

# Pacific Journal of Mathematics

**ON A SPLITTING FIELD OF REPRESENTATIONS OF A FINITE  
GROUP**

TOSHIHIKO YAMADA

## ON A SPLITTING FIELD OF REPRESENTATIONS OF A FINITE GROUP

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**The theorem of P. Fong about a splitting field of representations of a finite group  $G$  will be improved to the effect that the order of  $G$  mentioned in it will be replaced by the exponent of  $G$ . The proof depends on the Brauer-Witt theorem and properties of cyclotomic algebras.**

Let  $Q$  denote the rational field. For a positive integer  $n$ ,  $\zeta_n$  is a primitive  $n$ th root of unity. Let  $\chi$  be an irreducible character of a finite group  $G$  (an irreducible character means an absolutely irreducible one). Let  $K$  be a field of characteristic 0. Then  $m_K(\chi)$  denotes the Schur index of  $\chi$  over  $K$ . The simple component of the group algebra  $K[G]$  corresponding to  $\chi$  is denoted by  $A(\chi, K)$ . Its index is exactly  $m_K(\chi)$ . If  $L/K$  is normal,  $\mathcal{S}(L/K)$  is the Galois group of  $L$  over  $K$ .

In this paper we will prove the following:

**THEOREM.** *Let  $G$  be a finite group of exponent  $s = l^n n$ , where  $l$  is a rational prime and  $(l, n) = 1$ . Let  $k = Q(\zeta_n)$  if  $l$  is odd, let  $k = Q(\zeta_n, \zeta_4)$  if  $l = 2$ . Then,  $m_k(\chi) = 1$  for every irreducible character  $\chi$  of  $G$ .*

**REMARK.** In Fong [2, Theorem 1], the above  $s$  denoted the order of  $G$  (instead of the exponent of  $G$ ).

First we review

**BRAUER-WITT THEOREM.** *Let  $\chi$  be an irreducible character of a finite group  $G$  of exponent  $s$ . Let  $q$  be a prime number. Let  $K$  be a field of characteristic 0 with  $K(\chi) = K$ . Let  $L$  be the subfield of  $K(\zeta_s)$  over  $K$  such that  $[K(\zeta_s):L]$  is a power of  $q$  and  $[L:K] \not\equiv 0 \pmod{q}$ . Then there is a subgroup  $F$  of  $G$  and an irreducible character  $\xi$  of  $F$  with the following properties: (1) there is a normal subgroup  $N$  of  $F$  and a linear character  $\psi$  of  $N$  such that  $\xi = \psi^F$  and  $L(\xi) = L$ , (2)  $F/N \cong \mathcal{S}(L(\psi)/L)$ , (3)  $m_L(\xi)$  is equal to the  $q$ -part of  $m_K(\chi)$ , (4) for every  $f \in F$  there is a  $\tau(f) \in \mathcal{S}(L(\psi)/L)$  such that  $\psi(fnf^{-1}) = \tau(f)(\psi(n))$  for all  $n \in N$ , and (5)  $A(\xi, L)$  is isomorphic to the crossed product  $(\beta, L(\psi)/L)$  where, if  $S$  is a complete set of coset representatives of  $N$  in  $F$  ( $1 \in S$ ) with  $ff' = n(f, f')f''$  for  $f, f', f'' \in S$ ,  $n(f, f') \in N$ , then  $\beta(\tau(f), \tau(f')) = \psi(n(f, f'))$ .*

*Proof.* See, for instance, [1] and [4].

REMARK. The above crossed product is called a cyclotomic algebra (cf. [3]).

COROLLARY. Let  $p$  be a prime number. Denote by  $Q_p$  the rational  $p$ -adic field. Suppose that  $p \nmid s$  if  $p \neq 2$ , and that  $4 \nmid s$  if  $p = 2$ ,  $s$  being the exponent of  $G$ . Then  $m_{Q_p}(\chi) = 1$  for every irreducible character  $\chi$  of  $G$ .

