 85) has shown that for \(p = 2\), \(M_z\) on \(L_2(S, \mathcal{B}(S), \nu)\) is cyclic in the sense of Definition 2.1. An examination of the proof of this result shows that it is not dependent on the properties of Hilbert space. By use of Remark 2.1, Lemma 2.1 and an obvious modification of Bram's proof, one obtains:

**Theorem 2.3.** A multiplication operator on \(L_p(\mu)\) is extended cyclic if and only if it is cyclic. (However, in general a cyclic multiplication operator has a larger collection of extended cyclic functions than cyclic functions.)

We now obtain immediately:

**Theorem 2.4.** A bounded operator \(T\) on \(L_p(X, \Sigma, \mu), p \neq 2\) corresponds to \(M_z\) on \(L_p(S, \mathcal{B}(S), \nu)\) if and only if each \(p\)-direct summand of \(L_p(\mu)\) is invariant for \(T\) and \(T\) is cyclic.

**Theorem 2.5.** An operator \(M_f \in \mathcal{M}_\mu\) is cyclic on \(L_p(\mu), p \neq 2\), if and only if \(M_f\) is cyclic on \(L_2(\mu)\).

**Proof.** We may assume that \((X, \Sigma, \mu)\) is a finite measure space.
with $\mu(X) = 1$. Then if $1 \leq p < p' < \infty$, we have $L_p'(\mu)$ is $p$-norm-dense in $L_p(\mu)$ and if $f \in L_p'(\mu)$, then $\| f \|_p \leq \| f \|_{p'}$.

Suppose $M_f$ is cyclic on $L_p(\mu)$, with cyclic function $g$ and $p > 2$. The set $\mathcal{P} = \{ q(M_f) | q \text{ is a polynomial in } z \}$ is norm-dense in $L_p(\mu)$. Thus it is norm-dense in $L_2(\mu)$.

If $p < 2$, there exists a surjective isometry $J : L_p(\mu) \rightarrow L_2(S, \mathcal{B}(S), \nu)$ such that $M_f J = J M_f$. Thus the dual mapping $J^* : L_2^*(\nu) \rightarrow L_2^*(\mu)$ is such that $J^* M_f^* = M_{f^*} J^*$. But $M_f^* = M_f$ on $L_2^*(\nu)$ and $M_f^* = M_f$ on $L_2^*(\mu)$. So $M_f$ is cyclic on $L_2(\mu)$ and hence on $L_2(\mu)$.

The converse is proved similarly.

From Theorems 2.2 and 2.5 we conclude immediately that Theorem 2.1 holds for $p = 2$.

3. Cyclicity and univalence. Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space.

**Definition 3.1.** A subset $A$ of $\Sigma$ is called a $\sigma$-algebra contained in $\Sigma$ (written $A \subset \Sigma$) if $A$ is itself a $\sigma$-algebra.

**Theorem 3.1.** Let $f \in L_\infty(\mu)$. There exists a $\sigma$-algebra $\Lambda_f \subset \Sigma$ depending on $f$ such that $f$ is in $L_\infty(X, \Lambda_f, \mu |_{\Lambda_f})$ and $M_f$ is cyclic on $L_p(X, \Lambda_f, \mu |_{\Lambda_f})$.

**Proof.** Consider $\Lambda_f = \{ f^{-1}(\beta) | \beta \in \mathcal{B}(C) \}$. Then $\Lambda \subset \Sigma$ is a $\sigma$-algebra. Restrict $\mu$ to $\Lambda_f$. There exists $\beta_0 \in \Lambda_f$ such that $\mu(\beta_0) = 0$ and $f |_{x, \beta_0} \subset S_f$. Define $\nu$ on $\mathcal{B}(S_f)$ by $\nu(\gamma) = \mu(f^{-1}(\gamma))$. Let $\nu'$ be a finite measure equivalent to $\nu$. Then we see that $f^{-1}$ induces a $\sigma$-isomorphism $I' : \mathcal{B}(S_f)/N_{\nu'} \rightarrow \Lambda_f$. Hence $M_f$ on $L_p(X, \Lambda_f, \mu |_{\Lambda_f})$ corresponds to $M_z$ on $L_2(S_f, \mathcal{B}(S_f), \nu')$ and thus $M_f$ is cyclic.

