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**SELF ADJOINT STRICTLY CYCLIC OPERATOR ALGEBRAS**

MARY RODRIGUEZ EMBRY

## SELF ADJOINT STRICTLY CYCLIC OPERATOR ALGEBRAS

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A *strictly cyclic operator algebra*  $\mathcal{A}$  on a Hilbert space  $X$  is a uniformly closed subalgebra of  $\mathcal{L}(X)$  such that  $\mathcal{A}x_0 = X$  for some  $x_0$  in  $X$ . In this paper it is shown that if  $\mathcal{A}$  is a strictly cyclic self-adjoint algebra, then (i) there exists a finite orthogonal decomposition of  $X$ ,  $X = \sum_{j=1}^n \oplus M_j$ , such that each  $M_j$  reduces  $\mathcal{A}$  and the restriction of  $\mathcal{A}$  to  $M_j$  is strongly dense in  $\mathcal{L}(M_j)$  and (ii) the commutant of  $\mathcal{A}$  is finite dimensional.

1. Notation and terminology. Throughout the paper  $X$  is a complex Hilbert space and  $\mathcal{L}(X)$  is the algebra of continuous linear operators on  $X$ .  $\mathcal{A}$  will denote a uniformly closed subalgebra of  $\mathcal{L}(X)$  which is *strictly cyclic* and  $x_0$  will be a *strictly cyclic vector* for  $\mathcal{A}$ : That is,  $\mathcal{A}x_0 = X$ . We do not insist that the identity element  $I$  of  $\mathcal{L}(X)$  be an element of  $\mathcal{A}$ . We say that  $\mathcal{A}$  is *self-adjoint* if  $A^* \in \mathcal{A}$  whenever  $A \in \mathcal{A}$ .

If  $\mathcal{B} \subset \mathcal{L}(X)$ , then the *commutant* of  $\mathcal{B}$  is  $\mathcal{B}' = \{E: E \in \mathcal{L}(X) \text{ and } EB = BE \text{ for all } B \text{ in } \mathcal{B}\}$ . A closed linear subspace  $M$  of  $X$  *reduces*  $\mathcal{B}$  if the projection of  $X$  onto  $M$  is in  $\mathcal{B}'$ . In this case  $M$  is a *minimal reducing subspace* of  $\mathcal{B}$  if  $M \neq \{\theta\}$  and  $\{\theta\}$  is the only reducing subspace of  $\mathcal{B}$  properly contained in  $M$ .

We say that a collection  $\{M_j\}_{j=1}^n$  of closed linear subspaces of  $X$  is an *orthogonal decomposition* of  $X$  if and only if the  $M_j$  are pairwise orthogonal and span  $X$ . A collection  $\{P_j\}_{j=1}^n$  of projections is a *resolution of identity* if and only if the collection  $\{P_j(X)\}_{j=1}^n$  of ranges of the  $P_j$  is an orthogonal decomposition of  $X$ .

2. Introduction. Strictly cyclic operator algebras have been studied by R. Bolstein, A. Lambert, the author of this paper and others. (See for example [1], [2], and [4].) In Lemma 1 of [1] Bolstein shows that if  $N$  is a normal operator on  $X$  and  $\{N\}'$  is strictly cyclic, then  $\{N\}''$  is finite dimensional. This raised questions about the nature of arbitrary self-adjoint, strictly cyclic operator algebras. In this paper we show that if  $\mathcal{A}$  is such an operator algebra, then there exists a finite orthogonal decomposition  $\{M_j\}$  of  $X$  such that each  $M_j$  reduces  $\mathcal{A}$  and  $\mathcal{A}|_{M_j}$  is strongly dense in  $\mathcal{L}(M_j)$ . From this it follows that  $\mathcal{A}'$  is finite dimensional; indeed we show that  $\mathcal{A}' = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$  (where  $P_j$  is the projection of  $X$  onto  $M_j$ ) and that for each  $j$  and  $k$ ,  $P_j \mathcal{A}' P_k$  is of dimension zero or one. If  $\mathcal{A}'$

is abelian, we are able to show more; namely that  $\mathcal{A}' = \{\sum_{j=1}^n \lambda_j P_j: \lambda_j \text{ complex}\}$ , giving us a complete generalization of Bolstein's result.

