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SELF ADJOINT STRICTLY CYCLIC OPERATOR ALGEBRAS

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A *strictly cyclic* operator algebra \mathcal{A} on a Hilbert space X is a uniformly closed subalgebra of $\mathcal{L}(X)$ such that $\mathcal{A}x_0 = X$ for some x_0 in X . In this paper it is shown that if \mathcal{A} is a strictly cyclic self-adjoint algebra, then (i) there exists a finite orthogonal decomposition of X , $X = \sum_{j=1}^n \oplus M_j$, such that each M_j reduces \mathcal{A} and the restriction of \mathcal{A} to M_j is strongly dense in $\mathcal{L}(M_j)$ and (ii) the commutant of \mathcal{A} is finite dimensional.

1. Notation and terminology. Throughout the paper X is a complex Hilbert space and $\mathcal{L}(X)$ is the algebra of continuous linear operators on X . \mathcal{A} will denote a uniformly closed subalgebra of $\mathcal{L}(X)$ which is *strictly cyclic* and x_0 will be a *strictly cyclic vector* for \mathcal{A} : That is, $\mathcal{A}x_0 = X$. We do not insist that the identity element I of $\mathcal{L}(X)$ be an element of \mathcal{A} . We say that \mathcal{A} is *self-adjoint* if $A^* \in \mathcal{A}$ whenever $A \in \mathcal{A}$.

If $\mathcal{B} \subset \mathcal{L}(X)$, then the *commutant* of \mathcal{B} is $\mathcal{B}' = \{E: E \in \mathcal{L}(X) \text{ and } EB = BE \text{ for all } B \text{ in } \mathcal{B}\}$. A closed linear subspace M of X *reduces* \mathcal{B} if the projection of X onto M is in \mathcal{B}' . In this case M is a *minimal reducing subspace* of \mathcal{B} if $M \neq \{\theta\}$ and $\{\theta\}$ is the only reducing subspace of \mathcal{B} properly contained in M .

We say that a collection $\{M_j\}_{j=1}^n$ of closed linear subspaces of X is an *orthogonal decomposition* of X if and only if the M_j are pairwise orthogonal and span X . A collection $\{P_j\}_{j=1}^n$ of projections is a *resolution of identity* if and only if the collection $\{P_j(X)\}_{j=1}^n$ of ranges of the P_j is an orthogonal decomposition of X .

2. Introduction. Strictly cyclic operator algebras have been studied by R. Bolstein, A. Lambert, the author of this paper and others. (See for example [1], [2], and [4].) In Lemma 1 of [1] Bolstein shows that if N is a normal operator on X and $\{N\}'$ is strictly cyclic, then $\{N\}''$ is finite dimensional. This raised questions about the nature of arbitrary self-adjoint, strictly cyclic operator algebras. In this paper we show that if \mathcal{A} is such an operator algebra, then there exists a finite orthogonal decomposition $\{M_j\}$ of X such that each M_j reduces \mathcal{A} and \mathcal{A}/M_j is strongly dense in $\mathcal{L}(M_j)$. From this it follows that \mathcal{A}' is finite dimensional; indeed we show that $\mathcal{A}' = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$ (where P_j is the projection of X onto M_j) and that for each j and k , $P_j \mathcal{A}' P_k$ is of dimension zero or one. If \mathcal{A}'

is abelian, we are able to show more; namely that $\mathcal{A}' = \{\sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex}\}$, giving us a complete generalization of Bolstein's result.

Each of the results mentioned above is a consequence of two basic facts concerning a self-adjoint strictly cyclic operator algebra \mathcal{A} : (1) (Lemma 1) each collection of pairwise orthogonal projections in \mathcal{A} is finite and (2) (Theorems 1 and 2 of [3]) \mathcal{A} has minimal reducing subspaces.

3. Decomposition theorem. The first lemma in this section demonstrates a very special characteristic of strictly cyclic operator algebras on a Hilbert space.

LEMMA 1. *Let \mathcal{A} be a strictly cyclic operator algebra on X . Each collection of mutually orthogonal projections in \mathcal{A} is finite.*

Proof. Let $\{P_j\}$ be a collection of mutually orthogonal projections in \mathcal{A}' . Without loss of generality we may assume that $\{P_j\}$ is countable. Let $Q_n = \sum_{j=1}^n P_j$ and note that Q_n converges strongly to $Q = \sum_{j \geq 1} P_j$. Thus by Lemma 2.1 in [2] Q_n converges uniformly to $Q = \sum_{j \geq 1} P_j$. However, $Q - Q_n$ is a projection and hence has norm zero or one. Thus for n sufficiently large $Q_n = Q$ and thus $\{P_j\}$ is finite.

This lemma and its proof were suggested by Robert Kallman, University of Florida.

COROLLARY 2. *Let \mathcal{A} be a strictly cyclic operator algebra on X . Each normal element of \mathcal{A}' has finite spectrum.*

Proof. By Lemma 3.6 in [2] if $E \in \mathcal{A}'$, then E has no continuous spectrum. Thus if E is a normal element of \mathcal{A}' , the spectrum of E consists entirely of point spectrum and by Lemma 1 E has only a finite number of distinct eigenspaces. Thus the spectrum of E is finite.