**Definition 3.2.** A measurable function $f$ is essentially univalent if it is univalent on the complement of some set of measure zero.

We observe that $M_z$ is cyclic on $L_2(S, \mathcal{B}(S), \nu)$ and $z$ is a univalent function. It is reasonable to ask whether all cyclic multiplication operators on $L_2(X, \Sigma, \mu)$ arise from essentially univalent $L_\infty$-functions and conversely. The answer in general is negative.

**Example 3.1.** Consider $([0, 1], \mathcal{B}([0, 1]), \lambda)$, the usual Borel measure space on $[0, 1]$. Let $f$ be defined by

$$f(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2} \\ t - \frac{1}{2}, & \frac{1}{2} < t \leq 1. \end{cases}$$
Then $f$ is in $L^2(\lambda)$ and $S_f = [0, 1/2]$. There exists $A_f \subset \mathcal{B}([0, 1])$ such that $M_f$ is cyclic on $L_2([0, 1], A_f, \lambda \mid_{A_f})$, but $f$ is not essentially univalent on $([0, 1], A_f, \lambda \mid_{A_f})$.

We shall show that essential univalence of $f \in L^\infty(\mu)$ need not imply that $M_f$ is cyclic on $L^\infty(\mu)$ after some preliminaries (see Example 3.2 ff.). In addition we will determine when essential univalence of an $L^\infty$-function $f$ is equivalent to the cyclicity of $M_f$ on $L^\infty$.

**Definition 3.3.** A $\sigma$-finite measure space $(X, \Sigma, \mu)$ is called proper if:

(i) it is complete and nonatomic;

(ii) there exists $\Lambda \subset \Sigma$ such that $(X, \Sigma, \mu)$ is properly separable with respect to $\Lambda$;

(iii) $\Lambda$ has a separating sequence.

The $\sigma$-algebra $\Lambda$ is called the Borel sets of $\Sigma$ and a $\Lambda$-measurable function is called a Baire function. All italicized terms are defined as in [3]. We denote a proper measure space by $(X, \Sigma, \Lambda, \mu)$.

**Definition 3.4.** A proper measure space $(X, \Sigma, \Lambda, \mu)$ is normal (c-normal) if to each real-valued (complex-valued) univalent Baire function $f$, there corresponds a set $X_0$ in $\Sigma$ depending on $f$ such that $\mu(X_0) = 0$ and such that $f(X \setminus X_0)$ is a Borel subset of $\mathcal{B}(of C)$.

**Remark 3.1.** By duplicating the proofs of Lemmas 1-4 which Halmos and von Neumann proved for real-valued functions on proper and normal measure spaces ([3], pp. 337-339), we obtain:

**Theorem 3.2.** A proper measure space $(X, \Sigma, \Lambda, \mu)$ is normal if and only if it is c-normal.

Let $(X, \Sigma, \mu)$ and $(Y, \Phi, \nu)$ be measure spaces.

**Definition 3.5.** A bijective mapping $\theta: X \setminus X_0 \to Y \setminus Y_0$, where $\mu(X_0) = \nu(Y_0) = 0$, is a point isomorphism if $\theta$ and $\theta^{-1}$ are measurable and $\theta$ induces a $\sigma$-isomorphism $\Gamma: \Sigma' \to \Phi'$. In this event we say that $(X, \Sigma, \mu)$ and $(Y, \Phi, \nu)$ are point isomorphic. If in addition we have $\mu(\sigma) = \nu(\Gamma(\sigma))$ for $\sigma \in \Sigma$, then $\theta$ and $\Gamma$ are said to be measure preserving.

Let $(X, \Sigma, \mu)$ be a measure space. Then we shall denote by $(X, \tilde{\Sigma}, \tilde{\mu})$ the measure space where $\tilde{\Sigma}$ is the completion of $\Sigma$ and $\tilde{\mu}$ is the completion of $\mu$.