Each of the results mentioned above is a consequence of two basic facts concerning a self-adjoint strictly cyclic operator algebra  $\mathcal{A}$ : (1) (Lemma 1) each collection of pairwise orthogonal projections in  $\mathcal{A}'$  is finite and (2) (Theorems 1 and 2 of [3])  $\mathcal{A}$  has minimal reducing subspaces.

**3. Decomposition theorem.** The first lemma in this section demonstrates a very special characteristic of strictly cyclic operator algebras on a Hilbert space.

**LEMMA 1.** *Let  $\mathcal{A}$  be a strictly cyclic operator algebra on  $X$ . Each collection of mutually orthogonal projections in  $\mathcal{A}'$  is finite.*

*Proof.* Let  $\{P_j\}$  be a collection of mutually orthogonal projections in  $\mathcal{A}'$ . Without loss of generality we may assume that  $\{P_j\}$  is countable. Let  $Q_n = \sum_{j=1}^n P_j$  and note that  $Q_n$  converges strongly to  $Q = \sum_{j \geq 1} P_j$ . Thus by Lemma 2.1 in [2]  $Q_n$  converges uniformly to  $Q = \sum_{j \geq 1} P_j$ . However,  $Q - Q_n$  is a projection and hence has norm zero or one. Thus for  $n$  sufficiently large  $Q_n = Q$  and thus  $\{P_j\}$  is finite.

This lemma and its proof were suggested by Robert Kallman, University of Florida.

**COROLLARY 2.** *Let  $\mathcal{A}$  be a strictly cyclic operator algebra on  $X$ . Each normal element of  $\mathcal{A}'$  has finite spectrum.*

*Proof.* By Lemma 3.6 in [2] if  $E \in \mathcal{A}'$ , then  $E$  has no continuous spectrum. Thus if  $E$  is a normal element of  $\mathcal{A}'$ , the spectrum of  $E$  consists entirely of point spectrum and by Lemma 1  $E$  has only a finite number of distinct eigenspaces. Thus the spectrum of  $E$  is finite.

Corollary 2 was proven by R. Bolstein in [1] in the special case in which  $\mathcal{A}$  is the commutant of a normal operator  $N$ .

Before considering further the nature of the commutant of a self-adjoint, strictly cyclic operator algebra  $\mathcal{A}$ , we shall study the algebra  $\mathcal{A}$  itself.

**THEOREM 3.** *If  $\mathcal{A}$  is a self-adjoint strictly cyclic operator algebra on  $X$ , then there exists a finite orthogonal decomposition  $\{M_k\}_{k=1}^n$  of  $X$  such that each  $M_k$  reduces  $\mathcal{A}$  and  $\mathcal{A}|_{M_k}$  is strongly dense in  $\mathcal{L}(M_k)$ .*

*Proof.* By Theorem 1 of [3] if  $X$  and  $\{\theta\}$  are the only reducing

subspaces of  $\mathcal{A}$ , then  $\mathcal{A}$  is strongly dense in  $\mathcal{L}(X)$  and the trivial decomposition  $\{X\}$  of  $X$  satisfies the requirements of the theorem.

Assume that  $\{M_k\}_{k=1}^p$  is a collection of mutually orthogonal subspaces of  $X$  such that each  $M_k$  reduces  $\mathcal{A}$  and  $\mathcal{A}/M_k$  is strongly dense in  $\mathcal{L}(M_k)$ . If the  $M_k$  span  $X$ , the conclusion of the theorem is satisfied. Otherwise consider  $\mathcal{A}_1 = \mathcal{A}/\{M_1, \dots, M_p\}^\perp$ . If  $P$  is the orthogonal projection of  $X$  onto  $\{M_1, \dots, M_p\}^\perp$ , then  $P \in \mathcal{A}'$ , and if  $x_0$  is a strictly cyclic vector for  $\mathcal{A}$ , then  $\mathcal{A}_1 P x_0 = \mathcal{A} P x_0 = P \mathcal{A} x_0 = P(X) = \{M_1, \dots, M_p\}^\perp$ . Thus  $\mathcal{A}_1$  is strictly cyclic. Again by Theorem 1 of [3], if  $\mathcal{A}_1$  has only trivial reducing subspaces,  $\mathcal{A}_1$  is strongly dense in  $\mathcal{L}(\{M_1, \dots, M_p\}^\perp)$  and the construction is complete. Otherwise  $\mathcal{A}_1$  has a nontrivial reducing subspace. Then by Theorem 2 of [3]  $\mathcal{A}_1$  has a minimal reducing subspace  $M_{p+1}$  and by Theorem 3 of [3]  $\mathcal{A}_1/M_{p+1}$  is strongly dense in  $\mathcal{L}(M_{p+1})$ . Thus  $M_1, \dots, M_{p+1}$  are pairwise orthogonal reducing subspaces for  $\mathcal{A}$  and  $\mathcal{A}/M_k$  is strongly dense in  $\mathcal{L}(M_k)$  for  $k = 1, \dots, p + 1$ . By Lemma 1 the construction will terminate with a finite number of pairwise orthogonal reducing subspaces.