Corollary 2 was proven by R. Bolstein in [1] in the special case in which \mathcal{A} is the commutant of a normal operator N .

Before considering further the nature of the commutant of a self-adjoint, strictly cyclic operator algebra \mathcal{A} , we shall study the algebra \mathcal{A} itself.

THEOREM 3. *If \mathcal{A} is a self-adjoint strictly cyclic operator algebra on X , then there exists a finite orthogonal decomposition $\{M_k\}_{k=1}^n$ of X such that each M_k reduces \mathcal{A} and \mathcal{A}/M_k is strongly dense in $\mathcal{L}(M_k)$.*

Proof. By Theorem 1 of [3] if X and $\{\theta\}$ are the only reducing

subspaces of \mathcal{A} , then \mathcal{A} is strongly dense in $\mathcal{L}(X)$ and the trivial decomposition $\{X\}$ of X satisfies the requirements of the theorem.

Assume that $\{M_k\}_{k=1}^p$ is a collection of mutually orthogonal subspaces of X such that each M_k reduces \mathcal{A} and \mathcal{A}/M_k is strongly dense in $\mathcal{L}(M_k)$. If the M_k span X , the conclusion of the theorem is satisfied. Otherwise consider $\mathcal{A}_1 = \mathcal{A}/\{M_1, \dots, M_p\}^\perp$. If P is the orthogonal projection of X onto $\{M_1, \dots, M_p\}^\perp$, then $P \in \mathcal{A}'$, and if x_0 is a strictly cyclic vector for \mathcal{A} , then $\mathcal{A}_1 P x_0 = \mathcal{A} P x_0 = P \mathcal{A} x_0 = P(X) = \{M_1, \dots, M_p\}^\perp$. Thus \mathcal{A}_1 is strictly cyclic. Again by Theorem 1 of [3], if \mathcal{A}_1 has only trivial reducing subspaces, \mathcal{A}_1 is strongly dense in $\mathcal{L}(\{M_1, \dots, M_p\}^\perp)$ and the construction is complete. Otherwise \mathcal{A}_1 has a nontrivial reducing subspace. Then by Theorem 2 of [3] \mathcal{A}_1 has a minimal reducing subspace M_{p+1} and by Theorem 3 of [3] \mathcal{A}/M_{p+1} is strongly dense in $\mathcal{L}(M_{p+1})$. Thus M_1, \dots, M_{p+1} are pairwise orthogonal reducing subspaces for \mathcal{A} and \mathcal{A}/M_k is strongly dense in $\mathcal{L}(M_k)$ for $k = 1, \dots, p + 1$. By Lemma 1 the construction will terminate with a finite number of pairwise orthogonal reducing subspaces.

In view of Theorem 3 it is tempting to write $\mathcal{A} = \bigoplus \sum_{k=1}^n \mathcal{L}(M_k)$. However, this is misleading since \mathcal{A} may not be the full direct sum of the $\mathcal{L}(M_k)$. The following simple finite dimensional example demonstrates this:

$$\mathcal{A} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \text{ a } 2 \times 2 \text{ complex matrix} \right\}.$$

Here \mathcal{A} is a strictly cyclic self-adjoint operator algebra on \mathcal{C}^4 .

We shall use the decomposition of \mathcal{A} developed in Theorem 3 to study the commutant of \mathcal{A} . It is worthwhile noting at this point that the decomposition in Theorem 3 may not be unique. We shall investigate this further in Corollary 7.

THEOREM 4. *Let \mathcal{A} be a self-adjoint strictly cyclic operator algebra and $\{M_k\}_{k=1}^n$ a decomposition of X as required in Theorem 3. Let P_k be the orthogonal projection of X onto M_k . Then $\mathcal{A}' = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$ and for each value of j and of k , $P_j \mathcal{A}' P_k$ is of dimension one or zero. In particular \mathcal{A}' is finite dimensional.*

Proof. We note that $\sum_{k=1}^n P_k = I$ and that since M_k is a minimal reducing subspace of \mathcal{A} , then P_k is a minimal projection in \mathcal{A}' . Further $\mathcal{A}' = (\sum_{j=1}^n P_j) \mathcal{A}' (\sum_{k=1}^n P_k) = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$.

We first show that $P_j \mathcal{A}' P_j = \{\lambda P_j\}$. Assume that $C = P_j \mathcal{A}' P_j$ is a projection. Note that $C \in \mathcal{A}'$ and $C = P_j C P_j \ll P_j$. Thus since P_j is minimal, either $C = 0$ or $C = P_j$ and the only projections in $P_j \mathcal{A}' P_j$

are 0 and P_j . Therefore $P_j \mathcal{A}' P_j = \{\lambda P_j\}$.