Throughout the remainder, the usual Borel measure space $([0, 1], \mathcal{B}([0, 1]), \lambda)$ shall be denoted by $[0, 1]$ and the usual (normal) Lebesgue measure space $([0, 1], \mathcal{L}([0, 1]), \mathcal{B}([0, 1]), \lambda)$ will be denoted by $[0, 1]^\sim$. 
REMARK 3.2. Halmos and von Neumann have proved the following:

**THEOREM.** A proper measure space \((X, \Sigma, \Lambda, \mu)\) with \(\mu(X) = 1\) is normal if and only if it is measure preserving point isomorphic to \([0, 1]^-\). (See [3], Theorem 2, p. 339.)

It follows from this and the fact that all measure preserving set automorphisms on \([0, 1]^-\) are induced by measure preserving point isomorphisms ([3], p. 340) that if \((X, \Sigma, \Lambda, \mu)\) and \((Y, \Phi, \Psi, \nu)\) are normal measure spaces, then each \(\sigma\)-isomorphism \(\Gamma: \Sigma' \to \Phi'\) is induced by a point isomorphism \(\gamma\).

REMARK 3.3. It follows from a Theorem proved by Halmos and von Neumann ([3], Lemma 3, p. 338) that if \((X, \Sigma, \Lambda, \mu)\) and \((Y, \Phi, \Psi, \nu)\) are normal measure spaces, then any point isomorphism between them may be so constructed as to take Borel sets to Borel sets.

**DEFINITION 3.6.** A measure space \((X, \Sigma, \mu)\) is pre-normal if there exists \(\Lambda \subset \tilde{\Sigma}\) such that \((X, \tilde{\Sigma}, \Lambda, \tilde{\mu})\) is normal.

REMARK 3.4. The measure space \((S, \mathcal{B}(S), \nu)\), where \(\nu\) is a non-atomic measure on \(\mathcal{B}(S)\) with \(\text{supp}(\nu) = S\), is pre-normal and \(\mathcal{B}(S)\) serves as the Borel sets ([5], Theorem XIII, p. 304).

**LEMMA 3.1.** Let \((X, \Sigma, \mu)\) be pre-normal. Then \(f \in L_\infty(\mu)\) is essentially univalent if and only if \(M_f\) is cyclic.

**Proof.** Suppose \(f\) in \(L_\infty(\mu)\) is essentially univalent. Since \((X, \Sigma, \mu)\) is pre-normal, there exists a \(\sigma\)-algebra \(\Lambda \subset \tilde{\Sigma}\) such that \((X, \tilde{\Sigma}, \Lambda, \tilde{\mu})\) is normal; and thus clearly \(\tilde{\Lambda} = \tilde{\Sigma}\). The function \(f\) is \(\tilde{\Sigma}\)-measurable. So there exists \(X_0 \in \Lambda\) with \(\tilde{\mu}(X_0) = 0\) and \(f|_{X \times X_0}\) is \(\Lambda\)-measurable univalent, and \(\text{rge}(f|_{X \times X_0}) \subset S_f\). We conclude that there exists \(X_1 \in \Lambda\) such that \(X_1 \supset X_0\), \(\tilde{\mu}(X_1) = 0\), and \(f(\tau)\) is a Borel subset of \(S_f\) when \(\tau \in \Lambda\) and \(\tau\) is a subset of \(X \setminus X_1\) ([3], Lemma 3, p. 338).

Define a function \(\nu'\) on \(\mathcal{B}(S_f)\) with range in the extended real numbers by

\[\nu'(\beta) = \tilde{\mu}(f^{-1}(\beta) \cap (X \setminus X_1)) = \tilde{\mu}(f^{-1}(\beta)) .\]

Then \(\nu'\) is a measure on \(\mathcal{B}(S_f)\). Let \(\nu\) be a finite measure on \(\mathcal{B}(S_f)\) equivalent to \(\nu'\). We see that \(f^{-1}\) induces a \(\sigma\)-isomorphism \(\Gamma: \mathcal{B}'(S_f) \to \Lambda' = \tilde{\Sigma}'\). Thus \(M_f\) on \(L_p(X, \tilde{\Sigma}, \tilde{\mu})\) corresponds to \(M_z\) on \(L_p(S_f, \mathcal{B}(S_f), \nu)\). Hence \(M_f\) is cyclic on \(L_p(X, \Sigma, \mu)\).

