LOCAL COMPACTNESS OF FAMILIES OF CONTINUOUS POINT-COMPACT RELATIONS

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The purpose of this paper is to prove that the pointwise closure of an equicontinuous family of point-compact relations from a compact Hausdorff space to a locally compact Hausdorff uniform space is locally compact in the topology of uniform convergence. This is a generalization of a recent result of R. V. Fuller.

1. Introduction. The purpose of this paper is to study conditions under which a family of continuous point-compact relations is locally compact. Our theorem generalizes a theorem of Fuller [2]. For the most part we use the concepts and results of Smithson ([5], [6], [7]), and Michael [4].

We use the term relation where other authors use multivalued function or multifunction. If \( F \) is a relation from \( X \) to \( Y \) and \( B \subset Y \), we write

\[
F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.
\]

A relation \( F \) from a topological space \( X \) to a topological space \( Y \) is called continuous iff

(a) \( F^{-1}(A) \) is closed in \( X \) whenever \( A \) is closed in \( Y \), and

(b) \( F^{-1}(B) \) is open in \( X \) whenever \( B \) is open in \( Y \).

\( F \) is point-closed (respectively point-compact) iff \( F(x) \) is closed (respectively compact) for each \( x \in X \).

We recall three topologies defined on the collection of all non-empty subsets of a topological space \( X \) (see Michael [4]). The collection of all sets of the form \( \{B \subset X : B \subset U\} \) where \( U \) is open in \( X \), is a base for the upper semi-finite (u.s.f.) topology. The collection of all sets of the form \( \{B \subset X : B \cap U \neq \emptyset\} \) where \( U \) is open in \( X \), is a subbase for the lower semi-finite (l.s.f.) topology. The finite topology is the supremum of the l.s.f. and u.s.f. topologies. Equivalently, the finite topology has as a basis all sets of the form \( \langle U_i, \cdots, U_n \rangle = \{A \subset X : A \cap U_i \neq \emptyset, 1 \leq i \leq n \text{ and } A \subset \bigcup_{i=1}^n U_i\} \), where \( U_i, \cdots, U_n \) are open in \( X \). A relation \( F : X \to Y \) is continuous if and only if the function \( F : X \to P(Y) \) (the power set of \( Y \) with the finite topology) is continuous, c.f. remark following Theorem 2.7 of [7].

Let \( X \) and \( Y \) be topological spaces and let \( \mathcal{F} \) be a set of relations from \( X \) to \( Y \). The pointwise topology \( \mathcal{P} \) [7] on \( \mathcal{F} \) has a subbase consisting of the sets of the form \( \{F \in \mathcal{F} : F(x) \cap U \neq \emptyset\} \) or \( \{F \in \mathcal{F} : F(x) \subset V\} \) where \( x \in X \), and \( U, V \) are open in \( Y \). We
note that the projections \( \{\Pi_x: x \in X\} \) defined by \( \Pi_x(F) = F(x) \) are continuous functions into the collection of all nonempty subsets of \( Y \) with the finite topology.

Let \( X \) be a topological space, \((Y, \mathcal{U})\) a uniform space and let \( \mathcal{F} \) be a set of relations from \( X \) to \( Y \). For \( V \in \mathcal{U} \), let \( \mathcal{W}(V) = \{(F, G) \in \mathcal{F} \times \mathcal{F} : \text{for all } x \in X, (y, G(x)) \cap V \neq \emptyset \text{ for all } y \in F(x), \text{ and } (F(x), y') \cap V \neq \emptyset \text{ for all } y' \in G(x)\} \). Let \( \mathcal{W} \) be the uniformity on \( \mathcal{F} \) generated by the collection of all such encourages as \( \mathcal{W}(V) \). The topology generated by \( \mathcal{W} \) is the topology of uniform convergence \([5]\) and is denoted by \( \mathcal{U}_C \). If \((Y, \mathcal{U})\) is a uniform space, \( \mathcal{F} \) is called equicompact at \( x \in X \) \([5]\) iff for every \( V \in \mathcal{U} \) there is a nbhd. \( U \) of \( x \) such that for all \( F \in \mathcal{F} \),

\[
\begin{align*}
(a) & \quad F(U) \subseteq V(F(x)), \text{ and} \\
(b) & \quad F(z) \cap V(y) \neq \emptyset \text{ for all } z \in U \text{ and for all } y \in F(x).
\end{align*}
\]

We now state a theorem of Smithson which we use in the final section.

