ON A SPLITTING FIELD OF REPRESENTATIONS OF A FINITE GROUP

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The theorem of P. Fong about a splitting field of representations of a finite group $G$ will be improved to the effect that the order of $G$ mentioned in it will be replaced by the exponent of $G$. The proof depends on the Brauer-Witt theorem and properties of cyclotomic algebras.

Let $Q$ denote the rational field. For a positive integer $n$, $\zeta_n$ is a primitive $n$th root of unity. Let $\chi$ be an irreducible character of a finite group $G$ (an irreducible character means an absolutely irreducible one). Let $K$ be a field of characteristic 0. Then $m_\chi(\chi)$ denotes the Schur index of $\chi$ over $K$. The simple component of the group algebra $K[G]$ corresponding to $\chi$ is denoted by $A(\chi, K)$. Its index is exactly $m_\chi(\chi)$. If $L/K$ is normal, $\mathcal{E}(L/K)$ is the Galois group of $L$ over $K$.

In this paper we will prove the following:

**THEOREM.** Let $G$ be a finite group of exponent $s = l^n$, where $l$ is a rational prime and $(l, n) = 1$. Let $k = Q(\zeta_n)$ if $l$ is odd, let $k = Q(\zeta_n, \zeta_4)$ if $l = 2$. Then, $m_k(\chi) = 1$ for every irreducible character $\chi$ of $G$.

**REMARK.** In Fong [2, Theorem 1], the above $s$ denoted the order of $G$ (instead of the exponent of $G$).

First we review

**BRAUER-WITT THEOREM.** Let $\chi$ be an irreducible character of a finite group $G$ of exponent $s$. Let $q$ be a prime number. Let $K$ be a field of characteristic 0 with $K(\chi) = K$. Let $L$ be the subfield of $K(\zeta_n)$ over $K$ such that $[K(\zeta_n): L]$ is a power of $q$ and $[L: K] \neq 0 \pmod{q}$. Then there is a subgroup $F$ of $G$ and an irreducible character $\xi$ of $F$ with the following properties: (1) there is a normal subgroup $N$ of $F$ and a linear character $\psi$ of $N$ such that $L(\xi) = L$, (2) $F/N \cong \mathcal{E}(L(\psi)/L)$, (3) $m_\psi(\xi)$ is equal to the $q$-part of $m_\chi(\chi)$, (4) for every $f \in F$ there is a $\tau(f) \in \mathcal{E}(L(\psi)/L)$ such that $\psi(f n f^{-1}) = \tau(f)(\psi(n))$ for all $n \in N$, and (5) $A(\xi, L)$ is isomorphic to the crossed product $(\beta, L(\psi)/L)$ where, if $S$ is a complete set of coset representatives of $N$ in $F$ ($1 \in S$) with $f f' = n(f, f') f''$ for $f, f', f'' \in S$, $n(f, f') \in N$, then $\beta(\tau(f), \tau(f')) = \psi(n(f, f'))$. 

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Proof. See, for instance, [1] and [4].

REMARK. The above crossed product is called a cyclotomic algebra (cf. [3]).

COROLLARY. Let $p$ be a prime number. Denote by $Q_p$ the rational $p$-adic field. Suppose that $p \nmid s$ if $p \neq 2$, and that $4 \nmid s$ if $p = 2$, $s$ being the exponent of $G$. Then $m_{Q_p}(\chi) = 1$ for every irreducible character $\chi$ of $G$.

Proof. Set $K = Q_p(\chi)$. Then $m_K(\chi) = m_{Q_p}(\chi)$. Let $q$ be any prime number. By the Brauer-Witt theorem, the $q$-part of $m_K(\chi)$ equals the index of some cyclotomic algebra of the form $(\beta, L(\psi)/L)$, where $Q_p \subset K \subset L \subset L(\psi) \subset Q_p(\zeta_s)$. It follows from the assumption that the extension $Q_p(\zeta_s)/Q_p$ is unramified, a fortiori, $L(\psi)/L$ is unramified. Because the values of the factor set $\beta$ are roots of unity, it follows that $(\beta, L(\psi)/L) \sim L$. As $q$ is an arbitrary prime, we conclude that $m_K(\chi) = 1$.

For the remainder of the paper we will use the same notation as in the theorem. Recall that $m_k(\chi)$ is the index of $A(\chi, k(\chi))$. Hence it suffices to prove $A(\chi, k(\chi)) \otimes_{k(\chi)} k(\chi) \sim k(\chi)$, for every prime $p$ of $k(\chi)$, where $k(\chi)_p$ is the completion of $k(\chi)$ with respect to $p$. For simplicity, set $K = k(\chi)$. Because $A(\chi, k(\chi)) \otimes_{k(\chi)} K$ is $K$-isomorphic to $A(\chi, K)$, we need to show $A(\chi, K) \sim K$, i.e., $m_K(\chi) = 1$. Note that $k(\chi)$ is a cyclotomic extension of the rational field $Q$. If $M$ is a cyclotomic extension of $Q$ containing $k(\chi)$, then $M^\flat$ represents the isomorphy type of the completion $M_\wp$, $\wp$ being any prime of $M$ dividing $p$.