Conversely suppose \(M_f\) is cyclic on \(L_p(X, \Sigma, \mu)\). Then \(M_f\) corresponds to \(M_z\) on \(L_p(S, \mathcal{B}(S), \nu)\). So \(f^{-1}\) induces a \(\sigma\)-isomorphism...
Γ: Σ → Σ'. Now Γ extends to a σ-isomorphism \( \tilde{\Gamma}: \widetilde{\mathcal{B}}'(S) \to \Sigma' \) defined by \( \tilde{\Gamma}([\eta]) = \Gamma([\beta]) \) where \( \eta = \beta \cup \tau \) for \( \beta \in \mathcal{B}(S) \) and \( \tau \) a subset of a set of \( \nu \)-measure zero. Since \( \nu \) is nonatomic, it follows that \((S, \widetilde{\mathcal{B}}(S), \mathcal{B}(S), \nu)\) is normal. Since \((X, \Sigma, \mu)\) is pre-normal, there exist \( X_0 \in \widetilde{\Sigma} \) and \( N_0 \in \mathcal{B}(S) \) with \( \tilde{\mu}(X_0) = \nu(N_0) = 0 \) and a point isomorphism \( \theta: X \setminus X_0 \to S \setminus N_0 \) such that \( \theta^{-1} \) induces \( \tilde{\Gamma} \).

Now let \( \tilde{\Sigma} \)-measurable on \( X \setminus X_0 \). There exists \( X_1 \in \Sigma \) with \( X_1 \supset X_0 \) and \( \mu(X_1) = 0 \) such that \( \theta |_{X \setminus X_1} \) is \( \Sigma \)-measurable, univalent and \( (\theta|_{X \setminus X_1})^{-1} \) induces \( \tilde{\Gamma} \) and \( \Gamma \).

There exists a surjective isometry \( J: L_p(\nu) \to L_p(\mu) \) inducing \( \Gamma \) such that \( M_\nu J = JM_\mu \). There exists \( k \in L_\infty(\mu) \) with \( |k| = 1 \) a.e. such that \( J = M_\nu J \). Thus for \( g \in L_p(S, \mathcal{B}(S), \nu) \), we have \( J(g)(x) = k(x)h(x)^{1/p}g(f(x)) = k(x)h(x)g(\theta(x)) \) a.e. \( \mu \) on \( X \setminus X_2 \) where \( X_2 \supset X_1 \), \( \mu(X_2) = 0 \) and \( \text{rge} (f |_{X \setminus X_1}) \subset S \). In particular, with \( g = z \), we conclude that \( f(x) = \theta(x) \) a.e. \( \mu \) on \( X \setminus X_2 \).

Let \((X, \Sigma, \mu)\) be a measure space.

**Definition 3.7.** Suppose \( Y \) is a (not necessarily measurable) subset of \( X \) with inner measure \( \mu_*(X \setminus Y) = 0 \). Let \( \Sigma_\gamma = \{\tau | \tau = \sigma \cap Y \text{ for some } \sigma \in \Sigma\} \). Then \( \Sigma_\gamma \) is a \( \sigma \)-algebra and the extended real-valued function \( \mu_\gamma \) on \( \Sigma_\gamma \) defined by \( \mu_\gamma(\tau) = \mu(\sigma) \), where \( \sigma \in \Sigma \) and \( \sigma \cap Y = \tau \), is a well defined measure on \( \Sigma_\gamma \). The triple \((Y, \Sigma_\gamma, \mu_\gamma)\) is called the induced measure space on \( Y \).