*Proof.* Set  $K = Q_p(\chi)$ . Then  $m_K(\chi) = m_{Q_p}(\chi)$ . Let  $q$  be any prime number. By the Brauer-Witt theorem, the  $q$ -part of  $m_K(\chi)$  equals the index of some cyclotomic algebra of the form  $(\beta, L(\psi)/L)$ , where  $Q_p \subset K \subset L \subset L(\psi) \subset Q_p(\zeta_s)$ . It follows from the assumption that the extension  $Q_p(\zeta_s)/Q_p$  is unramified, a fortiori,  $L(\psi)/L$  is unramified. Because the values of the factor set  $\beta$  are roots of unity, it follows that  $(\beta, L(\psi)/L) \sim L$ . As  $q$  is an arbitrary prime, we conclude that  $m_K(\chi) = 1$ .

For the remainder of the paper we will use the same notation as in the theorem. Recall that  $m_k(\chi)$  is the index of  $A(\chi, k(\chi))$ . Hence it suffices to prove  $A(\chi, k(\chi)) \otimes_{k(\chi)} k(\chi)_\mathfrak{p} \sim k(\chi)_\mathfrak{p}$  for every prime  $\mathfrak{p}$  of  $k(\chi)$ , where  $k(\chi)_\mathfrak{p}$  is the completion of  $k(\chi)$  with respect to  $\mathfrak{p}$ . For simplicity, set  $K = k(\chi)_\mathfrak{p}$ . Because  $A(\chi, k(\chi)) \otimes_{k(\chi)} K$  is  $K$ -isomorphic to  $A(\chi, K)$ , we need to show  $A(\chi, K) \sim K$ , i.e.,  $m_K(\chi) = 1$ . Note that  $k(\chi)$  is a cyclotomic extension of the rational field  $Q$ . If  $M$  is a cyclotomic extension of  $Q$  containing  $k(\chi)$ , then  $M^\mathfrak{p}$  represents the isomorphy type of the completion  $M_\mathfrak{p}$ ,  $\mathfrak{p}$  being any prime of  $M$  dividing  $\mathfrak{p}$ .

(i) Suppose that  $\mathfrak{p}$  is an infinite prime. Denote by  $R$  (resp.  $C$ ) the field of real numbers (resp. complex numbers). If  $k(\chi)$  is not real, then  $\mathfrak{p}$  is a complex prime, and so  $m_K(\chi) = 1$ . Suppose that  $k(\chi)$  is real. Then  $K = k(\chi)_\mathfrak{p} = R$ ,  $l \neq 2$ , and  $n = 1$  or  $2$ , i.e.,  $k = Q(\zeta_n) = Q$  and  $\chi$  is real valued. Therefore, 4 does not divide  $s$ , the exponent of  $G$ . If  $s = 1$  or  $2$ , then  $G$  is abelian, and so  $m_k(\chi) = 1$ . Hence we assume that  $s > 2$ , so that the field  $Q(\zeta_s)$  is imaginary and  $R = K \subset Q(\zeta_s)^\mathfrak{p} = C$ . Note that  $m_K(\chi) = 1$  or  $2$ . By the Brauer-Witt theorem there are subgroups  $F$  and  $N$  of  $G$  and a linear character  $\psi$  of  $N$  such that  $F \triangleright N$  and  $R(\psi^F) = R(\chi) = R$  and that  $m_R(\chi)$  is equal to the index of a cyclotomic algebra of the form  $(\beta, R(\psi)/R)$ . Recall that  $\mathcal{S}(R(\psi)/R) \cong F/N$ . If  $R(\psi) = R$ , then  $(\beta, R(\psi)/R) \sim R$ . If  $R(\psi) = C$ , then  $[F:N] = 2$ . Set  $F = N \cup Nf$ . We have

$$(\beta, R(\psi)/R) = (\psi(f^2), C/R, \rho), \quad (\rho(\sqrt{-1}) = -\sqrt{-1})$$

where the right side denotes a cyclic algebra over  $R$  and  $\psi(f^2)$  is a root of unity contained in  $R$  so that  $\psi(f^2) = \pm 1$ . If  $\psi(f^2) = -1$ , then the order of  $f$  would be divisible by 4, which is a contradiction. Consequently,  $\psi(f^2) = 1$  and so  $(\psi(f^2), C/R, \rho) \sim R$ , yielding that  $m_K(\chi) = 1$ .