In view of Theorem 3 it is tempting to write  $\mathcal{A} = \bigoplus \sum_{k=1}^n \mathcal{L}(M_k)$ . However, this is misleading since  $\mathcal{A}$  may not be the full direct sum of the  $\mathcal{L}(M_k)$ . The following simple finite dimensional example demonstrates this:

$$\mathcal{A} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \text{ a } 2 \times 2 \text{ complex matrix} \right\}.$$

Here  $\mathcal{A}$  is a strictly cyclic self-adjoint operator algebra on  $\mathcal{E}^4$ .

We shall use the decomposition of  $\mathcal{A}$  developed in Theorem 3 to study the commutant of  $\mathcal{A}$ . It is worthwhile noting at this point that the decomposition in Theorem 3 may not be unique. We shall investigate this further in Corollary 7.

**THEOREM 4.** *Let  $\mathcal{A}$  be a self-adjoint strictly cyclic operator algebra and  $\{M_k\}_{k=1}^n$  a decomposition of  $X$  as required in Theorem 3. Let  $P_k$  be the orthogonal projection of  $X$  onto  $M_k$ . Then  $\mathcal{A}' = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$  and for each value of  $j$  and of  $k$ ,  $P_j \mathcal{A}' P_k$  is of dimension one or zero. In particular  $\mathcal{A}'$  is finite dimensional.*

*Proof.* We note that  $\sum_{k=1}^n P_k = I$  and that since  $M_k$  is a minimal reducing subspace of  $\mathcal{A}$ , then  $P_k$  is a minimal projection in  $\mathcal{A}'$ . Further  $\mathcal{A}' = (\sum_{j=1}^n P_j) \mathcal{A}' (\sum_{k=1}^n P_k) = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$ .

We first show that  $P_j \mathcal{A}' P_j = \{\lambda P_j\}$ . Assume that  $C = P_j \mathcal{A}' P_j$  is a projection. Note that  $C \in \mathcal{A}'$  and  $C = P_j C P_j \ll P_j$ . Thus since  $P_j$  is minimal, either  $C = 0$  or  $C = P_j$  and the only projections in  $P_j \mathcal{A}' P_j$

are 0 and  $P_j$ . Therefore  $P_j \mathcal{A}' P_j = \{\lambda P_j\}$ .

Secondly we show that either  $P_j \mathcal{A}' P_k = 0$  or  $P_j \mathcal{A}' P_k = \{\lambda U_{jk}\}$  where  $U_{jk}$  is the partial isometry with initial set  $P_k(X)$  and final set  $P_j(X)$ . Let  $F = P_j E P_k$ ,  $E \in \mathcal{A}'$ . Then  $FF^* \in P_j \mathcal{A}' P_j$  and hence by the preceding paragraph  $FF^* = \lambda P_j$  for some complex  $\lambda$ . Therefore,  $FF^*F = \lambda F$ . If  $P_j \mathcal{A}' P_k \neq 0$ , then some  $F \neq 0$ . Since  $FF^*F = \lambda F = \lambda P_j E P_k$ ,  $F$  is a scalar multiple of the partial isometry with initial set  $P_k(X)$  and final set  $P_j(X)$ .

The proof of Theorem 4 was provided by T. Hoover.