**Definition 3.8.** A (not necessarily measurable) subset \( Y \) of \( X \) with \( \mu_*(X \setminus Y) = 0 \) is restrictive if each essentially univalent function \( f \in L_\infty(\nu, \Sigma_\nu, \mu) \) is the restriction of an essentially univalent function \( f' \in L_\infty(X, \Sigma, \mu) \).

**Remark 3.5.** If \( Y \) is as in Definition 3.7, then the mapping \( \Gamma': \Sigma' \to \Sigma_\gamma \) defined by \( \Gamma'([\sigma]) = [\sigma \cap Y] \) is a \( \sigma \)-isomorphism. Hence the canonical mapping \( J_\gamma: L_p(\mu) \to L_p(\mu_\gamma) \) is a surjective isometry and under \( J_\gamma \) we see that if \( M_\nu \in \mathcal{M}_\nu \), then \( M_\nu |_{\Sigma_\gamma} \in \mathcal{M}_{\mu_\gamma} \) corresponds to \( M_\gamma \).

**Example 3.2.** In the measure space \([0, 1]\) it is known that there exists a non-Lebesgue measurable subset \( \sigma \) such that \( \lambda_*(\sigma) = 1 \) and \( \lambda_*(\sigma) = 0 \) (see e.g. [3], Lemma 10, p. 342). Thus we see that

\[ \lambda^*([0, 1]\setminus \sigma) = 1 \text{ and } \lambda_*(([0, 1]\setminus \sigma) = 0 . \]

Let \( \tau = [0, 1]\setminus \sigma \). The map \( \varphi: [0, 1] \to [0, 1/2] \) defined by \( \varphi(t) = t/2 \) is a homeomorphism which preserves Borel and Lebesgue measurability. Thus \( \varphi(\sigma) \) and \( \varphi(\tau) \) are non-Lebesgue measurable subsets of \(([0, 1/2], \mathcal{B}([0, 1/2]), \lambda)\) with \( \varphi(\sigma) = [0, 1/2]\setminus \varphi(\tau) \), and \( \lambda^*(\varphi(\sigma)) = \lambda^*(\varphi(\tau)) = 1/2 \).
while $\lambda_*(\mathcal{P}(\sigma)) = \lambda_*(\mathcal{P}(\tau)) = 0$. The map $\omega: [0, 1] \to [0, 1]$ defined by $\omega(t) = 1 - t$ is a homeomorphism which preserves Borel and Lebesgue measurability. Let $Y = \mathcal{P}(\sigma) \cup \omega(\mathcal{P}(\tau))$. Then we see that $Y \subset [0, 1]$ with $\lambda_*(Y) = 1$ and $\lambda_*(Y) = 0$. In addition, it follows from the construction of $Y$ that if $t \in [0, 1] \cap Y$, then $1 - t \in [0, 1] \setminus Y$, for $t \neq 1/2$.

Let

$$f(t) = \begin{cases} 
2t , & 0 \leq t \leq \frac{1}{2} \\
2 - 2t , & \frac{1}{2} \leq t \leq 1.
\end{cases}$$

Then $f(t)$ is a bounded $\mathcal{B}([0, 1])$-measurable function on $[0, 1]$ which is not essentially univalent. However, the function $f|_Y$ is bounded, univalent, and $\mathcal{B}([0, 1])_p$-measurable.

Since $[0, 1]$ is pre-normal, $M_f$ is not cyclic on $L_p([0, 1])$. Thus $M_{f|_Y}$ is not cyclic on $L_p(Y, \mathcal{B}([0, 1])_p, \lambda_\nu)$.

**Definition 3.9.** Let $(X, \Sigma, \mu)$ and $(Y, \Phi, \nu)$ be measure spaces. Then $(X, \Sigma, \mu)$ is almost point isomorphic to $(Y, \Phi, \nu)$ if there exists $X_0 \in \Sigma$ with $\mu(X_0) = 0$, and an injective measurable map $\rho: X \setminus X_0 \to Y$ such that $\rho^{-1}$ induces a $\sigma$-isomorphism $\Gamma: \Phi' \to \Sigma'$, and if $X_1 \in \Sigma$, $X_1 \supset X_0$, and $\mu(X_1) = 0$, then $\rho(X \setminus X_1)$ is a restrictive subset of $Y$. The map $\rho$ is called an $\alpha$-point isomorphism.

