PERTURBATION THEORY FOR GENERALIZED FREDHOLM OPERATORS

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Successful development of the theory of Fredholm and semi-Fredholm operators and a general recognition of the importance of the subject provides an impetus for the study of various generalizations. In this paper, a study is made of operators whose ranges and null spaces are closed complemented subspaces. In particular, if \( T \) is such an operator, a rather general sufficient condition is obtained to ensure that \( T - U \) is of the same kind. This perturbation theorem includes, as special cases, previous results due to Dieudonné, Saphar, and Crownover.

1. Definitions and basic properties. If \( X \) is a Banach space, write \( B(X) \) to denote the space of continuous linear operators defined on \( X \). We shall study the class of operators \( T \) in \( B(X) \) which have the property that both the null space \( N(T) \) and the range \( R(T) \) are closed complemented subspaces of \( X \). This class of operators contains all the Fredholm operators and will therefore be called the generalized Fredholm operators, denoted \( GF(X) \). We observe immediately that when \( X \) is a Hilbert space, then \( GF(X) \) consists merely of those operators which have closed range and that in all cases, \( GF(X) \) contains all projections. Such an observation would discourage one's optimism about paralleling the theory of Fredholm operators to any reasonable extent. However, the fact that \( GF(X) \) has already been studied under another guise encourages further scrutiny; more precisely, we introduce the notion of generalized inverse as follows: an operator \( T \) in \( B(X) \) is said to have a generalized inverse \( S \) in \( B(X) \) if the following equations are valid:

\[
\begin{align*}
(1) & \quad TST = T \\
(2) & \quad STS = S.
\end{align*}
\]

The above notion has attracted a great deal of interest in the finite dimensional case and has been the subject of at least three recent monographs ([3], [8], [9]). It is perhaps appropriate to remark that conditions (1) and (2) are usually augmented by others leading to a variety of notions with a rather variable terminology. Some extensions to infinite dimensional situations have been considered by F. J. Beutler [2], and W. T. Reid [10] and others. Particular attention should be paid to the early paper of Atkinson [1] in which he calls operators satisfying (1) and (2) relatively regular and derives
many of their properties. The fact that generalized Fredholm operators are exactly those which have a generalized inverse has been known for a long time but since its proof is usually given in a finite dimensional setting, it is included here in the general case.

**Lemma.** $T$ is a generalized Fredholm operator if and only if $T$ has a generalized inverse.

**Proof.** Let $T$ belong to $GF(X)$ so that there exist closed subspaces $X_1$ and $X_2$ such that $X$ can be decomposed as $N(T) \oplus X_1$ and $R(T) \oplus X_2$. Let $P$ denote the projection onto $R(T)$ parallel to $X_2$. Now write $T_1$ for the restriction of $T$ to $X_1$ so that $T_1$ has a continuous inverse defined on $R(T)$. Therefore, $T_1^{-1}P$ is well defined in $B(X)$; it is easy to check that $T_1^{-1}P$ is a generalized inverse of $T$.

Conversely, if $S$ is a generalized inverse of $T$, then equations (1) and (2) imply that $TS$ is a projection of $X$ onto $R(T)$ parallel to $N(S)$ and $I - ST$ is a projection of $X$ onto $N(T)$ parallel to $R(S)$. Hence $T$ belongs to $GF(X)$.

2. Perturbation theory. In general, $GF(X)$ is not an open set nor is it stable under compact perturbations. To see this, we recall a construction due to R. J. Whitley ([7], V.2.6): if $T$ is an operator with closed range of infinite codimension and $N(T)$ is infinite dimensional, then it is possible to construct a compact operator $B$ such that $T + \lambda B$ does not have closed range for any $\lambda \neq 0$. The perturbation problem which we study therefore, gives a sufficient condition on the operators concerned and our result will include various interesting special cases including those of Saphar [11] and Crownover [4].

**Theorem.** Let $T$ be a generalized Fredholm operator defined on Banach space $X$ and suppose that $S$ is a generalized inverse of $T$. Let $U$ be an operator in $B(X)$ such that $\|U\| < \|S\|^{-1}$ and $(I - US)^{-1}U$ maps $N(T)$ into $R(T)$. Then $S(I - US)^{-1}$ and $(I - SU)^{-1}S$ are equal and their common value $V$ is a generalized inverse of $T - U$. Moreover, $N(T - U)$ is a complementary subspace to $R(S)$ and $R(T - U)$ is a complementary subspace to $N(S)$ so that $N(T - U)$ and $R(T - U)$ are linearly homeomorphic, respectively to $N(T)$ and $R(T)$.

