GENERALIZED $\omega - L$-UNIPOTENT BISIMPLE SEMIGROUPS

RONSON JOSEPH WARNE
Let $S$ be a bisimple semigroup and let $E(S)$ be the set of idempotents of $S$. If $E(S)$ is an $\omega$-chain of rectangular bands ($E_n; n \in \mathbb{N}$, the nonnegative integers) and $\mathcal{L}$, Green's equivalence relation, is a left congruence on $E(S)$, we term $S$ a generalized $\omega$-\$\mathcal{L}$-unipotent bisimple semigroup. We characterize $S$ in terms of $(I, o)$, an $\omega$-chain of left zero semigroups ($I_k; k \in \mathbb{N}$), $(J, *)$ an $\omega$-chain of right groups ($J_k; k \in \mathbb{N}$); a homomorphism $(n, r) \to \alpha_{(n, r)}$ of $C$, the bicyclic semigroup, into $\text{End}(I, o)$, the semigroup of endomorphisms of $(I, o)$ (iteration); a homomorphism $(n, r) \to \beta_{(n, r)}$ of $C$ into $\text{End}(J, *)$; and an (upper) anti-homomorphism $j \to A_j$ of $(J, *)$ into $T$, the full transformation semigroup on $I$ ($A \text{ is } \text{almost} \text{ an endomorphism}$). In fact, $S^* \equiv (i, (n, k), j; i \in I_n, j \in J_k, n, k \in \mathbb{N}) \text{ under the multiplication } (i, (n, k), j)(u, (r, s), v) = (i \circ (uA \alpha_{(k, n)}), (n + r - \min(k, r), k + s - \min(k, r)), j \beta_{(s, r)}v)$ (Theorem 4.1). We then characterize $(J, *)$ as a semi-direct product of an $\omega$-chain of right zero semigroups by an $\omega$-chain of groups. Finally, we specialize Theorem 4.1 to obtain our previous characterization of $\omega$-\$\mathcal{L}$-unipotent bisimple semigroups $S(E(S)$ is an $\omega$-chain of right zero semigroups).

We will use the definitions of Clifford and Preston [1] unless otherwise specified. In particular, $\mathcal{R}, \mathcal{L}, \mathcal{H}, \text{ and } \mathcal{D}$ will denote Green's equivalence relations on a semigroup $S$, i.e., $((a, b) \in \mathcal{R}$ if $a \cup aS = b \cup bS; (a, b) \in \mathcal{L}$ if $a \cup Sa = b \cup Sb; \mathcal{H} = \mathcal{R} \cap \mathcal{L}; \mathcal{D} = \mathcal{R} \circ \mathcal{L} ((a, b) \in \mathcal{D}$ if there exists $x \in S$ such that $(a, x) \in \mathcal{R}$ and $(x, b) \in \mathcal{R})$. $R_s$ will denote the $\mathcal{R}$-class containing $a \in S$. A semigroup consisting of a single $\mathcal{D}$-class is termed a bisimple semigroup. This bicyclic semigroup is $C = N \times N$ under the multiplication $(n, m)(p, q) = (n + p - \min(m, p), m + q - \min(m, p))$. A semigroup $S$ which is a union of a collection of pairwise disjoint subsemigroups $(S_y; y \in Y)$ where $Y$ is a semilattice and $S_xS_y \subseteq S_{x\wedge y}$ for all $y, t \in Y$ is termed a semilattice $Y$ of the semigroups $(S_y; y \in Y)$.

If $Y = \mathbb{N}$ with $n \land m = \max(n, m)$, $S$ is termed an $\omega$-chain of the semigroups $(S_n; n \in \mathbb{N})$. A semigroup is termed regular if $a \in aS$ for every $a \in S$. A rectangular band is the algebraic direct product of a left zero semigroup $U(x, y \in U \text{ implies } xy = x)$ and a right zero semigroup. A right group is a semigroup $X$ such that $a, b \in X$ implies there exists a unique $x \in S$ such that $ax = b$. If $V$ is a subset of a semigroup $S$, $E(V)$ will always denote the set of idempotents of $V$.

In [4], we defined a generalized $\mathcal{L}$-unipotent semigroup to be a
regular semigroup $S$ such that $E(S)$ satisfy the condition: $e, f \in E(S)$ and $ef = e$ imply that $gef = ge$ for all $g \in E(S)$. Combining [4, Lemma 1] and a result of Clifford and McLean [2, 1, p. 129, Exercise 1], a regular semigroup $S$ is generalized $\mathcal{L}$-unipotent if and only if $E(S)$ is a semilattice $Y$ of rectangular bands $(E_y; y \in Y)$ and $\mathcal{L}$ is a left congruence on $E(S)$. Since any bisimple semigroup containing an idempotent is regular by a result of Clifford and Miller [1, Theorem 2.11], the reason for the terminology “generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroup” is clear. We introduced the term $\mathcal{L}$-unipotent in [3] to denote a semigroup in which each $\mathcal{L}$-class contains precisely one idempotent. By [3, Proposition 5], a semigroup $S$ is $\mathcal{L}$-unipotent if and only if $S$ is regular and $E(S)$ is a semilattice $Y$ of right zero semigroups $(E_y; y \in Y)$. Hence, the terminology “$\omega$-$\mathcal{L}$-unipotent bisimple semigroup” is also clear.

Let $S$ be a generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroup. In §1, we define a congruence $t$ on $S$ such that $S/t = C$, the bicyclic semigroup, and give an explicit multiplication for $(E(C)/t^{-1})$, the kernel of $t(ker t)$. In §2, we describe $S$ as an “extension” of ker $t$ by $S/t$ (the converse of Theorem 4.1). In §3, we prove the direct part of Theorem 4.1. In §4, we state Theorem 4.1 and characterize an $\omega$-chain of right groups as a semi-direct product of an $\omega$-chain of right zero semigroups by an $\omega$-chain of groups (Theorem 4.3). Combining Theorem 4.1, Theorem 4.3, and Clifford’s characterization of semilattices of groups [1; theorem 4.11], we have characterized generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroups in terms of groups, $\omega$-chains of left zero semigroups, $\omega$-chains of right zero semigroups, and ‘homomorphisms’. In §5, we obtain our characterization of $\omega$-$\mathcal{L}$-unipotent bisimple semigroups [5, Theorem 7.11] as a corollary of Theorem 4.1.

1. The congruence $t$. In this section, $S$ will denote a generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroup, i.e., $S$ is a bisimple semigroup such that $E(S)$ is an $\omega$-chain of rectangular bands $(E_n; n \in N)$ and $\mathcal{L}$ is a left congruence on $E(S)$. Recall $S$ is a regular semigroup. Thus, for each $a \in S$, there exists $y \in S$ such that $aya = a$ and $yay = y$ (for example, if $a = axa$, let $y = xax$ [1, Lemma 1.14]). The element $y$ is termed an inverse of $a$. We will denote the set of all inverses of $a$ by $\mathcal{F}(a)$.

Let $t = [(x, y) \in S^2; xx' \in E_m$ and $x'y' \in E_n$ for some $m, n \in N$, $x' \in \mathcal{F}(x)$, and $y' \in \mathcal{F}(y))$. We first show that $t$ is a congruence on $S$ and that $S/t \cong C$, the bicyclic semigroup. We also note that $C$ may be taken as a set of representative elements for the $t$-classes of $S$ and that $T = ker t$ (the union of the collection of $t$-classes of $S$ containing idempotents) is an $\omega$-chain of rectangular groups.

Finally, we describe $T$ in terms of $(I, o)$, an $\omega$-chain of left zero
semigroups \((I_n; \, n \in \mathbb{N}); \, (J_n; \, n \in \mathbb{N})\) and an anti-homomorphism \(j \rightarrow A_j\) of \(J\) into \(T\), the full transformation semigroups on \(I\). In fact, \(T \cong U(I_n \times J_n; \, n \in \mathbb{N})\) under the multiplication \((i, j)(p, q) = (i \circ pA_j, \, j^*q)\).

**Lemma 1.1.** If \(e_0 \in E_{e_0} \, R_{e_0}\) is a semigroup.

**Proof.** Lemma 1.1 is a special case of [5, Lemma 3.1].

**Remark.** Immediately below, we write Theorem 1.2 [5, Theorem 3.3)]. This means Theorem 1.2 is obtained by taking "\(Y\)" to be a one element set in [5, Theorem 3.3]. \((D_\delta; \, \delta \in \mathbb{Y})\) is the collection of \(\mathcal{D}\)-classes in the semigroup of [5, Theorem 3.3]. We do the same thing in Note 1.3, Propositions 1.4 and 1.5, and Lemmas 1.9-1.12.

**Theorem 1.2** [5, Theorem 3.3]. \(t\) is a congruence on \(S\) and \(S/t \cong C\).

**Note 1.3** [5, Note 3.4]. If we let \(t_{(n, k)} = (n, k)t\), the \(t\)-classes of \(S\) are \((t_{(n, k)}; \, n, \, k \in \mathbb{N})\) with \(t_{(n, k)}t_{(r, s)} \subseteq t_{(n + r \cdot \min(k, r), \, k + \cdot \min(k, r))}\). We may write \(E(S) = \bigcup (E_{(k, k)}; \, k \in \mathbb{N})\) where \(E_{(k, k)}\) is a rectangular band and \(E(t_{(k, k)}) = E_{(k, k)}\). Actually, \(E_{(k, k)} = E_k\).