(i) Suppose that $p$ is an infinite prime. Denote by $R$ (resp. $C$) the field of real numbers (resp. complex numbers). If $k(\chi)$ is not real, then $p$ is a complex prime, and so $m_K(\chi) = 1$. Suppose that $k(\chi)$ is real. Then $K = k(\chi)_p = R$, $l \neq 2$, and $n = 1$ or $2$, i.e., $k = Q(\zeta_n) = Q$ and $\chi$ is real valued. Therefore, $4$ does not divide $s$, the exponent of $G$. If $s = 1$ or $2$, then $G$ is abelian, and so $m_K(\chi) = 1$. Hence we assume that $s > 2$, so that the field $Q(\zeta_s)$ is imaginary and $R = K \subset Q(\zeta_s)^v = C$. Note that $m_k(\chi) = 1$ or $2$. By the Brauer-Witt theorem there are subgroups $F$ and $N$ of $G$ and a linear character $\psi$ of $N$ such that $F \supset N$ and $R(\psi^\flat) = R(\chi) = R$ and that $m_\psi(\chi)$ is equal to the index of a cyclotomic algebra of the form $(\beta, R(\psi)/R)$. Recall that $z(R(\psi)/R) \equiv F/N$. If $R(\psi) = R$, then $(\beta, R(\psi)/R) \sim R$. If $R(\psi) = C$, then $[F: N] = 2$. Set $F = N \cup N\bar{\psi}$. We have

$$(\beta, R(\psi)/R) = (\psi(\tau^\flat), C/R, \rho), \quad (\rho(\sqrt{-1}) = -\sqrt{-1})$$
where the right side denotes a cyclic algebra over $R$ and $\psi(f^2)$ is a root of unity contained in $R$ so that $\psi(f^2) = \pm 1$. If $\psi(f^2) = -1$, then the order of $f$ would be divisible by 4, which is a contradiction. Consequently, $\psi(f^2) = 1$ and so $(\psi(f^2), C/R, \rho) \sim R$, yielding that $m_K(\chi) = 1$.

(ii) Suppose that $p$ does not divide $s = l^n$. Then the corollary implies that $m_K(\chi) = 1$.

(iii) Suppose that $p \mid l$ and $l = 2$. Then $\zeta_4 \in K$, and so $\zeta_4 \in K$. It follows from [3, Satz 12] that $m_K(\chi) = 1$.

(iv) Suppose that $p \mid l$ and $l \neq 2$. Let $q$ be a prime number. Let $L$ be the subfield of $M = \mathbb{Q}(\zeta_{12}, \zeta_n)$ over $K = k(\chi) = \mathbb{Q}(\zeta_n, \chi)$, such that $q \nmid [L: K]$ and $[M: L]$ is a power of $q$. By the Brauer-Witt theorem there exist subgroups $F$ and $N$ of $G$ and a linear character $\psi$ of $N$ such that $G \supset F \supset N$, $\mathcal{E}(L(\psi)/L) \cong F/N$, $[F: N]$ is a power of $q$, and the $q$-part of $m_K(\chi)$ is equal to the index of a cyclotomic algebra of the form $(\beta, L(\psi)/L)$. Since $l \neq 2$ and $\mathcal{E}(M/K)$ is canonically isomorphic to a subgroup of $\mathcal{E}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, it follows that $M/K$ is cyclic, and so $L(\psi)/L$ is cyclic. Let $\rho = [F: N] = [L(\psi): L]$, $\langle \sigma \rangle = \mathcal{E}(L(\psi)/L)$ and $F = \bigcup_{i=0}^{q^i} N_i$. Then we have

$$(\beta, L(\psi)/L) = (\psi(f^{q^i}), L(\psi)/L, \sigma), \quad \psi(f^{q^i}) \in L.$$ 

As $\psi$ is a linear character, $\psi(f^{q^i})$ is a primitive $t$th root of unity for some integer $t$. Let $t = q^i h, (q, h) = 1$. Then we can write $\psi(f^{q^i}) = \zeta_{q^i h}^{\zeta_h}$, which implies that the order of $f$ is divisible by $q^{i+d}$. Consequently, $q^{i+d}$ divides $n$, and so a primitive $q^{i+d}$th root of unity $\zeta_{q^{i+d}}$ belongs to $L$. We may assume that $\zeta_{q^{i+d}} = \zeta_h$. Let $r$ be an integer satisfying $rq^i = 1 \pmod{h}$. Since both $\zeta_{q^{i+d}}$ and $\zeta_h$ belong to $L$, it follows that

$$N_{L(\psi)/L}(\zeta_{q^{i+d}}^{\zeta_h}) = \zeta_{q^{i+d}}^{r \cdot \zeta_h} = \zeta_h^{r \cdot \zeta_h},$$

which yields that $(\psi(f^{q^i}), L(\psi)/L, \sigma) \sim L$. Therefore, the $q$-part of $m_K(\chi)$ is equal to 1. As $q$ is an arbitrary prime, it follows that $m_K(\chi) = 1$.

(v) Suppose that $p \mid n$ and $p \nmid 2$. Then $k$ contains a primitive $p$th root of unity $\zeta_p$, $p$ being the rational prime divided by $p$. It follows from [3, Satz 12] that $m_K(\chi) = 1$.

(vi) Suppose that $p \mid n$ and $p \mid 2$. Then $k = \mathbb{Q}(\zeta_n)$. If $4 \mid n$ then $\zeta_n \in K$ and so $m_K(\chi) = 1$. If $4 \nmid n$, then $4 \nmid s$. It follows from the corollary that $m_K(\chi) = 1$.

The theorem is completely proved.

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Received February 21, 1973. This research was done while the author was a Visiting Associate Professor of Queen's University for 1971/72.

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