(ii) Suppose that  $p$  does not divide  $s = l^n$ . Then the corollary implies that  $m_K(\chi) = 1$ .

(iii) Suppose that  $p \mid l$  and  $l = 2$ . Then  $\zeta_4 \in k$ , and so  $\zeta_4 \in K$ . It follows from [3, Satz 12] that  $m_K(\chi) = 1$ .

(iv) Suppose that  $p \mid l$  and  $l \neq 2$ . Let  $q$  be a prime number. Let  $L$  be the subfield of  $M = Q(\zeta_{l^a}, \zeta_n)^p$  over  $K = k(\chi)_p = Q(\zeta_n, \chi)_p$  such that  $q \nmid [L:K]$  and  $[M:L]$  is a power of  $q$ . By the Brauer-Witt theorem there exist subgroups  $F$  and  $N$  of  $G$  and a linear character  $\psi$  of  $N$  such that  $G \supset F \triangleright N$ ,  $\mathcal{G}(L(\psi)/L) \cong F/N$ ,  $[F:N]$  is a power of  $q$ , and the  $q$ -part of  $m_K(\chi)$  is equal to the index of a cyclotomic algebra of the form  $(\beta, L(\psi)/L)$ . Since  $l \neq 2$  and  $\mathcal{G}(M/K)$  is canonically isomorphic to a subgroup of  $\mathcal{G}(Q(\zeta_{l^a})/Q)$ , it follows that  $M/K$  is cyclic, and so  $L(\psi)/L$  is cyclic. Let  $q^e = [F:N] = [L(\psi):L]$ ,  $\langle \sigma \rangle = \mathcal{G}(L(\psi)/L)$  and  $F = \bigcup_{i=0}^{q^e-1} Nf^i$ . Then we have

$$(\beta, L(\psi)/L) = (\psi(f^{q^e}), L(\psi)/L, \sigma), \quad \psi(f^{q^e}) \in L.$$

As  $\psi$  is a linear character,  $\psi(f^{q^e})$  is a primitive  $t$ th root of unity for some integer  $t$ . Let  $t = q^d h$ ,  $(q, h) = 1$ . Then we can write  $\psi(f^{q^e}) = \zeta_{q^d} \zeta_h$ , which implies that the order of  $f$  is divisible by  $q^{e+d}$ . Consequently,  $q^{e+d}$  divides  $n$ , and so a primitive  $q^{e+d}$ th root of unity  $\zeta_{q^{e+d}}$  belongs to  $L$ . We may assume that  $\zeta_{q^{e+d}} = \zeta_{q^d}$ . Let  $r$  be an integer satisfying  $rq^e \equiv 1 \pmod{h}$ . Since both  $\zeta_{q^{e+d}}$  and  $\zeta_h$  belong to  $L$ , it follows that

$$N_{L(\psi)/L}(\zeta_{q^{e+d}} \zeta_h^r) = \zeta_{q^{e+d}}^{q^e} \zeta_h^{r q^e} = \zeta_{q^d} \zeta_h,$$

which yields that  $(\psi(f^{q^e}), L(\psi)/L, \sigma) \sim L$ . Therefore, the  $q$ -part of  $m_K(\chi)$  is equal to 1. As  $q$  is an arbitrary prime, it follows that  $m_K(\chi) = 1$ .

(v) Suppose that  $p \mid n$  and  $p \nmid 2$ . Then  $k$  contains a primitive  $p$ th root of unity  $\zeta_p$ ,  $p$  being the rational prime divided by  $p$ . It follows from [3, Satz 12] that  $m_K(\chi) = 1$ .

(vi) Suppose that  $p \mid n$  and  $p \mid 2$ . Then  $k = Q(\zeta_n)$ . If  $4 \mid n$  then  $\zeta_4 \in K$  and so  $m_K(\chi) = 1$ . If  $4 \nmid n$ , then  $4 \nmid s$ . It follows from the corollary that  $m_K(\chi) = 1$ .

The theorem is completely proved.

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