**Proposition 1.4** [5, Proposition 3.5]. \(t_{(n, k)} = (a \in S; \, a' \in E_{(n, n)}\) and \(a' \in E_{(k, k)}\) for some \(a' \in \mathcal{S}(a) = \bigcup (R_s \cap L_f; \, e \in E_{(n, n)}\) and \(f \in E_{(k, k)}\). A rectangular group is the algebraic direct product of a group and a rectangular band.

**Proposition 1.5** [5, Proposition 3.6]. For each \(k \in \mathbb{N}\), \(t_{(k, k)}\) is a rectangular group. In fact, \(t_{(k, k)} \cong G \times E_{(k, k)}\) where \(G\) is a fixed maximal subgroup of \(S\). Furthermore, \(t_{(k, k)} \subseteq t_{(\max(k, s), \max(k, s))}\).

**Remark 1.6.** If \(b \in R_s \cap L_f(e, \, f \in E(S))\), there exists \(x \in S\) such that \(bx = e\). It is shown in the proof of [1, Theorem 2.18] that \(b^{-1} = fxe\) is the unique inverse of \(b\) contained in \(R_f \cap L_e\) and that \(bb^{-1} = e\) and \(b^{-1}b = f\).

**Note 1.7.** Let \(e_0\) be a fixed element of \(E_{e_0}\) and fix an element \(e_1 \in E_{(1,1)}\) such that \(e_1 < e_0\). For example, select any \(f \in E_{(1,1)}\) and let \(e_1 = e_0fe_0\). Hence, \(e_1 \in E_{(1,1)}\) by Note 1.3 and \(e_1 < e_0\).

**Note 1.8.** Select and fix \(a \in R_{e_0} \cap L_{e_0}\). By Remark 1.6, there exists a unique \(a^{-1} \in \mathcal{S}(a) \cap R_{e_1} \cap L_{e_0}\) with \(aa^{-1} = e_0\) and \(a^{-1}a = e_1\). Define
\[ a^{-n} = (a^{-1})^n \] for all positive integers \( n \) and define \( a^0 = e_0 \). Utilizing Proposition 1.4 and Note 1.3, \( a^{-n}a^k \in t_{(n,k)} \) for all \( n,k \in \mathbb{N} \).

**Lemma 1.9** [5, Lemma 3.9]. \( a^k a^{-k} = e_0 \) for all \( k \in \mathbb{N} \).

**Lemma 1.10** [5, Lemma 3.10].

\[
a^k a^{-r} = \begin{cases} 
 a^{k-r} & \text{if } k > r \\
 a^{-(r-k)} & \text{if } r > k \\
 e_0 & \text{if } r = k 
\end{cases}
\]

**Lemma 1.11** [5, Lemmas 3.11, 3.12].

1. \( a^{-k}a^p a^{-r}a^s = a^{-(k+r-min\{r,p\})}a^{p+s-min\{r,p\}} \)  
2. \( a^{-r}a^r \in E_{(r,r)} \) for all \( r \in \mathbb{N} \).

For brevity, let \( T_k = t_{(k,k)} \) and let \( T = \bigcup (T_k : k \in \mathbb{N}) \). Hence, \( T \) is an \( \omega \)-chain of the rectangular groups \( (T_k : k \in \mathbb{N}) \) by Proposition 1.5. Since \( E(S) = E(T) \) by Note 1.3, \( T \) is generalized \( \mathcal{L} \)-unipotent. Utilizing Proposition 1.5, \( T_k = G \times M_k \times N_k \) where \( G \) is a group, \( M_k \) is a left zero semigroup, and \( N_k \) is a right zero semigroup. By Lemma 1.11, \( a^{-k}a^k \in E(T_k) \). Let \( I_k \) denote the set of idempotents of the \( \mathcal{L} \)-class of \( T_k \) containing \( a^{-k}a^k \) and let \( J_k \) denote the \( \mathcal{R} \)-class of \( T_k \) containing \( a^{-k}a^k \). We may suppose that \( l_k \in M_k \cap N_k \), \( a^{-k}a^k = (e, l_k, l_k) \) where \( e \) is the identity of \( G \), \( I_k = (e) \times M_k \times (l_k) \), and \( J_k = G \times (l_k) \times N_k \). For brevity, let \( e_k = (e, l_k, l_k) \). Hence, using Lemma 1.11, \( e_m e_n = e_{\max\{m,n\}} \).

Let \( I = \bigcup (I_n : n \in \mathbb{N}) \) and let \( J = \bigcup (J_n : n \in \mathbb{N}) \).

**Lemma 1.12.** \( I \) is an \( \omega \)-chain of left zero semigroups \( (I_n : n \in \mathbb{N}) \).

**Proof.** By a direct calculation, \( I_n \) is a left zero semigroup for each \( n \in \mathbb{N} \). Let \( x \in I_k \) and let \( y \in I_l \). Hence, \( x \mathcal{L} e_k \) and \( y \mathcal{L} e_n \). Since \( T \) is generalized \( \mathcal{L} \)-unipotent, \( xy \mathcal{L} xe_n \). Thus, since

\[ xe_n \mathcal{L} e_k e_n, xy \mathcal{L} e_{\max\{k,n\}} \]

Hence, \( xy \in I_{\max\{k,n\}} \).

**Lemma 1.13.** For each \( n \in \mathbb{N} \), \( J_n \) is a right group. If \( x \in J_n \), \( y \in J_m \), and \( n \geq m \), \( xy \in J_n \).

**Proof.** By [1, Theorem 1.27], \( J_n \) is a right group for each \( n \in \mathbb{N} \). Let \( x \in J_n \), \( y \in J_m \), and \( n \geq m \). Hence, \( y \mathcal{R} e_m \) implies \( xy \mathcal{R} xe_m \). Since \( e_n(xe_m) = (e_n x)e_m = xe_m \) and \( xe_m \in T_n, xe_m \in J_n \) by a simple calculation.
Thus, \( xy \in J_n \).

**Lemma 1.14.** Every element of \( T \) may be uniquely expressed in the form \( x = ij \) with \( i \in I_n \) and \( j \in J_n \) for some \( n \in \mathbb{N} \).

**Proof.** If \( x = (g, i, j) \in T_n, x = (e, i, l_n)(g, l_n, j). \)

If \( X \) is a set, \( T_X \) will denote the semigroup (iteration) of mappings of \( X \) into \( X \).

**Lemma 1.15.** There exists a mapping \( j \rightarrow A_j \) of \( J \) into \( T_{I_{\max(m,n)}} \) and a mapping \( p \rightarrow B_p \) of \( I \) into \( T_J \) such that \( I_nA_j \subseteq I_{\max(n,m)} \) for \( j \in J_m \) and \( J_mB_p \subseteq J_{\max(m,n)} \) for \( p \in I_m \). If \( j \in J \) and \( p \in I \), \( jp = pA_jjB_p \). Furthermore, \( jp \mathcal{R} pA_j(e T) \) and \( jp \mathcal{L} jB_p(e T) \).

**Proof.** Let \( j \in J_m \) and \( p \in I_m \). Thus, \( jp \in T_{\max(m,n)}. \) Hence, by Lemma 1.14, there exists a unique \( u \in I_{\max(m,n)} \) and \( v \in I_{\max(m,n)} \) such that \( jp = uv. \) Let \( u = pA_j \) and \( v = jB_p \). The last statement is valid by a simple calculation.

**Lemma 1.16.** If \( j \in J, jB_v = e \cdot jv. \) If \( j \in J, and r \geq v, jB_v = je_v. \)

**Proof.** Let \( j \in J_r \) and suppose that \( v > r. \) Thus, \( je_v \in T_v \) and \( (je_v)e_v = je_v. \) Hence, \( je_v = (g, i, l_v) \) for some \( g \in G \) and \( i \in M_v. \) By Lemma 1.15, \( je_v = e_rA_jB_v \) with \( jB_v \mathcal{L} je_v(e T). \) Hence, \( jB_v = (g, l_v, l_v). \) Thus, \( jB_v = (e, l_v, l_v)(g, i, l_v) = e \cdot je_v. \) Next, suppose that \( r \geq v. \) Hence, \( je_v \in J_r \) by Lemma 1.13. Thus, utilizing Lemma 1.15, \( e \cdot (je_v) = je_v = e_rA_jB_v \) where \( e_rA_j \in I_r \) and \( jB_v \in J_r. \) Hence, \( jB_v = je_v \) by Lemma 1.14. This establishes the second sentence of the lemma. Since, for \( r \geq v, e \cdot j = e, e \cdot j = e, j = j, \) \( jB_v = e \cdot je_v \) for \( r \geq v. \)

**Lemma 1.17.** \( (e, f) \in \mathcal{L} \cap (E(T))^2 \) and \( p \in T \) imply \( (pe, pf) \in \mathcal{L} \cap (E(T))^2 \).