We observe that if $\rho$ is a point isomorphism between $(X, \Sigma, \mu)$ and $(Y, \Phi, \nu)$ then $\rho$ is also an $\alpha$-point isomorphism, because $\rho$ preserves measurable sets.

**Lemma 3.2.** Let $(X, \Sigma, \mu)$ and $(Y, \Phi, \nu)$ be measure spaces. Suppose that $f$ an essentially univalent function in $L_\infty(\mu)$ implies that $M_f$ is cyclic on $L_p(\mu)$, and suppose that $M_g$ cyclic on $L_p(\nu)$ implies that $g \in L_\infty(\nu)$ is essentially univalent. If there exists $X_0 \in \Sigma$, with $\mu(X_0) = 0$, and a measurable injective mapping $\rho: X \setminus X_0 \to Y$ such that $\rho^{-1}$ induces a $\sigma$-isomorphism $\Gamma: \Phi' \to \Sigma'$, then $\rho$ is an $\alpha$-point isomorphism.

**Proof.** Let $X_1 \in \Sigma$ be such that $X_1 \supset X_0$ and $\mu(X_1) = 0$. Let $W = \rho(X \setminus X_0)$ and let $(W, \Phi_w, \nu_w)$ be the induced measure space on $W$. We observe that $\rho|_{X \setminus X_1}$ is measurable and $(\rho|_{X \setminus X_1})^{-1}$ induces the $\sigma$-isomorphism $\Gamma_w: \Phi_w' \to \Sigma'$ defined by $\Gamma_w([\tau]) = \Gamma([\mathcal{P}])$ where $\mathcal{P} \in \Phi$ and $\tau = \mathcal{P} \cap W$.

Let $g \in L_\infty(\nu_w)$ be essentially univalent. The composition $g \circ \rho$ is a $\Sigma$-measurable function on $X \setminus X_1$.

Define the function
h(x) = \begin{cases} \ g \circ \rho(x), & x \in X \setminus X_1 \\ 0, & x \in X_1 \end{cases}

an essentially univalent function in \( L_\omega(\mu) \). Thus \( M_h \) is cyclic on \( L_p(\mu) \). It follows that \( M_h \) corresponds to \( M_z \) on \( L_p(S_h, \mathcal{B}(S_h), \omega) \). In addition \( h^{-1} \) induces a \( \sigma \)-isomorphism \( \chi: \mathcal{B}'(S_h) \to \Sigma' \). It follows that \( g^{-1} \) induces the \( \sigma \)-isomorphism \( \Gamma^{-1}_{w^{-1}}: \mathcal{B}'(S_h) \to \varphi'_{w^{-1}} \). Thus \( M_g \) is cyclic on \( L_p(\nu_w) \). There exists \( g' \in L_\omega(\nu) \) such that \( g' \mid_w = g \) a.e. \( \nu_w \) and \( M_{g'} \) corresponds to \( M_g \) as constructed in Remark 3.5. Thus \( M_{g'} \) is cyclic on \( L_p(\nu) \). It follows that \( g' \) is an essentially univalent \( L_\omega(\nu) \) function. Thus \( W \) is restrictive subset of \( Y \).

**Theorem 3.3.** Let \((X, \Sigma, \mu)\) be a separable, nonatomic measure space. The following are equivalent:

(i) a function \( f \in L_\omega(\mu) \) is essentially univalent if and only if \( M_f \) is cyclic on \( L_p(\mu) \);

(ii) \((X, \Sigma, \mu)\) is almost point isomorphic to \([0, 1]\).

