ON A SPLITTING FIELD OF REPRESENTATIONS OF A FINITE GROUP

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The theorem of P. Fong about a splitting field of representations of a finite group $G$ will be improved to the effect that the order of $G$ mentioned in it will be replaced by the exponent of $G$. The proof depends on the Brauer-Witt theorem and properties of cyclotomic algebras.

Let $Q$ denote the rational field. For a positive integer $n$, $\zeta_n$ is a primitive $n$th root of unity. Let $\chi$ be an irreducible character of a finite group $G$ (an irreducible character means an absolutely irreducible one). Let $K$ be a field of characteristic 0. Then $m_K(\chi)$ denotes the Schur index of $\chi$ over $K$. The simple component of the group algebra $K[G]$ corresponding to $\chi$ is denoted by $A(\chi, K)$. Its index is exactly $m_K(\chi)$. If $L/K$ is normal, $G(L/K)$ is the Galois group of $L$ over $K$.

In this paper we will prove the following:

**Theorem.** Let $G$ be a finite group of exponent $s = l^n$, where $l$ is a rational prime and $(l, n) = 1$. Let $k = Q(\zeta_n)$ if $l$ is odd, let $k = Q(\zeta_n, \zeta_4)$ if $l = 2$. Then, $m_K(\chi) = 1$ for every irreducible character $\chi$ of $G$.

**Remark.** In Fong [2, Theorem 1], the above $s$ denoted the order of $G$ (instead of the exponent of $G$).

First we review

**Brauer-Witt Theorem.** Let $\chi$ be an irreducible character of a finite group $G$ of exponent $s$. Let $q$ be a prime number. Let $K$ be a field of characteristic 0 with $K(\chi) = K$. Let $L$ be the subfield of $K(\zeta_s)$ over $K$ such that $[K(\zeta_s): L]$ is a power of $q$ and $[L: K] \equiv 0 \pmod{q}$. Then there is a subgroup $F$ of $G$ and an irreducible character $\zeta$ of $F$ with the following properties: (1) there is a normal subgroup $N$ of $F$ and a linear character $\varphi$ of $N$ such that $\zeta = \varphi^F$ and $L(\zeta) = L$, (2) $F/N \cong G(L(\varphi)/L)$, (3) $m_L(\zeta)$ is equal to the $q$-part of $m_K(\chi)$, (4) for every $f \in F$ there is a $\tau(f) \in G(L(\varphi)/L)$ such that $\varphi(f n f^{-1}) = \tau(f)(\varphi(n))$ for all $n \in N$, and (5) $A(\zeta, L)$ is isomorphic to the crossed product $(\beta, L(\varphi)/L)$ where, if $S$ is a complete set of coset representatives of $N$ in $F$ ($1 \in S$) with $f f' = n(f, f') f''$ for $f, f', f'' \in S$, $n(f, f') \in N$, then $\beta(\tau(f), \tau(f')) = \varphi(n(f, f'))$. 

649
Proof. See, for instance, [1] and [4].

Remark. The above crossed product is called a cyclotomic algebra (cf. [3]).

Corollary. Let \( p \) be a prime number. Denote by \( Q_p \) the rational \( p \)-adic field. Suppose that \( p \nmid s \) if \( p \neq 2 \), and that \( 4 \nmid s \) if \( p = 2 \), \( s \) being the exponent of \( G \). Then \( m_{Q_p}(\chi) = 1 \) for every irreducible character \( \chi \) of \( G \).

Proof. Set \( K = Q_p(\chi) \). Then \( m_K(\chi) = m_{Q_p}(\chi) \). Let \( q \) be any prime number. By the Brauer-Witt theorem, the \( q \)-part of \( m_K(\chi) \) equals the index of some cyclotomic algebra of the form \( (\beta, L(\psi)/L) \), where \( Q_p \subset K \subset L \subset L(\psi) \subset Q_p(\zeta_s) \). It follows from the assumption that the extension \( Q_p(\zeta_n)/Q_p \) is unramified, a fortiori, \( L(\psi)/L \) is unramified. Because the values of the factor set \( \beta \) are roots of unity, it follows that \( (\beta, L(\psi)/L) \sim L \). As \( q \) is an arbitrary prime, we conclude that \( m_K(\chi) = 1 \).

For the remainder of the paper we will use the same notation as in the theorem. Recall that \( m_k(\chi) \) is the index of \( A(\chi, k(\chi)) \). Hence it suffices to prove \( A(\chi, k(\chi)) \otimes_{k(\chi)} k(\chi)_p \sim k(\chi)_p \) for every prime \( p \) of \( k(\chi) \), where \( k(\chi)_p \) is the completion of \( k(\chi) \) with respect to \( p \). For simplicity, set \( K = k(\chi)_p \). Because \( A(\chi, k(\chi)) \otimes_{k(\chi)} K \) is \( K \)-isomorphic to \( A(\chi, K) \), we need to show \( A(\chi, K) \sim K \), i.e., \( m_K(\chi) = 1 \). Note that \( k(\chi) \) is a cyclotomic extension of the rational field \( Q \). If \( M \) is a cyclotomic extension of \( Q \) containing \( k(\chi) \), then \( M^p \) represents the isomorphy type of the completion \( M_\mathfrak{p} \), \( \mathfrak{p} \) being any prime of \( M \) dividing \( p \).