**Proof.** Suppose \( (e, f) \in \mathcal{L} \cap (E(T))^2. \) Hence, for \( p \in T, (p^{-1}pe, p^{-1}pf) \in \mathcal{L} \) \( (p^{-1} \) is the group inverse of \( p \) in the group containing \( p). \) Thus, \( p^{-1}pep^{-1}pf = p^{-1}pe \) and \( p^{-1}pf \cdot p^{-1}pe = p^{-1}pf. \) Hence, \( (pep^{-1})pf = pe \) and \( (pfp^{-1})pe = pf. \) Thus, \( (pe, pf) \in \mathcal{L} \cap (E(T))^2. \)

**Lemma 1.18.** If \( p \in I_r, B_p = B_{e_r}. \)

**Proof.** If \( n, m \in \mathbb{N}, \) let \( nm = \max(n, m) \) in this proof. Let \( j \in J_s \)
and \( p \in I \). Hence \( e_{rs}j = (g, l_{rs}, j') \) for some \( g \in G \) and \( j' \in N_{rs} \) by Lemma 1.13. By Lemma 1.12, \( pe_{rs} = (e, n, l_{rs}) \) for some \( n \in M_{rs} \). Thus,
\[
e_{rs}jpe_{rs} = (g, l_{rs}, l_{rs}) = e_{rs}je_{rs}.
\]
Hence, if \( jp = (w, m, n) \) and \( je_{rs} = (e, c, n) \), then \( w = u \). Since \( (p, e_r) \in \mathcal{L}_r \) and \( (j, e_r) \in \mathcal{L}_r \) by Lemma 1.17. Hence, \( n = d \). Thus, \( jp = (w, m, n) = (e, m, l_{rs}) \) while \( je_{rs} = (w, c, n) = (e, c, l_{rs}) \). Hence, utilizing Lemmas 1.14 and 1.15, \( jB_p = jB_{ep} \).

**Lemma 1.19.** Let \( r \in J_u, s \in J_v, v \leq u, \) and \( z \in N \). Then, (a) \((rs)\beta_z = r\beta_{\max(z,v)}s\beta_z \) (b) if \( x \in I \), \( xA_r = xA_s \).

**Proof.** Let \( r \in J_u, s \in J_v, u \geq v \), and \( x \in I \). Hence, utilizing Lemmas 1.13 and 1.15, \((rs)x = xA_r(rs)B_z \) while
\[
r(xs) = r(xA_r)sB_z = (r(xA_s))(sB_z) = xA_sA_r(rB_{zA_r}sB_z).
\]
Thus, utilizing Lemma 1.14, \( xA_r = xA_s \) and \((rs)B_z = rB_{\max(z,v)}sB_z \). Utilizing Lemmas 1.15 and 1.18, \((rs)B_z = rB_{\max(z,v)}sB_z \).

If \( x \in J_u \) and \( y \in J_v \), define \( x^* y = xB_{z'A}y \).

**Lemma 1.20.** If \( x \in J_u \) and \( y \in J_v \), \( x^* y = e, xy \). If \( u \geq v \), \( x^* y = xy \).

**Proof.** Let \( x \in J_u \) and \( y \in J_v \). Hence, utilizing Lemma 1.16,
\[
x^* y = xB_{z'A}y = (e, xe) y = e, x(e, y) = e, xy.
\]
If \( u \geq v \), again utilizing Lemma 1.16, \( x^* y = xB_{z'A}y = (xe) y = x(e, y) = xy \).

**Lemma 1.21.** \((J, *)\) is an \( \omega \)-chain of right groups \((J_n: n \in N)\).

**Proof.** Utilizing Lemmas 1.13 and 1.20, \((J, *)\) is a right group for each \( n \in N \) and \( J^* J_n \subseteq J_{\max(n, m)} \). We must just establish associativity. Let \( i \in J_n, p \in J_y, \) and \( w \in J_s \). Hence, utilizing Lemmas 1.15 and 1.13, \( i^*(p^* w) = i^*(pB_{s'w}) = iB_{\max(y, z)}pB_{z'w} \) while
\[
(i^* p)^* w = (iB_{s'y})^* w = (iB_{s'y})B_{z'w}.
\]
Utilizing Lemma 1.19 (a) \( iB_{s'y}B_{\max(y, z)} = iB_{s'y}B_{\max(y, z)} \). However, utilizing Lemma 1.16,
\[
iB_{s'y}B_{\max(y, z)} = e \max(y, z)e_{e\max(y, z)} = e \max(y, z)i\max(y, z) = iB_{\max(y, z)}.
\]
Hence, \( (i^* p)^* w = iB_{\max(y, z)}pB_{s'y} w = i^*(p^* w) \).

**Definition 1.22.** Let the semigroup \( X \) be an \( \omega \)-chain of semi-
groups \((X_n: n \in \mathbb{N})\) and let \(\phi\) be a mapping of \(X\) into a semi-group \(Y\). If \(r \in X_n, s \in X_m, \) and \(n \geq m\) imply \((rs)\phi = s\phi r\phi, \phi\) is termed an upper anti-homomorphism of \(X\) into \(Y\).

**Lemma 1.23.** \(r \rightarrow A_r\) is an upper anti-homomorphism of \((J, \ast)\) into \(T_1\).

**Proof.** Combine Lemmas 1.20 and 1.19 (b).

**Lemma 1.24.** If \(j \in J_s\) and \(i \in I_u, ji = iA_jj^*e_u = iA_j^*e_u.\)

**Proof.** Let \(j \in J_s\) and \(i \in I_u\). Hence, \(ji = iA_jj^*e_u\) by Lemma 1.15. However, utilizing Lemmas 1.18 and 1.16, \(j^*e_u = e_jj^*e_u\). Since \(iA_j \in I_{\text{max}(t, z)}, iA_j = iA_je_{\text{max}(t, z)}\). Hence, \(ji = iA_je_{\text{max}(t, z)}j^*e_u = iA_j^*e_u.\)

**Lemma 1.25.** If \(r, s \in I\) with \(r \in I_u\), \((rs)A_x = rA_xsA_{x^*u}\) for all \(x \in J.\)

**Proof.** Let \(r, s \in I\) with \(r \in I_u\) and let \(x \in J.\) Hence, utilizing Lemmas 1.15 and 1.12, \(x(rs) = (rs)A_xx_B,\) while

\[(xr)^s = (rA_xx_B,s) = rA_x(x_B,s) = rA_x(sA_{x^*u},x_B,B_x) = rA_xsA_{x^*u},x_B,B_x.\]

Hence, utilizing Lemmas 1.15, 1.12, and 1.14, \((rs)A_x = rA_xsA_{x^*u}\). Utilizing Lemmas 1.18, 1.16, and 1.20, \(x_B = x_{B^*u} = e_uxe_u = x^*e_u.\)

**Remark 1.26.** Results of \([6]\) could have been applied to characterize \(T.\)

2. Structure theorem for generalized \(\omega-L^-\)unipotent bisimple semigroups. (Proof of converse.) In this section, we complete the proof of the converse of our structure theorem for generalized \(\omega-L^-\)unipotent bisimple semigroups (Theorem 2.21).

We will use a sequence of twenty entries to establish Theorem 2.21. \(S\) will denote a generalized \(\omega-L^-\)unipotent bisimple semigroup.

**Lemma 2.1.** Every element of \(S\) may be uniquely expressed in the form \(x = i^\alpha a^k j\) where \(i \in I_u\) and \(j \in J_u.\)

**Proof.** Let \(x \in t_{(a, b)}.\) Hence, \((x, e) \in B\) for some \(e \in E_u\) by Proposition 1.4. Thus, \((x, i) \in B\) for some \(i \in I_u.\) Thus, since \(a^{-*}a^* \in I_u,\) a left zero semigroup, \(x = ix = (a^{-*}a^*)x = ia^{-*}a^*x.\) Since \(a^* \in R_{e_0}\) by Note 1.8 and Lemma 1.1 and \(a^*a^{-*}a^* = e_0\) by Lemma 1.9, \(a^*a^{-*}a^* = e_0.\)
\[ \alpha^n. \] Hence \( x = \alpha^{-n} a^n \alpha x = (\alpha^{-n} a^n)(\alpha^{-k} a^n x) \). However, \( \alpha^{-k} a^n x \) \( \in \) \( t(h,k) \) by Notes 1.8 and 1.3. Thus, since \( \alpha^{-k} a^n(\alpha^{-k} a^n x) = \alpha^{-k} a^n x \) and \( \alpha^{-k} a^n \in J_k \), \( \alpha^{-k} a^n x \) \( \in \) \( J_k \) by Proposition 1.5. Hence, \( x = \alpha^{-n} a^n j \) where \( i \in I_k \) and \( j \in J_k \). We next establish uniqueness. Suppose that \( x = \alpha^{-n} a^n j \) where \( i \in I_k \) and \( j \in J_k \). Thus, using Note 1.3, \( \alpha^{-n} a^n j = v a^{-n} a^k v \). Hence, \( \alpha^{-n} a^n a^{-n} a^k j = \alpha^{-n} a^n a^{-n} a^k v \). Thus, since \( \alpha^{-n} a^n \), \( i \in I_k \), \( u \in I_k \), and \( v \in I_k \). Thus, using Note 1.3, \( \alpha^{-n} a^n j = v a^{-n} a^k v \). Hence, \( \alpha^{-n} a^n j = v a^{-n} a^k v \). Since \( \alpha^{-n} a^n \in E(J_k) \) and \( j, v \in J_k \), a right group, \( j = v \). Thus, \( \alpha^{-n} a^n j = v a^{-n} a^k v \). Hence, \( \alpha^{-n} a^n = v a^{-n} a^k v \). Thus, since \( \alpha^{-n} a^n \), a left zero semigroup, \( \alpha^{-n} a^n = \alpha^{-n} a^n \). Thus, \( \alpha^{-n} a^n = v a^{-n} a^k v \). Hence, \( \alpha^{-n} a^n = u a^{-n} a^k v \). Thus, \( \alpha^{-n} a^n = u a^{-n} a^k v \). Hence \( \alpha^{-n} a^n = u a^{-n} a^k v \). Thus, \( \alpha^{-n} a^n = u a^{-n} a^k v \).