**Proof.** (i) \( \Rightarrow \) (ii) There exists a function \( f \in L_\omega(\mu) \) such that \( M_f \) is cyclic on \( L_p(\mu) \). Thus \( f \) is essentially univalent and \( M_f \) corresponds \( M_z \) on \( L_p(S_f, \mathcal{B}(S_f), \nu) \). In addition \( f^{-1} \) induces a \( \sigma \)-isomorphism \( \Gamma: \mathcal{B}'(S_f) \to \Sigma' \) and there exists \( X_0 \in \Sigma \) such that \( \mu(X_0) = 0 \) and \( f \mid_{X \setminus X_0} \) has range in \( S_f \).

Since \((S_f, \mathcal{B}(S_f), \nu)\) is nonatomic, the measure space \((S_f, \mathcal{B}(S_f), \mathcal{B}(S_f), \nu)\) is normal. There exists a point isomorphism \( \rho: S_f \to [0, 1] \setminus N_0 \) where \( \nu(S_0) = \lambda(N_0) = 0 \), \( \rho \) preserves Borel sets, and \( \rho^{-1} \) induces a \( \sigma \)-isomorphism \( \chi: \mathcal{B}'([0, 1]) \to \mathcal{B}'(S_f) \). There exists \( X_1 \supset X_0 \) such that \( \mu(X_1) = 0 \) and \( \rho \circ f \mid_{X_1 \setminus x_1} \) is defined, univalent, and \( (\rho \circ f \mid_{X_1 \setminus x_1})^{-1} \) induces the \( \sigma \)-isomorphism \( \Gamma: \mathcal{B}'([0, 1]) \to \Sigma' \). Since \([0, 1]\) is pre-normal, it follows from Lemmas 3.1 and 3.2 that \((X, \Sigma, \mu)\) is almost point isomorphic to \([0, 1] \).

(ii) \( \Rightarrow \) (i) There exists \( X_1 \in \Sigma \) with \( \mu(X_1) = 0 \) and an \( a \)-point isomorphism \( \theta: X \setminus X_1 \to [0, 1] \) such that \( \theta^{-1} \) induces a \( \sigma \)-isomorphism \( \Gamma: \mathcal{B}'([0, 1]) \to \Sigma' \). So we construct \( J_f: L_p([0, 1]) \to L_p(\mu) \), as in Remark 2.2. Then under \( J_f \), for each \( M_f \in \mathcal{M}_f \), there exists \( M_k \in \mathcal{M}_k \) such that \( M_f J_f = J_f M_k \). The function \( \chi([0, 1]) \) is in \( L_p([0, 1]) \) and we conclude that \( M_k J_f(\chi([0, 1])) = J_f M_k(\chi([0, 1])) \) a.e. \( \mu \). Thus there exists \( X_1 \supset X_0, \mu(X_1) = 0 \), and \( f(x) = k(\theta(x)) \) on \( X \setminus X_1 \).

Suppose \( M_f \) is cyclic on \( L_p(\mu) \). Then \( M_k \) is cyclic on \( L_p([0, 1]) \). Since \([0, 1]\) is pre-normal, \( k \) is essentially univalent. Thus \( f = k \circ \theta \) is essentially univalent.

Conversely suppose that \( f \) is essentially univalent. There exists \( X_2 \in \Sigma \) such that \( X_2 \supset X_1 \supset X_0, \mu(X_2) = 0 \) and \( f = k \circ \theta \) is univalent on
$X \setminus X_2$. Since $\theta(X \setminus X_2) = Y$ is a restrictive subset of $[0, 1]$, and $k|_Y$ is univalent, it follows that $k$ is an essentially univalent $L_\infty([0, 1])$ function. Since $[0, 1]$ is pre-normal, $M_k$ is cyclic on $L_\rho([0, 1])$. Thus $M_f$ is cyclic on $L_\rho(\mu)$.

This work is part of the author's doctoral dissertation at the University of California, Irvine. The author wishes to thank his advisor, Professor Gerhard K. Kalisch for his invaluable discussions and advice.

**References**


Received December 6, 1972. This research was supported in part by National Science Foundation Grants 12635, 13288, 21081, and 21334 and by the Air Force Office of Scientific Research Grant AFOSR70–1870. The referee's suggestions have enabled the author to strengthen his results in §§ 2 and 3.

University of Toronto