**Theorem 1.1.** \([5]\). If \( \mathcal{F} \) is an equicompact family of point-compact relations from a compact space \( X \) to a uniform space \( Y \), then on \( \mathcal{F}, \mathcal{P} = \mathcal{U}_C \).

For further details and a survey the reader is referred to Smithson \([7]\).

2. Local compactness of a space of relations. We begin by proving two lemmas.

**Lemma 2.1.** Let \( F \) be a point-compact relation from \( X \) to \( Y \), and let \( A = \{([x], F(x)) : x \in X\} \) be compact in \( P(X) \times P(Y) \), where each of \( P(X), P(Y) \) has the finite topology. Then \( F \) is a compact subset of \( X \times Y \).

**Proof.** Let \( \mathcal{O} \) be an open cover of \( F \) in \( X \times Y \). For each \( x \in X \), \([x] \times F(x)\) is compact; so there is a finite subcollection \( V_i^x \times U_i^x \), \( 1 \leq i \leq n \) of \( \mathcal{O} \) which covers the set. We can assume that \( x \in V_i^x \) for each \( i \) and that \( F(x) \cap U_i^x \neq \emptyset \). \( \langle V_1^x, \ldots, V_n^x \rangle \times \langle U_1^x, \ldots, U_n^x \rangle \) is an open set in \( P(X) \times P(Y) \). For each \( x \in X \), we obtain such a set, and this leads to an open cover of \( A \). Since \( A \) is compact, there is a finite subcover \( \langle V_1^z, \ldots, V_k^z \rangle \times \langle U_1^z, \ldots, U_k^z \rangle, 1 \leq i \leq k \). Finally, \( \{V_j^x \cap U_i^x : 1 \leq j \leq n, 1 \leq i \leq k\} \) is a cover of \( F \) and so \( F \) is compact.

**Lemma 2.2.** If \( F \) is a continuous relation from \( X \) to \( Y \), then the function \( g: X \to P(X) \times P(Y) \) defined by \( g(x) = ([x], F(x)) \) is continuous. \((P(X) \text{ and } P(Y) \text{ both have finite topology.})\)
Proof. Let \( \langle V \rangle \times \langle U_1, \ldots, U_n \rangle \) be a basic open nbhd. of \( \{x\}, F(x) \). The function \( F: X \to P(Y) \) is continuous, hence there exists a nbhd. \( N \subset V \) of \( x \) such that \( F(N) \subset \langle U_1, \ldots, U_n \rangle \). Clearly, \( g(N) \subset \langle V \rangle \times \langle U_1, \ldots, U_n \rangle \).

From the above lemmas it follows that if \( X \) is compact, \( Y \) is \( T_2 \) and \( F \) is a continuous point-compact relation, then \( F \) is a compact subset of \( X \times Y \).

The proof of the following lemma is straightforward.

**Lemma 2.3.** Let \( \mathcal{F} \) be a family of relations from a topological space \( X \) to a topological space \( Y \). Then the l.s.f. topology on \( \mathcal{F} \) is contained in \( \mathcal{P} \).

**Lemma 2.4.** Let \( X \) be compact Hausdorff, \( (Y, \mathscr{V}) \) a uniform space, and \( \mathcal{F} \) a family of continuous point-compact relations from \( X \) to \( Y \). Then on \( \mathcal{F} \) the u.s.f. topology is smaller than \( \mathcal{U} \subset \mathcal{E} \).

**Proof.** If \( F \in \mathcal{F} \), then \( F \) is a compact subset of \( X \times Y \). Suppose \( F \subset N \) is an open subset of \( X \times Y \). Let \( \mathcal{U} \) be the (unique) uniformity on \( X \). Then from [3] page 199, it follows that there exist \( U \in \mathcal{U}, V \in \mathcal{F} \) such that \( F \subset \bigcup \{U(x) \times V(y) : x \in X, y \in F(x)\} \subset N \). Then \( W(V)[F] \subset N \), thus completing the proof.

**Lemma 2.5.** Let \( \mathcal{F} \) be an equicontinuous family of relations from a \( T_1 \)-space \( X \) to a uniform space \( (Y, \mathscr{V}) \). Then on \( \mathcal{F}, \mathcal{P} \subset \mathcal{F} \) the finite topology.