Since \( i, u, a^{-n} a^n \in I_n \), a left zero semigroup, \( i = u \).

**Definition 2.2.** If \( u \in T \) and \( k \in \mathbb{N} \), define \( \nu_{(k,n)} = a^{-n} a^n u a^{-n} a^n \).

**Lemma 2.3.** \( T, \nu_{(k,n)} \subseteq T_{n+r-min(k,r)} \).

**Proof.** Let \( g \in T \). Hence, utilizing Note 1.3, \( g \nu_{(k,n)} = a^{-n} a^n g a^{-n} a^n \) \( \in \) \( t(s,k) \cap t(r,s) \) and, hence, \( n = r \) and \( k = s \). Thus, \( \alpha^{-n} a^n j = v a^{-n} a^k v \). Hence, \( \alpha^{-n} a^n i a^{-n} a^n j = \alpha^{-n} a^n u a^{-n} a^k v \). Thus, since \( \alpha^{-n} a^n \), \( i, u \in I_k \), and \( v \in I_k \).

**Lemma 2.4.** Let \( g, h \in T \) and \( h \in T \). If \( k \geq r, s \) or \( r = s \geq k \), \( (g,h) \nu_{(k,n)} = g \nu_{(k,n)} h \nu_{(k,n)} \nu_{(k,n)} \). In particular, \( \nu_{(k,n)} \) is a homomorphism of \( T \) into \( T_{n+r-min(k,r)} \).

**Proof.** Let \( g, h \in T \) and \( h \in T \) with \( k \geq r, s \). Hence,

\[ (g,h) \nu_{(k,n)} = a^{-n} a^n g h a^{-n} a^n = a^{-n} a^n (a^{-k} a^n g) u a^{-k} a^n f_k (h, a^{-k} a^n a^{-k} a^n) \]

where \( (v_k, a^{-k} a^n g) \in \mathcal{L} \) with \( v_k \in E(J_k) \) and \( (f_k, h, a^{-k} a^n) \in \mathcal{K} \) with \( f_k \in I_k \). Hence, \( (g,h) \nu_{(k,n)} = a^{-n} a^n g, a^{-k} a^n h, a^{-k} a^n = (a^{-n} a^n h, a^{-k} a^n) = g, \nu_{(k,n)} h, \nu_{(k,n)} \). Next suppose that \( r = s \geq k \). Then,

\[ (g,h) \nu_{(k,n)} = a^{-n} a^n g, v, a^{-k} a^n f_r h, a^{-k} a^n \]

where \( (v_r, g) \in \mathcal{L} \) with \( v_r \in E(J_k) \) and \( (f_r, h) \in \mathcal{K} \) with \( f_r \in I_k \). Hence, \( (g,h) \nu_{(k,n)} = (a^{-n} a^n g, a^{-k} a^n)(a^{-n} a^n h, a^{-k} a^n) = g, \nu_{(k,n)} h, \nu_{(k,n)} \).

**Definition 2.5.** Let \( \nu_{(k,n)} \mid I = \alpha_{(k,n)} \) and \( \nu_{(k,n)} \mid J = \beta_{(k,n)} \).

**Lemma 2.6.** (a) \( I, \alpha_{(k,n)} \subseteq I_{n+r-min(k,r)} \) \( \quad \) (b) \( J, \beta_{(k,n)} \subseteq J_{n+r-min(k,r)} \).

**Proof.** (a) By Lemma 2.3, \( I, \nu_{(k,n)} \subseteq T_n \) if \( k \geq r \) and \( I, \nu_{(k,n)} \subseteq T_{n+r-k} \) if \( k < r \).
if \( r \geq k \). If \( k \geq r \), \( \nu_{(k,n)} \) is a homomorphism of \( T_r \) into \( T_n \) by Lemma 2.4. Hence, \( I_r \nu_{(k,n)} \subseteq E(T_n) \). Let \( g_r \in I_r \). Hence, \( g_r \mathcal{L} a^{-r}a'(\in T_r) \). Thus, \( g_r \nu_{(k,n)} = \mathcal{L} a^{-n}a^k a^{-r} a^s a^k a^r \). However,

\[
a^{-n}a^k a^{-r} a^s a^k = a^{-n}a^s
\]

by Lemma 1.11. Hence, \( g_r \nu_{(k,n)} \in I_n \) if \( k \geq r \). The case \( r \geq k \) is treated similarly. To prove (b), just replace "\( I \)" by "\( J \)" and "\( \mathcal{L} \)" by "\( \mathcal{R} \)" in the proof of (a).

**Definition 2.7.** If \( X \) is a semigroup \( \text{End} \ X \) will denote the semigroup of endomorphisms of \( X \) (iteration).

**Lemma 2.8.** \( \alpha_{(k,n)} \in \text{End} \ I \) for each \( n, k \in N \).

*Proof.* Let \( i_r \in I_r \) and \( i_s \in I_s \). If \( r \geq k \), \( i_r a^{-k} i_r = i_s \). Hence,

\[
(i_r i_s) \alpha_{(k,n)} = a^{-n} a^k i_r a^{-k} a^n = a^{-n} a^k i_r a^{-k} a^n
\]

Thus, \( a^{-n} a^k i_r a^{-k} a^n = i_r \alpha_{(k,n)} i_r \alpha_{(k,n)} \).

Next, suppose that \( k > r \). Since \( S \) is generalized \( \mathcal{L} \)-unipotent, \( i_r \mathcal{L} a^{-r} a^s \) implies \( a^{-k} a^k i_r \mathcal{L} a^{-k} a^s a^{-r} a^s \). Thus, \( a^{-k} a^k i_r \mathcal{L} a^{-k} a^k \) by Lemma 1.11. Hence, \( a^{-k} a^k i_r \in I_h \). Thus,

\[
(i_r i_s) \alpha_{(k,n)} = a^{-n} a^k i_r a^{-k} a^n = a^{-n} a^k i_r a^{-k} a^n
\]

Thus, \( (a^{-n} a^k i_r a^{-k} a^n) = i_r \alpha_{(k,n)} i_r \alpha_{(k,n)} \).

**Lemma 2.9.** \( (n, k) \rightarrow \alpha_{(n,k)} \) is a homomorphism of \( C \) into \( \text{End} \ I \).

*Proof.* Let \( g \in I \). We will employ Lemma 1.11. Thus,

\[
g \alpha_{(r,s)} \alpha_{(n,p)} = a^{-r} a^s a^{-r} a^s a^{-n} a^p
\]

\[
= a^{-r} a^s a^{-n} a^p a^{-(r+s-min(s,e))} a^{-(r+s-min(s,e))} a^{-(r+s-min(s,e))} a^{s+p-min(s,e)}
\]

We next establish that \( \beta_{(s,k)} \in \text{End} \ (J, *) \). This will be accomplished by Lemmas 2.10–2.15.