**Proof.** Since $\|U\| < \|S\|^{-1}$, the Neumann series of $(I - US)^{-1}$ and $(I - SU)^{-1}$ converge and it is easy to see, by writing them out, that $S(I - US)^{-1}$ and $(I - SU)^{-1}S$ are equal. Then
\[ V(T - U)V = (I - SU)^{-1}S(T - U)S(I - US)^{-1} \]
\[ = (I - SU)^{-1}(STS - SUS)(I - US)^{-1} \]
\[ = (I - SU)^{-1}S(I - US)(I - US)^{-1} \]
\[ = V. \]

Now consider
\[ T - U - (T - U)V(T - U) = [I - (T - U)S(I - US)^{-1}](T - U) \]
\[ = [I - US - (T - U)S](I - US)^{-1}(T - U) \]
\[ = (I - TS)(I - US)^{-1}(T - UST + UST - U) \]
\[ = (I - TS)(I - US)^{-1}[I - US]T + U(ST - I)] \]
\[ = (I - TS)(I - US)^{-1}U(ST - I) \]
\[ = 0 \]
if \((I - US)^{-1}U\) maps \(R(ST - I)\) into \(N(I - TS)\). But, as noted in the proof of the Lemma, \(R(ST - I)\) is \(N(T)\) and \(N(I - TS)\) is \(R(T)\). Hence the first part of the proof is complete.

Now
\[ R(V) = R[S(I - US)^{-1}] = R(S) \]
so that
\[ X = N(T - U) \oplus R(V) = N(T - U) \oplus R(S). \]

Similarly
\[ N(V) = N[(I - SU)^{-1}] = N(S) \]
so that
\[ X = R(T - U) \oplus N(V) = R(T - U) \oplus N(S). \]

**Corollaries.** 1. If \(N(U) \supseteq N(T)\) or \(R(U) \subseteq R(T)\) with \(\|U\| \leq ||S||^{-1}\), then \(T - U\) is generalized Fredholm.

Proof. If \(N(U) \supseteq N(T)\), then \((I - US)^{-1}U\) maps \(N(T)\) onto \([0]\). If \(R(U) \subseteq R(T)\), then we observe that \((I - US)^{-1}U\) can be written as \(U(I - SU)^{-1}\) so that its range lies inside \(R(U)\) and hence it maps \(N(T)\) into \(R(T)\).

2. The class of left invertible operators is given by \(\{T \in GF(X): N(T) = [0]\}\); the class of right invertible operators is given by \(\{T \in GF(X): R(T) = X\}\). Hence from Corollary 1, we obtain the well-known result (Dieudonné [5]) that these classes are open sets in \(B(X)\).

3. Suppose \(T\) is a generalized Fredholm operator with \(R(T) \supseteq N(T)\). Let \(M\) be closed subspace of \(X\) such that \(R(T) \supseteq M \supseteq N(T)\)
and $TM = M$. Then if $UM \subseteq M$ and $||U|| < ||S||^{-1}$, the conclusions of the Theorem are valid.

Proof. $(TS)M = (TST)M = TM = M$ so that $SM \subseteq T^{-1}M$. Now if $m \in T^{-1}M$, then $Tm \in M = TM$ so that there exists $m' \in M$ such that $Tm = Tm'$. But $m - m' \in N(T) \subseteq M$ so that $m \in M$. Hence $T^{-1}M \subseteq M$. Therefore, $SM \subseteq M$ and so $(I - US)^{-1}UM \subseteq M$. Hence we can write

$$R[(I - TS)(I - US)^{-1}U(ST - I)] = (I - TS)(I - US)^{-1}UN(T) \subseteq (I - TS)(I - US)^{-1}UM \subseteq (I - TS)M \subseteq (I - TS)R(T) = \{0\}.$$ 

Hence the result.

4. If $R(T^n) \supseteq N(T)$ for each $n$ and all $R(T^n)$ are closed, then $M = \bigcap^n R(T^n)$ satisfies the conditions of Corollary 3. This gives Saphar's main result in [11] (Théorèmes 1 and 2).

5. If $N(T) = \{0\}$ and $R(T)$ has codimension 1 (and is therefore closed) and $\bigcap^n R(T^n) = \{0\}$, then Crownover [3] calls such an operator a shift on $X$. Clearly, for any such shift, all $R(T^n)$ are closed since $T$ is Fredholm. Therefore, shifts satisfy the condition of Corollary 4. Crownover's Theorem 2 ([4], p. 236) considers $T - \lambda I$ when $T$ is a shift and derives a result essentially the same as ours in this very special case.

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