(i) Suppose that \( p \) is an infinite prime. Denote by \( R \) (resp. \( C \)) the field of real numbers (resp. complex numbers). If \( k(\chi) \) is not real, then \( p \) is a complex prime, and so \( m_K(\chi) = 1 \). Suppose that \( k(\chi) \) is real. Then \( K = k(\chi)_p = R \), \( l \neq 2 \), and \( n = 1 \) or \( 2 \), i.e., \( k = Q(\zeta_n) = Q \) and \( \chi \) is real valued. Therefore, \( 4 \) does not divide \( s \), the exponent of \( G \). If \( s = 1 \) or \( 2 \), then \( G \) is abelian, and so \( m_k(\chi) = 1 \). Hence we assume that \( s > 2 \), so that the field \( Q(\zeta_s) \) is imaginary and \( R = K \subset Q(\zeta_s)^r = C \). Note that \( m_K(\chi) = 1 \) or \( 2 \). By the Brauer-Witt theorem there are subgroups \( F \) and \( N \) of \( G \) and a linear character \( \psi \) of \( N \) such that \( F \triangleright N \) and \( R(\psi^F) = R(\chi) = R \) and that \( m_R(\chi) \) is equal to the index of a cyclotomic algebra of the form \( (\beta, R(\psi)/R) \). Recall that \( \otimes (R(\psi)/R) \cong F/N \). If \( R(\psi) = R \), then \( (\beta, R(\psi)/R) \sim R \). If \( R(\psi) = C \), then \( [F: N] = 2 \). Set \( F = N \cup Nf \). We have

\[
(\beta, R(\psi)/R) = (\psi(f^*), C/R, \rho), \quad (\rho(\sqrt{-1}) = -\sqrt{-1})
\]
where the right side denotes a cyclic algebra over \( R \) and \( \psi(f^r) \) is a root of unity contained in \( R \) so that \( \psi(f^r) = \pm 1 \). If \( \psi(f^r) = -1 \), then the order of \( f \) would be divisible by 4, which is a contradiction. Consequently, \( \psi(f^r) = 1 \) and so \( (\psi(f^r), C/R, \rho) \sim R \), yielding that \( m_X(\chi) = 1 \).

(ii) Suppose that \( p \) does not divide \( s = l^e n \). Then the corollary implies that \( m_X(\chi) = 1 \).

(iii) Suppose that \( p \mid l \) and \( l = 2 \). Then \( \zeta_t \in k \), and so \( \zeta_t \in K \). It follows from [3, Satz 12] that \( m_X(\chi) = 1 \).

(iv) Suppose that \( p \nmid l \) and \( l \neq 2 \). Let \( q \) be a prime number. Let \( L \) be the subfield of \( M = \mathbb{Q}(\zeta_{ta}, \zeta_n)^p \) over \( K = k(\chi)_s = \mathbb{Q}(\zeta_n, \chi)_s \) such that \( q \nmid [L: K] \) and \( [M: L] \) is a power of \( q \). By the Brauer-Witt theorem there exist subgroups \( F \) and \( N \) of \( G \) and a linear character \( \psi \) of \( N \) such that \( G \supset F \supset N \), \( \mathcal{G}(L(\psi)/L) \cong F/N \), \([F: N]\) is a power of \( q \), and the \( q \)-part of \( m_X(\chi) \) is equal to the index of a cyclotomic algebra of the form \((\beta, L(\psi)/L)\). Since \( l \neq 2 \) and \( \mathcal{G}(M/K) \) is canonically isomorphic to a subgroup of \( \mathcal{G}(\mathbb{Q}(\zeta_t)/\mathbb{Q}) \), it follows that \( M/K \) is cyclic, and so \( L(\psi)/L \) is cyclic. Let \( q^e = [F: N] = [L(\psi): L], \langle \sigma \rangle = \mathcal{G}(L(\psi)/L) \) and \( F = \bigcup_{i=0}^{q^e-1} Nf^i \). Then we have

\[
(\beta, L(\psi)/L) = (\psi(f^{q^e}), L(\psi)/L, \sigma), \quad \psi(f^{q^e}) \in L.
\]

As \( \psi \) is a linear character, \( \psi(f^{q^e}) \) is a primitive \( t \)-th root of unity for some integer \( t \). Let \( t = q^d h \), \((q, h) = 1\). Then we can write \( \psi(f^{q^e}) = \zeta_{q^e}^{q^d} \zeta_h \), which implies that the order of \( f \) is divisible by \( q^{e+d} \). Consequently, \( q^{e+d} \) divides \( n \), and so a primitive \( q^{e+d} \)-th root of unity \( \zeta_{q^e+q^d} \) belongs to \( L \). We may assume that \( \zeta_{q^e+q^d} = \zeta_{q^d} \). Let \( r \) be an integer satisfying \( rq^e \equiv 1 \pmod{h} \). Since both \( \zeta_{q^e+q^d} \) and \( \zeta_h \) belong to \( L \), it follows that

\[
N_{L(\psi)/L}(\zeta_{q^e+q^d} \zeta_h^{q^e}) = \zeta_{q^e}^{q^d} \zeta^{q^e} \zeta_h = \zeta_{q^d} \zeta_h,
\]

which yields that \( (\psi(f^{q^e}), L(\psi)/L, \sigma) \sim L \). Therefore, the \( q \)-part of \( m_X(\chi) \) is equal to 1. As \( q \) is an arbitrary prime, it follows that \( m_X(\chi) = 1 \).

(v) Suppose that \( p \mid n \) and \( p \nmid 2 \). Then \( k \) contains a primitive \( p \)-th root of unity \( \zeta_p \), \( p \) being the rational prime divided by \( p \). It follows from [3, Satz 12] that \( m_X(\chi) = 1 \).

(vi) Suppose that \( p \mid n \) and \( p \mid 2 \). Then \( k = \mathbb{Q}(\zeta_n) \). If \( 4 \mid n \) then \( \zeta_t \in K \) and so \( m_X(\chi) = 1 \). If \( 4 \nmid n \), then \( 4 \nmid s \). It follows from the corollary that \( m_X(\chi) = 1 \).

The theorem is completely proved.

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* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

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