**Lemma 2.10.** \( \beta_{(1,0)} \in \text{End} \ (J, *) \).

*Proof.* Let \( w \in J_p \) and \( u \in J_s \). If \( p = s \), \( \beta_{(1,0)} \in \text{End} \ (J, *) \) by Lemmas 2.4, 1.20, and 2.6(b), and Definition 2.5. Let us first suppose \( s = 0 \). Utilizing Lemmas 1.13, 1.11, Note 1.8, and Definition 2.5,

\[
(wu_0)a^{-1} = a^{-r} a^s a^{-k} a(wu_0)a^{-1} = a^{-r} a^s a^{-k} a(wu_0)\beta_{(1,0)}
\]

\[
= a^{-r} a^s a^{-k} a(wu_0)\beta_{(1,0)} = e a^{-r} a^s a^{-k} a(wu_0)\beta_{(1,0)} .
\]
We note that \((wu_0)\beta(1,0) \in J_{p-1}\) by Lemma 2.6(b). Utilizing Note 1.8, Lemmas 1.15, 1.16, and Definition 2.5, \(u_0 e^{-1} = u_0 e^{-1} = e_1 A_{w_0} e u_0 e^{-1} = e_1 A_{w_0} e^{-1} u_0 e^{-1} = e_1 A_{w_0} e^{-1} (u_0 \beta(1,0))\). Hence, utilizing Lemmas 1.15, 1.23, 1.18, 1.16, and 1.11, and Definition 2.5,

\[
(wu_0) e^{-1} = w(e_1 A_w a^{-1} (u_0 \beta(1,0))) = e_1 A_w e\ e_1 a^{-1} (u_0 \beta(1,0)) = e_1 A_w e^{-1} (w \beta(1,0)) u_0 \beta(1,0) = e_1 A_w e^{-1} (w \beta(1,0)) u_0 \beta(1,0) = e_1 A_w e^{-1} (w \beta(1,0)) u_0 \beta(1,0).
\]

Utilizing Lemmas 1.15, 2.6(b), and 1.13, \(e_1 A_{w_0} \in I_p\) and \(w \beta(1,0) u_0 \beta(1,0) \in J_{p-1}\). Hence, \((wu_0) \beta(1,0) = w \beta(1,0) u_0 \beta(1,0)\) by Lemma 2.1. Thus, utilizing Lemma 1.20 and 2.6(b), \((wu_0) \beta(1,0) = w \beta(1,0) u_0 \beta(1,0)\). Next, we assume that \(p \geq s \geq 1\). Hence, utilizing Lemmas 1.11, 2.6(b), and 1.12,

\[
(w^* u_0) \beta(1,0) = (wu_0) \beta(1,0) = awu a^{-1} = (aw^{-1} a^{-1} u a^{-1}) = (wu a^{-1}) u a^{-1} = w \beta(1,0) u_0 \beta(1,0) = w \beta(1,0) u_0 \beta(1,0).
\]

Finally, we assume \(s > p\). Utilizing Lemmas 1.10, 1.13, 1.12 and the case \((p, s)\) just established,

\[
(w^* u_0) \beta(1,0) = (w^* u_0) \beta(1,0) = w^* u_0 \beta(1,0) = (u_0 w) \beta(1,0) u_0 \beta(1,0) = e_1 A_{w_0} e^* u_0 \beta(1,0) = e_1 A_{w_0} e^* u_0 \beta(1,0).
\]

Since \(u_0 \beta(1,0) \in J_{r-1}\) by Lemma 2.6(b), \(e_1 A_{w_0} \in I_p\) and \(w \beta(1,0) u_0 \beta(1,0) \in J_{p-1}\). Hence, \((w^* u_0) \beta(1,0) = w \beta^*(1,0) u_0 \beta(1,0)\) by Lemma 1.20. Hence, \((w^* u_0) \beta(1,0) = w \beta^*(1,0) u_0 \beta(1,0)\).

**Lemma 2.11.** \(\beta_{(0,0)} \in \text{End}(J, *)\).

**Proof.** Let \(u_0 \in J_r\) and \(v_0 \in J_s\). Utilizing Lemmas 1.20 and 2.6(b),

\[
(w^* v_0) \beta_{(0,0)} = (e u_0 \beta_{(0,0)} = e \beta_{(1,0)} u_0 \beta_{(1,0)} = e_1 A_{w_0} e^* u_0 \beta_{(1,0)} = e_1 A_{w_0} e^* u_0 \beta_{(1,0)} = e^* u_0 \beta_{(1,0)} u_0 \beta_{(1,0)} = u_0 \beta_{(1,0)} u_0 \beta_{(1,0)} = u_0 \beta_{(1,0)} u_0 \beta_{(1,0)}.
\]

**Lemma 2.12.** \((n, k) \rightarrow \beta_{(n, k)}\) is a homomorphism of \(C\) into \(T_J\).

**Proof.** Replace “I” by “J” and “a” by “b” in the proof of Lemma 2.9.

**Lemma 2.13.** \(\beta_{(k,0)} \in \text{End}(J, *)\) for all \(k \in N\).

**Proof.** We have shown that \(\beta_{(0,0)}\) in \(\text{End}(J, *)\)(Lemma 2.11) and that \(\beta_{(1,0)} \in \text{End}(J, *)\) (Lemma 2.10). Suppose that \(\beta_{(m,0)} \in \text{End}(J, *)\).
We show that $\beta_{(n+1,0)} \in \text{End}(J, \ast)$. Let $g, h \in J$. Hence, utilizing Lemma 2.12,

$$(g \ast h)\beta_{(n+1,0)} = (g \ast h)\beta_{(n,0)} \ast \beta_{(1,0)} = (g\beta_{(n,0)} \ast h\beta_{(n,0)})\beta_{(1,0)} = g\beta_{(n,0)} \ast h\beta_{(n,0)} \ast \beta_{(1,0)} = g\beta_{(n+1,0)} \ast h\beta_{(n+1,0)}.$$ 

**Lemma 2.14.** $\beta_{(0,k)} \in \text{End}(J, \ast)$ for all $k \in N$.

**Proof.** Let $u_r \in J$, and $v_r \in J_r$. First, assume $s \geq r$. Utilizing Lemma 1.20, $(u_r \ast v_r)\beta_{(0,k)} = (e, u_r, v_r)\beta_{(0,k)}$. Since $u_r\beta_r e_r, e_r u_r \beta_r e_r = e_r$. Hence, utilizing Lemma 2.4, $(e, u_r)\beta_{(0,k)} = (e, u_r)\beta_{(0,k)} v_r \beta_{(0,k)}$. Utilizing Definition 2.5, Note 1.8, Lemma 1.1, and Lemma 1.9, $(u_r \ast v_r)\beta_{(0,k)} = a^{-k} e_r (e, u_r) a^k = a^{-k} e_r a^k u_r a^k = (a^{-k} a^k u_r a^{-k}) = (u_r \beta_{(0,k)}).$ Since $v_r \beta_{(0,k)} \in J_{s+k}$ by Lemma 2.6(b), $u_r \ast (u_r \ast v_r) \beta_{(0,k)} = u_r \ast (v_r \ast v_r) \beta_{(0,k)}$ by Lemma 1.20. Thus, $(u_r \ast v_r) \beta_{(0,k)} = u_r \beta_{(0,k)} \ast v_r \beta_{(0,k)}$. We utilize Lemmas 1.20 and 1.9, and Definition 2.5 for the case $r > s$.

**Lemma 2.15.** $\beta_{(n,k)} \in \text{End}(J, \ast)$ for all $n, k \in N$.

**Proof.** Let $g, h \in J$. Hence, utilizing Lemmas 2.12, 2.13, and 2.14,

$$(g \ast h)\beta_{(n,k)} = (g \ast h)\beta_{(n,0,(0,k)} = (g \ast h)\beta_{(n,0)} \beta_{(0,k)} = (g\beta_{(n,0)} \ast h\beta_{(n,0)})\beta_{(0,k)} = g\beta_{(n,0)} \ast h\beta_{(n,0)} \ast \beta_{(0,k)} = g\beta_{(n,k)} \ast h\beta_{(n,k)}.$$ 

**Lemma 2.16.** $(n, k) \rightarrow \beta_{(n,k)}$ is a homomorphism of $C$ into $\text{End}(J, \ast)$.

**Proof.** Combine Lemmas 2.12 and 2.15.

If $a, b \in I$, define $a \circ b = ab$.

**Lemma 2.17.** $S \cong ((i, (n, k), j) : i \in I, j \in J, n, k \in N)$ under the multiplication $(i, (n, k), j)(u, (r, s), v) = (i \circ (u A_j \alpha_{(k, n)}), (n + r - \min(k, r), k + s - \min(k, r)), j \beta_{(r, s)} \ast v).$

**Proof.** Let $i \in I, j \in J, u \in I, v \in J$. Hence, utilizing Lemmas 1.24, 1.15, 1.11, 1.9, 2.6(b), 1.20, and Definition 2.5,

$$(ia^{-k} a^k)(ja^{-k} a^k) = ia^{-k} a^k u A_j ja^{-k} a^k v = ia^{-k} a^k u A_j ja^{-k} a^k v = i((u A_j) \alpha_{(k, n)} a^{-[n+r-\min(k, r)]} a^{k+s-\min(k, r)} j \beta_{(r, s)} \ast v).$$

Utilizing Lemma 2.6, $i \circ (u A_j) \alpha_{(k, n)} \in J_{n+r-\min(k, r)}$ and

$$j \beta_{(r, s)} \ast v \in J_{k+s-\min(k, r)}.$$
Hence, \(((i, (n, k), j) : i \in I, j \in J, n, k \in N)\) under the multiplication given in the statement of the lemma is a groupoid. The required isomorphism is given by the mapping \((ia^-a^*j)\varphi = (i, (n, k), j)\) by virtue of the above and Lemma 2.1.