**COROLLARY 5.** *If  $\mathcal{A}$  is a self-adjoint strictly cyclic operator algebra with an abelian commutant, then  $\mathcal{A}' = \{\sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex}\}$  where  $\{P_j\}$  is a resolution of identity as required in Theorem 4. In particular  $\mathcal{A}'$  consists of normal operators with finite spectra.*

*Proof.* By Theorem 4  $\mathcal{A}' = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$ . Thus if  $\mathcal{A}'$  is abelian,  $\mathcal{A}' = \sum_{j=1}^n P_j \mathcal{A}' P_j$ . Moreover, by Theorem 4,  $P_j \mathcal{A}' P_j = \{\lambda_j P_j; \lambda_j \text{ complex}\}$ .

The following corollary due to Bolstein, inspired the ideas which have been developed in this paper. The techniques used by Bolstein in [1] to arrive at this result differ radically from those used in this paper.

**COROLLARY 6.** (Bolstein) *Let  $N$  be a normal operator with a strictly cyclic commutant  $\{N\}'$ . Then there exist orthogonal projections  $P_1, \dots, P_n$  such that*

$$\{N\}'' = \left\{ \sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex} \right\}.$$

*Proof.* By the Fuglede theorem  $\{N\}'$  is self-adjoint. Thus since  $\{N\}''$  is abelian, we can apply Corollary 5.

We return now to the question of the uniqueness of the decomposition  $\{M_k\}_{k=1}^n$  in Theorem 3 or equivalently the uniqueness of a resolution of identity  $\{P_k\}_{k=1}^n$  in  $\mathcal{A}'$ , consisting of minimal projections.

**COROLLARY 7.** *The decomposition  $\{M_k\}_{k=1}^n$  in Theorem 3 is unique if and only if  $\mathcal{A}'$  is abelian.*

*Proof.* Assume first that  $\mathcal{A}'$  is abelian. By Corollary 5  $\mathcal{A}' = \{\sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex}\}$ . If  $Q$  is any projection in  $\mathcal{A}'$ ,  $QP_j = P_j Q$  for each  $j$ . Hence  $QP_j$  is a projection and since  $P_j$  is minimal, either

$QP_j = 0$  or  $QP_j = P_j$ . Therefore, if  $Q$  is a minimal projection in  $\mathcal{A}'$ , or equivalently  $Q(X)$  is a minimal reducing subspace of  $X$ , then  $Q = P_j$  for some  $j$ . Thus the decomposition  $\{M_k\}_{k=1}^n$  is unique.

Now assume that the decomposition  $\{M_k\}_{k=1}^n$  of Theorem 3 is unique. Let  $P$  be any nonzero projection in  $\mathcal{A}'$  and  $P_0$  a minimal projection dominated by  $P$ . Since the decomposition is unique, necessarily  $P_0(X) = M_k$  for some  $k$ . Consequently  $P = \sum_{j=1}^n \lambda_j P_j$  where  $\lambda_j$  is zero or one. Thus all projections (and hence all elements) in  $\mathcal{A}'$  commute.

In conclusion we note that if  $\mathcal{A}$  is an arbitrary strictly cyclic operator algebra on  $X$ , then  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$  where  $\mathcal{A}_1$  is self-adjoint strictly cyclic and  $\mathcal{A}_2$  is strictly cyclic but has no reducing subspaces on which it is self-adjoint. To see this we argue as follows: Let  $\mathcal{H}$  be the class of all reducing subspaces of  $\mathcal{A}$  on which  $\mathcal{A}$  is self-adjoint. Order  $\mathcal{H}$  by inclusion and note that Lemma 1 implies that any linearly ordered subset of  $\mathcal{H}$  is finite. Thus the Maximal Principle can be applied and there exists a maximal reducing subspace  $M$  such that  $\mathcal{A}/M$  is self-adjoint. Finally if  $x_0$  is a strictly cyclic vector for  $\mathcal{A}$  and  $P$  the projection of  $X$  onto  $M$ , then  $Px_0$  is a strictly cyclic vector for  $\mathcal{A}/M$ .

ADDENDUM. The referee kindly pointed out that Rickart (Section 3, pp. 622-623, of "The uniqueness of norm problems in Banach spaces", *Annals of Mathematics*, 51 (1950), 615-628) showed that the commutant of a strictly cyclic transitive algebra consists only of scalars and that the algebra is  $n$ -transitive for every  $n$ . Thus  $\mathcal{A}$  is strongly dense in  $\mathcal{L}(X)$ . These facts make it unnecessary to quote Theorem 1 of [3] in the proof of Theorem 3 of this paper.

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