Secondly we show that either $P_j \mathcal{A}' P_k = 0$ or $P_j \mathcal{A}' P_k = \{\lambda U_{jk}\}$ where U_{jk} is the partial isometry with initial set $P_k(X)$ and final set $P_j(X)$. Let $F = P_j E P_k$, $E \in \mathcal{A}'$. Then $FF^* \in P_j \mathcal{A}' P_j$ and hence by the preceding paragraph $FF^* = \lambda P_j$ for some complex λ . Therefore, $FF^*F = \lambda F$. If $P_j \mathcal{A}' P_k \neq 0$, then some $F \neq 0$. Since $FF^*F = \lambda F = \lambda P_j E P_k$, F is a scalar multiple of the partial isometry with initial set $P_k(X)$ and final set $P_j(X)$.

The proof of Theorem 4 was provided by T. Hoover.

COROLLARY 5. *If \mathcal{A} is a self-adjoint strictly cyclic operator algebra with an abelian commutant, then $\mathcal{A}' = \{\sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex}\}$ where $\{P_j\}$ is a resolution of identity as required in Theorem 4. In particular \mathcal{A}' consists of normal operators with finite spectra.*

Proof. By Theorem 4 $\mathcal{A}' = \sum_{j,k=1}^n P_j \mathcal{A}' P_k$. Thus if \mathcal{A}' is abelian, $\mathcal{A}' = \sum_{j=1}^n P_j \mathcal{A}' P_j$. Moreover, by Theorem 4, $P_j \mathcal{A}' P_j = \{\lambda_j P_j; \lambda_j \text{ complex}\}$.

The following corollary due to Bolstein, inspired the ideas which have been developed in this paper. The techniques used by Bolstein in [1] to arrive at this result differ radically from those used in this paper.

COROLLARY 6. (Bolstein) *Let N be a normal operator with a strictly cyclic commutant $\{N\}'$. Then there exist orthogonal projections P_1, \dots, P_n such that*

$$\{N\}'' = \left\{ \sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex} \right\}.$$

Proof. By the Fuglede theorem $\{N\}'$ is self-adjoint. Thus since $\{N\}''$ is abelian, we can apply Corollary 5.

We return now to the question of the uniqueness of the decomposition $\{M_k\}_{k=1}^n$ in Theorem 3 or equivalently the uniqueness of a resolution of identity $\{P_k\}_{k=1}^n$ in \mathcal{A}' , consisting of minimal projections.

COROLLARY 7. *The decomposition $\{M_k\}_{k=1}^n$ in Theorem 3 is unique if and only if \mathcal{A}' is abelian.*

Proof. Assume first that \mathcal{A}' is abelian. By Corollary 5 $\mathcal{A}' = \{\sum_{j=1}^n \lambda_j P_j; \lambda_j \text{ complex}\}$. If Q is any projection in \mathcal{A}' , $QP_j = P_j Q$ for each j . Hence QP_j is a projection and since P_j is minimal, either

$QP_j = 0$ or $QP_j = P_j$. Therefore, if Q is a minimal projection in \mathcal{A}' , or equivalently $Q(X)$ is a minimal reducing subspace of X , then $Q = P_j$ for some j . Thus the decomposition $\{M_k\}_{k=1}^n$ is unique.

Now assume that the decomposition $\{M_k\}_{k=1}^n$ of Theorem 3 is unique. Let P be any nonzero projection in \mathcal{A}' and P_0 a minimal projection dominated by P . Since the decomposition is unique, necessarily $P_0(X) = M_k$ for some k . Consequently $P = \sum_{j=1}^n \lambda_j P_j$ where λ_j is zero or one. Thus all projections (and hence all elements) in \mathcal{A}' commute.

In conclusion we note that if \mathcal{A} is an arbitrary strictly cyclic operator algebra on X , then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ where \mathcal{A}_1 is self-adjoint strictly cyclic and \mathcal{A}_2 is strictly cyclic but has no reducing subspaces on which it is self-adjoint. To see this we argue as follows: Let \mathcal{H} be the class of all reducing subspaces of \mathcal{A} on which \mathcal{A} is self-adjoint. Order \mathcal{H} by inclusion and note that Lemma 1 implies that any linearly ordered subset of \mathcal{H} is finite. Thus the Maximal Principle can be applied and there exists a maximal reducing subspace M such that \mathcal{A}/M is self-adjoint. Finally if x_0 is a strictly cyclic vector for \mathcal{A} and P the projection of X onto M , then Px_0 is a strictly cyclic vector for \mathcal{A}/M .

ADDENDUM. The referee kindly pointed out that Rickart (Section 3, pp. 622-623, of "The uniqueness of norm problems in Banach spaces", *Annals of Mathematics*, 51 (1950), 615-628) showed that the commutant of a strictly cyclic transitive algebra consists only of scalars and that the algebra is n -transitive for every n . Thus \mathcal{A} is strongly dense in $\mathcal{L}(X)$. These facts make it unnecessary to quote Theorem 1 of [3] in the proof of Theorem 3 of this paper.

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