**Lemma 2.18.** \(\alpha_{(r, s)}A_j\alpha_{(s, r)} = A_{j\beta_{(s, r)}}\) for all \(j \in J\) and \(r, s \in N\).

**Proof.** Let \(j \in J_p\) and \(w \in I_q\). Utilizing Definitions 2.2 and 2.5, and Lemmas 1.9, 1.12, 1.15, and 2.6,

\[
(j(w\alpha_{(r, s)}))\nu_{(s, r)} = a^-a^*j(w\alpha_{(r, s)})a^-a^*
\]

\[
= a^-a^*j\alpha_{(r, s)}a^-a^* = a^-a^*j\beta_{(s, r)}a^-a^*
\]

\[
= a^-a^*j\alpha_{(r, s)}a^-a^* = j\beta_{(s, r)}w = wA_{j\beta_{(s, r)}}^j\beta_{(s, r)}B_w.
\]

Utilizing Lemmas 1.15 and 2.6, \(wA_{j\beta_{(s, r)}} \in I_{\max\{q, r+p-\min(s, p)\}}\) and

\[
j\beta_{(s, r)}B_w \in J_{\max\{q, r+p-\min(s, p)\}}.
\]

Utilizing Definitions 2.2 and 2.5, and Lemmas 1.15 and 2.6,

\[
(j(w\alpha_{(r, s)}))\nu_{(s, r)} = a^-a^*j(w\alpha_{(r, s)})a^-a^*
\]

\[
= a^-a^*(w\alpha_{(r, s)}A_j)(jB_{\alpha_{(r, s)}})a^-a^*
\]

\[
= a^-a^*(w\alpha_{(r, s)}A_j)a^-a^*(jB_{\alpha_{(r, s)}})a^-a^*
\]

\[
= a^-a^*(w\alpha_{(r, s)}A_j)a^-a^*(a^-a^*jB_{\alpha_{(r, s)}})a^-a^*)
\]

\[
= (w\alpha_{(r, s)}A_j)\alpha_{(s, r)}(jB_{\alpha_{(r, s)}})\beta_{(s, r)}.
\]

Utilizing Lemmas 1.15 and 2.6, \(w\alpha_{(r, s)}A_j\alpha_{(s, r)} \in I_{\max\{p, s+q-\min(r, q)\}+r-s}\) and \((jB_{\alpha_{(r, s)}})\beta_{(s, r)} \in J_{\max\{p, s+q-\min(r, q)\}+r-s}\). Hence, \(w\alpha_{(r, s)}A_j\alpha_{(s, r)} = wA_{j\beta_{(s, r)}}\) by Lemma 1.14.

**Lemma 2.19.** (a) \(g\alpha_{(s, s)} = e_s \circ g\) for all \(g \in I\). (b) \(g\beta_{(s, s)} = g^*e_s\) for all \(g \in J\).

**Proof.** (a) Let \(g \in I\). Utilizing Lemma 1.12 \(g\alpha_{(s, s)} = (e_s)g = e_s \circ g\). (b) Let \(g \in J\). Utilizing Lemma 1.20, \(g\beta_{(s, s)} = e_sg = g^*e_s\).

In the following definition, we will describe the objects we will use to represent generalized \(\omega-\mathcal{L}\)-unipotent bisimple semigroups.

**Definition 2.20.** Let \((I, o)\) be an \(\omega\)-chain of left zero semigroups \((I_k : k \in N)\); let \((n, r) \to \alpha_{(n, r)}\) be a homomorphism of \(C\) into \(\text{End}(I, o)\); let \((J^*, *)\) be an \(\omega\)-chain of right groups \((J_k : k \in N)\); let \((n, r) \to \beta_{(n, r)}\) be a homomorphism of \(C\) into \(\text{End}(J, *)\); let \(j \to A_j\) be an upper anti-isomorphism of \((J, *)\) into \(T_j\); and let \({I_k \cap J_k = (e_k)}\), a single idempotent, for each \(k \in N\) such that...
(1) \( g\beta(\pi, \eta) = g^*e_\pi \) for all \( g \in J \).

(2) \( I_\alpha(n, k) \subseteq I_{k+r-\min(n, r)} \) and \( J_\beta(n, k) \subseteq J_{k+r-\min(n, r)} \).

(3) \( I_\alpha \subseteq I_{\max(r, \pi)} \) if \( j \in J_k \).

(4) \((r \circ s) A_j = r A_s \circ s A_x \) for \( r, s \in I \) with \( r \in I_u \) and \( x \in J \).

(5) \( \alpha_{(r, s)} A_j \alpha_{(s, r)} = A_{j\beta(r, s)} \) for all \( j \in J \) and \( r, s \in N \).

We denote \( ((i, (n, k), j) : i \in I_n, j \in J_k) \) under the multiplication

(6) \((i, (n, k), j)(u, (r, s), v)\)

\(= (i \circ (u A_j \alpha_{(k, n)}), (n + r - \min(k, r), k + s - \min(k, r)), j\beta_{(r, s)} v)\)

by \((I, J, \alpha, \beta, A)\).

**Theorem 2.21.** Let \( S \) be a generalized \( \omega-\beta \)-unipotent bisimple semigroup. Then, \( S \) is isomorphic to some \((I, J, \alpha, \beta, A)\).

**Proof.** The theorem is a consequence of the definition of “\( \omega \)”, Lemmas 1.12, 2.9, 1.21, 2.16, 1.23, the choice of “\( e_\pi \)”, Lemmas 2.19, 2.6, 1.15, 1.25, 2.18, and 2.17.

We thank the referee for the following remark.

**Remark 2.22.** In Definition 2.20, the middle component \((m, n)\) of \((i, (m, n), j)\) serves only as a marker. Hence, \( S \) is actually represented by the cartesian product \( I \times J \) under the multiplication

\((i, j)(u, v) = (i \circ (u A_j \alpha_{(k, n)}), j\beta_{(r, s)} v)\)

where \( i \in I_n, j \in J_k, u \in I_r, \) and \( v \in J_s \).

3. Structure theorem for generalized \( \omega-\beta \)-unipotent bisimple semigroups (proof of direct half). In this section, we show that \((I, J, \alpha, \beta, A)\) is a generalized \( \omega-\beta \)-unipotent bisimple semigroup.

**Lemma 3.1.** \((I, J, \alpha, \beta, A)\) is a semigroup.

**Proof.** We use (2) and (3) of Definition 2.20 to establish closure. We next establish associativity. Let \((i, (n, k), j)_1 = i\) and \((i, (n, k), j)_2 = ((m, n), j)\). Let \( a = (i, (n, k), j), b = (u, (r, s), v), \) and \( c = (z, (p, q), w) \in (I, J, \alpha, \beta, A)\). Utilizing the fact that \((n, r) \rightarrow \beta_{(n, r)}\) is a homomorphism,

\[(ab)c = ((i \circ (u A_j \alpha_{(k, n)}), (n, k)(r, s), j\beta_{(r, s)} v)(z, (p, q), w)) \]

\[= ((n, k)(r, s)(p, q), (j\beta_{(r, s)} v)\beta_{(p, q)} w) \]

\[= ((n, k)(r, s)(p, q), j\beta_{(r, s)(p, q)} v\beta_{(p, q)} w) \]
while

\[(a(bc))_{28} = ((i, (n, k), j)(u \circ (zA_s \alpha_{(s, r)}), (r, s)(p, q), v \beta_{(p, q)} * w))_{28} = ((n, k)(r, s)(p, q), j \beta_{(r, s);(p, q)} * v \beta_{(p, q)} * w).
\]

Hence, \((ab)c_{28} = (a(bc))_{28}\). Utilizing the fact \((k, n) \rightarrow \alpha_{(k, n)}\) is a homomorphism of \(C\) into \(\text{End} (I, o)\), the fact \(j \rightarrow A_j\) is an upper anti-homomorphism of \((J, *)\) into \(T, (5), (1), \) and \((4)\),

\[
((\alpha \delta)c)_X = i \circ u \circ zA_s \alpha_{(s, r)} \circ A_j \alpha_{(r, s)} = i \circ (u \circ zA_s \alpha_{(s, r)} \circ A_j \alpha_{(r, s)}).
\]

Hence, \((ab)c = a(bc)\).

**Lemma 3.2.** Let \((i, (n, k), j), (w, (p, q), z) \in (I, J, \alpha, \delta, \beta, A)\).

(a) \((i, (n, k), j) \mathcal{H}(w, (p, q), z)\) if and only if \(i = w\) and \(n = p\).

(b) \((i, (n, k), j) \mathcal{L}(w, (p, q), z)\) if and only if \(k = q\) and \((j, z) \in \mathcal{H}(e J_k)\).