**Proof.** Let \( [x, U] = \{G \in \mathcal{F} : G(x) \subset U\} \), where \( x \in X \) and \( U \) is open in \( Y \). If \( F \in [x, U] \), then \( N = \langle X \times U \cup (X - \{x\}) \times Y \rangle \) is a nbhd. of \( F \) in the finite topology, and \( F \subset N \subset [x, U] \). Suppose \( F \in M = \{G \in \mathcal{F} : G(x) \cap W \neq \emptyset\} \) where \( x \in X \) and \( W \) is open in \( Y \). If \( F(x) \subset W \), then the above method works, and so we assume that \( F(x) \not\subset W \). Let \( p \in F(x) \cap W \) and let \( V \in \mathcal{V} \) such that \( \overline{V(p)} \subset W \). Since \( \mathcal{F} \) is equicontinuous at \( x \), there is a nbhd. \( U \) of \( x \) such that for all \( T \in \mathcal{F}, T(U) \subset V(T(x)) \). Now \( F \in [U \times \{V(p)\}], U \times (Y - \overline{V(p)}), U \times W, (X - \{x\}) \times Y \rangle \subset M \), which completes the proof.

**Lemma 2.6.** Let \( X \) be compact Hausdorff, and let \( (Y, \mathcal{V}) \) be a uniform space. Let \( \hat{\mathcal{F}} \) be an equicontinuous family of point-compact relations from \( X \) to \( Y \). If \( \hat{\mathcal{F}} \) is the \( \mathcal{P} \)-closure of \( \mathcal{F} \) in the space of all point-compact relations from \( X \) to \( Y \), then \( \hat{\mathcal{F}} \) is closed in \( \mathcal{E}(X \times Y) \), the space of all nonempty compact subsets of \( X \times Y \) with the finite topology.
Proof. Let \((F_a)\) be a net in \(\mathcal{F}\) converging to \(F \in \mathcal{C}(X \times Y)\). Note that the domain of \(F\) is \(X\); for if \(x \in X - \text{dom } F\), then \(\{G \in \mathcal{C}(X \times Y) : G \subset X \times Y - \{x\} \times Y\}\) is an u.s.f. nbhd. of \(F\) in \(\mathcal{C}(X \times Y)\), and \(F_a\) is eventually in this nbhd., a contradiction. Clearly \(F\) is point-compact. We now show that \((F_a) \rightarrow F\) in \(\mathcal{P}\). Let \(U\) be open in \(Y\), and suppose \(F(x) \subset U\). Then the set \(N = \langle X \times U \cup (X - \{x\}) \times Y \rangle\) is a nbhd. of \(F\) in the finite topology on \(\mathcal{C}(X \times Y)\). Since \(F_a\) is eventually in \(N\), it follows that \(F_a\) is eventually in \([x, U]\). If we are given a nbhd. \(M = \{G : G(x) \cap W = \emptyset\}\), \((W \text{ open in } Y)\) of \(F\) and \(F(x) \not\subset W\), then we employ the technique used in the last part of the proof of Lemma 2.5, and use the fact that \(\mathcal{F}\) is equicontinuous ([5], Lemma 6).

We now prove the main result.

THEOREM 2.7. Let \(\mathcal{F}\) be an equicontinuous family of point-compact relations from a compact Hausdorff space \(X\) to a locally compact Hausdorff uniform space \(Y\). Let \(\mathcal{F}\) be the \(\mathcal{P}\)-closure of \(\mathcal{F}\) in the space of all point-compact relations from \(X\) to \(Y\). Then \(\mathcal{F}\) is locally compact in \(\mathcal{U} \mathcal{C}\).

Proof. We first note that on \(\mathcal{F}\), \(\mathcal{P} \subset \mathcal{U} \mathcal{C}\) ([5], Lemma 1). From Lemmas 2.3 and 2.4 it follows that the finite topology is contained in \(\mathcal{U} \mathcal{C}\), and since \(\mathcal{F}\) is equicontinuous, we have by Theorem 1.1, \(\mathcal{P} = \mathcal{U} \mathcal{C}\) on \(\mathcal{F}\). From Lemma 2.5, it follows that the finite topology equals \(\mathcal{U} \mathcal{C}\) on \(\mathcal{F}\). Each member of \(\mathcal{F}\) is compact, and so \(\mathcal{F} \subset \mathcal{C}(X \times Y)\). By Lemma 2.6, \(\mathcal{F}\) is closed in the finite topology on \(\mathcal{C}(X \times Y)\). Since \(\mathcal{C}(X \times Y)\) is locally compact ([4], Prop. 4.4.1), \(\mathcal{F}\) is locally compact.

If in the above theorem, each \(F \in \mathcal{F}\) is a (single valued) function, then it is easy to verify that each member of \(\mathcal{F}\) is also a function. Hence a recent result of R. V. Fuller [2] on the local compactness of \(\mathcal{F}\) is a special case of the above theorem.

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