**Proof.** (a) Let us show that \((i, (n, k), j) \mathcal{H}(i, (n, q), z)\). Let \(u \in I_k\). Hence, \(uA_s \alpha_{(k, n)} \in I_n\) by (2) and (3). Thus, since \((I_n, o)\) is a left zero semigroup, \(i \circ uA_s \alpha_{(k, n)} = i\). By (2), \(j \beta_{(k, q)} \in J_q\). Hence, since \((J_q, *)\) is a right group, there exists \(v \in J_q\) such that \(j \beta_{(k, q)} * v = z\). Hence, utilizing (6), \((i, (n, k), j)(u, (k, q), v) = (i, (n, q), z)\). Similarly, there exists \(a \in I_q\) such that \((i, (n, q), z)(a, (q, k), b) = (i, (n, k), j)\). Utilizing (6), the converse follows from the fact that \(\mathcal{H}\) is the dentity on \((I, 0)\) and \((n, k) \mathcal{H}(p, q)\) in \(C\) implies \(n = p\). Let us show that \((i, (n, k), j) \mathcal{L}(w, (p, k), z)\) if \((j, z) \in \mathcal{H}(e J_k)\). Since \((j, z) \in \mathcal{H}(e J_k)\), there exists \(u \in J_k\) such that \(u \beta_{(k, n)} = z\). By (2), \(u \beta_{(k, n)} \in J_k\). Utilizing (1) and the fact \((n, k) \rightarrow \beta_{(n, k)}\) is a homomorphism of \(C\) into \(\text{End} (J, *)\), \(u \beta_{(k, n)} \beta_{(n, k)} = u \beta_{(k, k)} = u \beta_{(k, l)}\). Hence, \((u \beta_{(k, n)} \beta_{(n, k)} \beta_{(k, l)} = u \beta_{(k, k)} \beta_{(k, l)} = u \beta_{(k, l)}\). Thus, utilizing (2), (3), and (6), \((w, (p, k), z) \in \mathcal{L}(e J_k)\). Similarly, there exists \(u \in J_k\) such that \((i, (n, p), v \beta_{(k, p)})(w, (p, k), z) = (i, (n, l), j)\). Utilizing (6), the converse follows from the fact that \(\mathcal{L} = \mathcal{L}\) in \((J, *)\) and \((n, k) \mathcal{L}(p, q)\) in \(C\) implies \(k = q\).
**Lemma 3.3.** \((I, J, \alpha, \beta, A)\) is a bisimple semigroup.

**Proof.** Let \((i, (n, k), j), (u, (r, s), v) \in (I, J, \alpha, \beta, A)\). Hence, utilizing Lemma 3.2, \((i, (n, k), j) \mathcal{R}(i, (n, s), v) \mathcal{L}(u, (r, s), v)\). \((I, J, \alpha, \beta, A)\) is a semigroup by Lemma 3.1.

**Lemma 3.4.** \(E(I, J, \alpha, \beta, A) = \{(i, (n, n), j): j \in E(J_n), n \in N\}\).

**Proof.** Let \((i, (n, k), j) \in E(I, J, \alpha, \beta, A)\). Hence, \((i, (n, k), j)(i, (n, k), j) = (i, (n, k), j)\). Using (6), \(n = k\) since \((n, k)^* = (n, k)\) in \(C\). Hence, using (6) and (1), \(j = j^*\). Utilizing (6), (2), (3), and (1), \(j \in E(J_n)\) implies \((i, (n, n), j) \in E(I, J, \alpha, \beta, A)\) for \(n \in N\) and \(i \in I_n\).

**Lemma 3.5.** \((I, J, \alpha, \beta, A)\) is a regular bisimple semigroup.

**Proof.** It follows from a result of Clifford and Miller [1, Theorem 2.11] that any bisimple semigroup containing an idempotent is regular. Hence, we just apply Lemmas 3.3 and 3.4.

**Lemma 3.6.** \(E(I, J, \alpha, \beta, A)\) is a semigroup.

**Proof.** We will utilize Lemma 3.4. Let \(a = (i, (n, n), j), b = (u, (s, s), v) \in E(I, J, \alpha, \beta, A)\). Hence, \(j \in E(J_n)\) and \(v \in E(J_s)\). Thus, using (1), \(j^*v = j^*e^*v = j^*v\). However, \(E(T)\) is a semigroup for any chain of right groups \(T\). Thus, it follows that \(j^*v \in E(J_{\max(s,s)})\). Hence, \(ab \in E(I, J, \alpha, \beta, A)\) by Lemma 3.4.

**Lemma 3.7.** \(\mathcal{L}\) is a congruence on the semigroup \(E(I, J, \alpha, \beta, A)\).

**Proof.** Let \(X\) be any semigroup such that \(E(X)\) is a semigroup. Then, it is easily seen that if \(e, f \in E(X), (e, f) \in \mathcal{L}(\in E(X))\) if and only if \((e, f) \in \mathcal{L}(\in E(X))\). Let \(j \in E(J_n)\) and \(v \in E(J_s)\). Hence, utilizing Lemmas 3.4 and 3.2(b), \((i, (n, n), j) \mathcal{L}(u, (s, s), v) \in E(I, J, \alpha, \beta, A)\) if and only if \(n = s\) and \(j = v\). Thus, using (6), \(\mathcal{L}\) is a left congruence on \(E(I, J, \alpha, \beta, A)\) by a routine calculation.

**Lemma 3.8.** \(E(I, J, \alpha, \beta, A)\) is an \(\omega\)-chain of rectangular bands \((E_n: n \in N)\) where \(E_n = \{(i, (n, n), j): i \in I_n, j \in E(J_n)\}\).

**Proof.** Let \((i, (n, n), j), (u, (n, n), v) \in E_n\). Utilizing (6), (2), (3), and a routine calculation, \((i, (n, n), j)(u, (n, n), v) = (i, (n, n), v)\). Hence, \(E_n\) is a rectangular band. Again, utilizing (6), (2), (3), and a routine calculation, \(E_nE_h \subseteq E_{\max(n,k)}\).
**Theorem 3.9.** 

(I, J, α, β, A) is a generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroup.

**Proof.** Combine Lemmas 3.5–3.8.

4. Structure of generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroups. Combining Theorems 4.1 and 4.3 (below) will give a description of generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroups in terms of groups, $\omega$-chains of left zero semigroups, and $\omega$-chains of right zero semigroups.

**Theorem 4.1.** 

(I, J, α, β, A) is a generalized $\omega$-$\mathcal{L}$-unipotent bisimple semigroup, and conversely every such semigroup is isomorphic to some (I, J, α, β, A).

**Proof.** Combine Theorems 3.9 and 2.21.

**Remark.** In contrast to the structure theorem for generalized $\mathcal{L}$-unipotent semigroups given in [4], no factor systems are required in Theorem 4.1.

We will next characterize an $\omega$-chain $J$ of right groups ($J_n: n \in N$) as a semi-direct product of an $\omega$-chain $X$ of right zero semigroups ($X_n: n \in N$) by an $\omega$-chain $G$ of groups ($G_n: n \in N$).

We first need a definition.

**Definition 4.2.** Let the semigroup $U$ be an $\omega$-chain of semigroups ($U_n: n \in N$) and let $\theta$ be a mapping of $U$ into a semigroup $V$ such that $r \in U_n$, $s \in U_m$, and $m \geq n$ imply $(rs)\theta = r\theta s\theta$. We term $\theta$ a lower homomorphism of $U$ into $V$.

Let $(G, \circ)$ be an $\omega$-chain of groups ($G_n: n \in N$) and let $(X, \ast)$ be an $\omega$-chain of right zero semigroups ($X_n: n \in N$) such that $G_n \cap X_n = \{e_n\}$, a single idempotent element, for each $n \in N$. Let $g \rightarrow B_g$ be a lower homomorphism of $G$ into $T_x$ subject to the conditions (1) $X_nB_g \subseteq X_{\max(s,m)}$ if $g \in G_m$ (2) if $r \in X_m, s \in X_n$ and $m \geq n$, $(rs)B_g = rB_{r_n \circ s} B_g$.

Let $(G, X, B)$ denote $\bigcup (G_n \times X_n: n \in N)$ under the multiplication $(i,j)(p, q) = (i \circ p, jB_s q)$.

**Theorem 4.3.** $J$ is an $\omega$-chain of right groups if and only if $J \cong (G, X, B)$ for some collection $G, X, B$.

**Proof.** We just specialize [6, Theorem 7.2].

**Note 4.4.** The structure of $G$ is known mod groups and homomorphisms by a well known result of Clifford [1, Theorem 4.11].
5. \(\omega\alpha\)-unipotent bisimple semigroups. In this section, we specialize Theorem 4.1 to obtain [5, Theorem 7.11] (our previous structure theorem for \(\omega\alpha\)-unipotent bisimple semigroups).

A bisimple semigroup \(S\) is termed \(\omega\alpha\)-unipotent if \(E(S)\) is an \(\omega\)-chain of right zero semigroups.

**Theorem 5.1.** Let \(S\) be an \(\omega\alpha\)-unipotent bisimple semigroup. Then, there exists an \(\omega\)-chain \((J, *)\) of right groups \((J_n: n \in \mathbb{N})\) and a homomorphism \((n, r) \mapsto \beta_{(n, r)}\) of \(C\) into \(\text{End}(J, *)\) such that for each \(k \in \mathbb{N}\) there exists \(e_k \in E(J_k)\) and

1. \(g \beta_{(k, k)} = g^* e_k\) for all \(g \in J\).
2. \(J \beta_{(n, k)} \subseteq J_{k+r+\min(s, r)}\). Furthermore, \(S \cong (((n, k), j): j \in J_k, n, k \in \mathbb{N})\) under the multiplication.
3. \(((n, k), j)(r, s), v) = ((n, k)(r, s), j \beta_{(r, s)}^* v)\) where juxtaposition denotes multiplication in \(C\).

Conversely, let \((J, *)\) be an \(\omega\)-chain of right groups and let \((n, r) \mapsto \beta_{(n, r)}\) be a homomorphism of \(C\) into \(\text{End}(J, *)\) such that (1) and (2) are valid. Then, \(S = (((n, k), j): j \in J_k, n, k \in \mathbb{N})\) under (3) is an \(\omega\alpha\)-unipotent bisimple semigroup.

**Proof.** We first establish the converse. We employ Theorem 4.1 and its notation. Let \(I_v = (e_v)\) for each \(v \in \mathbb{N}\) and define \(e_v \circ e_v = e_{\max\{s, v\}}\). Let \(I = \bigcup (I_v: v \in \mathbb{N})\). Then, \((I, o)\) is an \(\omega\)-chain of left zero semigroups \((I_n: n \in \mathbb{N})\). Define \(e_n \alpha_{(r, s)} = e_{s+n-\min(s, r)}\) and \(e_n A_s = e_{\max\{s, m\}}\) if \(v \in J_s\). By a routine calculation, \((n, r) \mapsto \alpha_{(n, r)}\) is a homomorphism of \(C\) into \((I, o)\) and \(p \mapsto A_p\) is an upper anti-homomorphism of \((J, *)\) into \(T_l\) such that (2)–(5) of Theorem 4.1 is valid. The multiplication (6) of Theorem 4.1 becomes (6') \((e_n, (n, k), j)(e_r, (r, s), v) = (e_{n+r+\min(k, r)}, (n, k)(r, s), \beta_{(r, s)}^* v)\) where juxtaposition is multiplication in \(C\). Hence, \(U = (I, J, \alpha, \beta, A)\) (notation of §3) is a generalized \(\omega\alpha\)-unipotent bisimple semigroup by Theorem 4.1. Utilizing Lemma 3.4, \(E(U) = ((e_n, (n, n), j): j \in E(J_n), n \in \mathbb{N})\). Utilizing Lemma 3.2, \((e_n, (n, n), j)\alpha_l \langle e_m, (k, k), w)(j \in E(J_n)\) and \(u \in E(J_m)\)) implies \(n = k\) and \(j = u\). Hence, \(E(U)\) is an \(\omega\)-chain of right zero semigroups and, thus, \(U\) is an \(\omega\alpha\)-unipotent bisimple semigroup. Since \((e_n, (n, k), j) = ((n, k), j)\) define an isomorphism of \((U, (6'))\) onto \((S, (3))\). \(S\) is an \(\omega\alpha\)-unipotent bisimple semigroup.

Next, let \(T\) be an \(\omega\alpha\)-unipotent bisimple semigroup. Hence, \(T\) is a generalized \(\omega\alpha\)-unipotent bisimple semigroup and the structure of \(T\) is given by Theorem 4.1. Thus, utilizing Lemmas 3.8 and 3.2, \(I_n = (e_n)\) for each \(n \in \mathbb{N}\). Hence, utilizing (2) and (3) of Theorem 4.1, \(e_r \alpha_{(s, k)} = e_{s+r-\min(s, r)}\) and \(e_r A_s = e_{\max\{s, k\}}\) if \(j \in J_k\). Thus, (6) of Theorem 4.1 becomes (6') and \((U, (6')) \cong (S, (3))\). The conditions of Theorem 5.1 are given by Theorem 4.1 ((1) and (2)).
REFERENCES

5. ———, $\omega Y$-unipotent semigroups, to appear in JNANABHA.

Received February 1, 1973.

UNIVERSITY OF ALABAMA IN BIRMINGHAM
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)  J. DUGUNDJI*
University of California  Department of Mathematics
Los Angeles, California 90024  University of Southern California

R. A. BEAUMONT  D. GILBARG AND J. MILGRAM
University of Washington  Stanford University
Seattle, Washington 98105  Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH  B. H. NEUMANN  F. WOLF  K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY  STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA  UNIVERSITY OF TOKYO
MONTANA STATE UNIVERSITY  UNIVERSITY OF UTAH
UNIVERSITY OF NEVADA  WASHINGTON STATE UNIVERSITY
NEW MEXICO STATE UNIVERSITY  UNIVERSITY OF WASHINGTON
OREGON STATE UNIVERSITY  *
UNIVERSITY OF OREGON  *
OSAKA UNIVERSITY  *

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The Pacific of Journal Mathematics is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsuisha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>William George Bade, <em>Two properties of the Sorgenfrey plane</em></td>
<td>349</td>
</tr>
<tr>
<td>John Robert Baxter and Rafael Van Severen Chacon, <em>Functionals on continuous functions</em></td>
<td>355</td>
</tr>
<tr>
<td>Phillip Wayne Bean, <em>Helly and Radon-type theorems in interval convexity spaces</em></td>
<td>363</td>
</tr>
<tr>
<td>James Robert Boone, <em>On k-quotient mappings</em></td>
<td>369</td>
</tr>
<tr>
<td>Ronald P. Brown, <em>Extended prime spots and quadratic forms</em></td>
<td>379</td>
</tr>
<tr>
<td>William Hugh Cornish, <em>Crowley’s completion of a conditionally upper continuous lattice</em></td>
<td>397</td>
</tr>
<tr>
<td>Robert S. Cunningham, <em>On finite left localizations</em></td>
<td>407</td>
</tr>
<tr>
<td>Robert Jay Daverman, <em>Approximating polyhedra in codimension one spheres embedded in s^n by tame polyhedra</em></td>
<td>417</td>
</tr>
<tr>
<td>Burton I. Fein, <em>Minimal splitting fields for group representations</em></td>
<td>427</td>
</tr>
<tr>
<td>Peter Fletcher and Robert Allen McCoy, <em>Conditions under which a connected representable space is locally connected</em></td>
<td>433</td>
</tr>
<tr>
<td>Jonathan Samuel Golan, <em>Topologies on the torsion-theoretic spectrum of a noncommutative ring</em></td>
<td>439</td>
</tr>
<tr>
<td>Manfred Gordon and Edward Martin Wilkinson, <em>Determinants of Petrie matrices</em></td>
<td>451</td>
</tr>
<tr>
<td>Alfred Peter Hallstrom, <em>A counterexample to a conjecture on an integral condition for determining peak points (counterexample concerning peak points)</em></td>
<td>455</td>
</tr>
<tr>
<td>E. R. Heal and Michael Windham, <em>Finitely generated F-algebras with applications to Stein manifolds</em></td>
<td>459</td>
</tr>
<tr>
<td>Denton Elwood Hewgill, <em>On the eigenvalues of a second order elliptic operator in an unbounded domain</em></td>
<td>467</td>
</tr>
<tr>
<td>Charles Royal Johnson, <em>The Hadamard product of A and A</em>^*</td>
<td>477</td>
</tr>
<tr>
<td>Darrell Conley Kent and Gary Douglas Richardson, <em>Regular completions of Cauchy spaces</em></td>
<td>483</td>
</tr>
<tr>
<td>Alan Greenwell Law and Ann L. McKerracher, <em>Sharpened polynomial approximation</em></td>
<td>491</td>
</tr>
<tr>
<td>Bruce Stephen Lund, <em>Subalgebras of finite codimension in the algebra of analytic functions on a Riemann surface</em></td>
<td>495</td>
</tr>
<tr>
<td>Robert Wilmer Miller, <em>TTF classes and quasi-generators</em></td>
<td>499</td>
</tr>
<tr>
<td>Roberta Mura and Akbar H. Rhemtulla, <em>Solvable groups in which every maximal partial order is isolated</em></td>
<td>509</td>
</tr>
<tr>
<td>Isaac Namioka, <em>Separate continuity and joint continuity</em></td>
<td>515</td>
</tr>
<tr>
<td>Edgar Andrews Rutter, <em>A characterization of QF – 3 rings</em></td>
<td>533</td>
</tr>
<tr>
<td>Alan Saleski, <em>Entropy of self-homeomorphisms of statistical pseudo-metric spaces</em></td>
<td>537</td>
</tr>
<tr>
<td>Ryōtarō Satō, <em>An Abel-maximal ergodic theorem for semi-groups</em></td>
<td>543</td>
</tr>
<tr>
<td>H. A. Seid, <em>Cyclic multiplication operators on L_p-spaces</em></td>
<td>549</td>
</tr>
<tr>
<td>H. B. Skerry, <em>On matrix maps of entire sequences</em></td>
<td>563</td>
</tr>
<tr>
<td>John Brendan Sullivan, <em>A proof of the finite generation of invariants of a normal subgroup</em></td>
<td>571</td>
</tr>
<tr>
<td>John Griggs Thompson, <em>Nonsolvable finite groups all of whose local subgroups are solvable, VI</em></td>
<td>573</td>
</tr>
<tr>
<td>Ronson Joseph Warne, <em>Generalized ω – L-unipotent bisimple semigroups</em></td>
<td>631</td>
</tr>
<tr>
<td>Toshihiko Yamada, <em>On a splitting field of representations of a finite group</em></td>
<td>649</td>
</tr>
</tbody>
</table>